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THE VARIATIONAL PARABOLIC INEQUALITY OF HIGHER ORDER IN AN UNBOUNDED DOMAIN. UNIQUENESS

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In this paper we consider the variational parabolic inequality for the operator $u_{tt} + A_3 u + A_4 u_t + g(u_t)$ in unbounded domain, where A_3 is a linear elliptic operator of the fourth order and A_4 is a linear elliptic operator of the second order. We obtain some conditions of the uniqueness of solution. We prove the uniqueness using the method of introducing a parameter.

Л. Заремба. *Вариационное параболическое неравенство высшего порядка в неограниченной области. Единственность* // Математичні Студії. – 2004. – Т.22, №1. – С.57–66.

В статье рассматривается вариационное параболическое неравенство для оператора $u_{tt} + A_3 u + A_4 u_t + g(u_t)$ в неограниченной области, где A_3 — линейный эллиптический оператор четвертого порядка, а A_4 — линейный эллиптический оператор второго порядка. Используя метод введения параметра, получены некоторые условия единственности решения.

The variational parabolic inequality for the operator $u_t + Au$ in an unbounded domain, where A is a linear elliptic operator of the second order, has been first considered in the paper [1]. The existence and uniqueness of a solution of this inequality has been shown in the class of functions which do not grow faster than the function $e^{a|x|}$, $a > 0$, for $|x| \to \infty$. In the paper [2], the author has considered a variational inequality for the operator $u_t + Au + |u|^{p-2}u$, $p > 1$, in an unbounded domain. In particular, she has proved uniqueness for $p \in (1, 2]$ in the Tichonov class of functions E_2, namely in the class of functions which grow for $|x| \to \infty$ not faster than $e^{a|x|^2}$.

The variational inequalities for the operators $u_{tt} + A_1 u$ in bounded domains have been considered in the papers [3] and [4]. The authors have showed some conditions of existence and uniqueness of the solution of the variational parabolic inequality for the operator $u_{tt} + A_1 u + A_2 u_t$. Here, A_1, A_2 are linear elliptic operators of the high order.

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain and $\partial\Omega \in C^1$, $\Omega^R = \Omega \cap B_R$ be a domain for all $R > 0$, where $B_R = \{x \in \mathbb{R}^n, |x| < R\}$. By Γ_1^R we denote the set $\Gamma_1^R = \partial\Omega \cap \partial\Omega^R$.
Let
$$H^{2,0}(\Omega^R) = \Big\{ u \in H^2(\Omega^R) :\ u\big|_{\Gamma_1^R} = 0,\ \frac{\partial u}{\partial v}\Big|_{\Gamma_1^R} = 0 \Big\},$$

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$$H^{2,0}_{\mathrm{loc}}(\bar\Omega) = \bigcap_{R>0} H^{2,0}(\Omega^R), \quad H^2_{\mathrm{loc}}(\bar\Omega) = \bigcap_{R>0} H^2(\Omega^R),$$

$$L^r_{\mathrm{loc}}(\bar\Omega) = \bigcap_{R>0} L^r(\Omega^R), \quad r \in (1, +\infty).$$

The space V is such that the inclusions $H^{2,0}_{\mathrm{loc}}(\bar\Omega) \subset V \subset H^2_{\mathrm{loc}}(\bar\Omega)$ are continuous. Moreover, let K be a closed and convex subset of the set V such that $0 \in K$.

Let us denote

$$Q_\tau = \Omega \times (0,\tau),\ \tau \in (0,T],\ T>0, \quad Q^R_\tau = \Omega^R \times (0,\tau).$$

We shall consider the inequality of the form

$$\int_{Q_\tau} \Big[u_{tt}(v-u_t)\Psi(x) + \sum_{i,j,k,l=1}^n a^{kl}_{ij}(x,t) u_{x_i x_j}((v-u_t)\Psi(x))_{x_k x_l} + \\ + \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}((v-u_t)\Psi(x))_{x_j} + \\ + a(x,t)u(v-u_t)\Psi(x) + \sum_{i,j=1}^n b_{ij}(x,t) u_{tx_i}((v-u_t)\Psi(x))_{x_j} + \\ + g(x,t,u_t)(v-u_t)\Psi(x) - f(x,t)(v-u_t)\Psi(x) \Big] dxdt \ge 0, \quad \tau \in (0,T]. \tag{1}$$

For this inequality we put the following initial conditions

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x). \tag{2}$$

For inequality (1) we give the following system of assumptions:

(A₁) $a^{kl}_{ij}, a^{kl}_{jit} \in L^\infty(Q_T)$; $a^{kl}_{ij}(x,t) = a^{ij}_{kl}(x,t)$ for almost all $(x,t) \in Q_T$;

$$\sum_{i,j,k,l=1}^n a^{kl}_{ij}(x,t)\xi_{ij}\xi_{kl} \ge a_2 \sum_{i,j=1}^n \xi^2_{ij}$$

for almost all $(x,t) \in Q_T$ and for all $\xi \in \mathbb{R}^{n(n-1)/2}$ and $a_2 > 0$ is a constant;

(A₂) $a_{ij}, a_{ijx_i}, a_{ijt} \in L^\infty(Q_T)$, $i,j \in \{1,...,n\}$; $a_{ij}(x,t) = a_{ji}(x,t)$ for almost all $(x,t) \in Q_T$;

$$\sum_{ij=1}^n a_{ij}(x,t)\xi_i\xi_j \ge 0 \quad \text{and} \quad \sum_{ij=1}^n a_{ijt}(x,t)\xi_i\xi_j \le 0$$

for almost all $(x,t) \in Q_T$ and for all $\xi \in \mathbb{R}^n$;

(A₃) $a \in L^\infty(Q_T)$, $a(x,t) \ge a_0 > 0$ for almost all $(x,t) \in Q_T$, where a_0 is a constant;

(B) $b_{ij} \in L^\infty(Q_T)$,

$$\sum_{ij=1}^n b_{ij}(x,t)\xi_i\xi_j \ge b_0 \sum_{i=1}^n \xi_i^2$$

for almost all $(x,t) \in Q_T$ and for all $\xi \in \mathbb{R}^n$ and $b_0 > 0$ is a constant;

(G) The function $(x,t) \to g(x,t,\xi)$ is continuous for every $\xi \in \mathbb{R}$, the function $\xi \to g(x,t,\xi)$ is measurable for almost all $(x,t) \in Q_T$ and satisfies the following inequalities: $(g(x,t,\xi) - g(x,t,\mu))(\xi - \mu) \geq 0$ for almost all $(x,t) \in Q_T$ and for all $\xi, \mu \in \mathbb{R}$; $|g(x,t,\xi)| \leq g_1|\xi|^{p-1}$, $p > 1$ for almost all $(x,t) \in Q_T$ and for all $\xi \in \mathbb{R}$;

(F) $f \in L^{p'}\big((0,T);\, L^{p'}_{\mathrm{loc}}(\bar{\Omega})\big)$, $p' = \frac{p}{p-1}$;

(U) $u_0 \in V$, $u_1 \in K$.

Definition. We call the function u a *solution of Problem (1)–(2)* if

$$u \in L^2\big((0,T);V\big), \quad u_{tt} \in L^2\big((0,T);L^2_{\mathrm{loc}}(\overline{\Omega})\big), \quad u_t \in L^2\big((0,T);V\big) \cap L^p\big((0,T);L^p_{\mathrm{loc}}(\overline{\Omega})\big),$$

and $u_t \in K$ for almost all $t \in (0,T)$. Moreover, u satisfies inequality (1) and conditions (2) for all $v \in L^2\big((0,T);V\big) \cap L^p\big((0,T);L^p_{\mathrm{loc}}(\overline{\Omega})\big)$ such that $v \in K$ for almost all $t \in (0,T)$ and for all $\Psi \in C_0^2(\mathbb{R}^n)$ such that $\Psi(x) \geq 0$ in R^n and for all $\tau \in (0,T]$.

Theorem. *If the conditions* $(\mathbf{A_1})$–$(\mathbf{A_3})$, **(B), (G), (F), (U)** *hold, then Problem (1)–(2) has at most one solution in the class of the functions u such that*

$$\int\limits_{Q_T^R} \left[u^2 + \sum_{i,j=1}^{n} u^2_{x_i x_j} + u_t^2\right] dx dt \leq e^{aR^2}$$

for all $R > 0$, where $a > 0$ is a constant.

Proof. To obtain a contradiction, suppose that there exist two solutions u^1, u^2 of inequality (1) such that $u^1 \neq u^2$. Then the following inequalities are satisfied

$$\begin{aligned}\int\limits_{Q_\tau} \Bigg[& u^s_{tt}(v - u^s_t)\Psi(x) + \sum_{i,j,k,l=1}^{n} a^{kl}_{ij}(x,t) u^s_{x_i x_j}((v - u^s_t)\Psi(x))_{x_k x_l} + \\ & + \sum_{i,j=1}^{n} a_{ij}(x,t) u^s_{x_i}((v - u^s_t)\Psi(x))_{x_j} + a(x,t) u^s (v - u^s_t)\Psi(x) + \\ & + \sum_{i,j=1}^{n} b_{ij}(x,t) u^s_{t x_i}(x,t)((v - u^s_t)\Psi(x))_{x_j} + \\ & + g(x,t,u^s_t)(v - u^s_t)\Psi(x) - f(x,t)(v - u^s_t)\Psi(x)\Bigg] dx dt \geq 0, \quad s \in \{1,2\},\end{aligned} \tag{3}$$

in particular for the function

$$v = \frac{1}{2}(u^1_t + u^2_t)e^{-\gamma t}, \ \gamma > 0. \tag{4}$$

Adding (3) for $s = 1$ and $s = 2$ and choosing v of form (4) we obtain the following inequality

$$\begin{aligned}\int\limits_{Q_\tau} \Bigg[& u_{tt} u_t \Psi(x) + \sum_{i,j,k,l=1}^{n} a^{kl}_{ij}(x,t) u_{x_i x_j}(u_t \Psi(x))_{x_k x_l} + \\ & + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}(u_t\Psi(x))_{x_j} + a(x,t) u u_t \Psi(x) + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{t x_i}(u_t \Psi(x))_{x_j} + \\ & + (g(x,t,u^2_t) - g(x,t,u^1_t)) u_t \Psi(x)\Bigg] e^{-\gamma t} dx dt \leq 0,\end{aligned} \tag{5}$$

where $u = u^2 - u^1$, $\tau \in (0, T]$ and

$$u(x,0) = 0, \; u_t(x,0) = 0. \tag{6}$$

From initial conditions (6) we obtain

$$I_1 := \int_{Q_\tau} u_{tt} u_t \Psi(x) e^{-\gamma t} dxdt =$$
$$= \frac{\gamma}{2} \int_{Q_\tau} |u_t|^2 e^{-\gamma t} \Psi(x) dxdt + \frac{1}{2} \int_{\Omega_\tau} |u_t|^2 e^{-\gamma\tau} \Psi(x) dx.$$

Taking into account the assumptions of the theorem and the form of the function v we obtain the following equality

$$I_2 := \int_{Q_\tau} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{x_i x_j} (u_t \Psi(x))_{x_k x_l} e^{-\gamma t} dxdt = I_2^1 + I_2^2 + I_2^3,$$

where

$$I_2^1 = \int_{Q_\tau} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{x_i x_j}, u_{t x_k x_l} e^{-\gamma t} \Psi(x) dxdt =$$
$$= \frac{1}{2} \int_{\Omega_\tau} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{x_i x_j}, u_{x_k x_l} e^{-\gamma\tau} \Psi(x) dx +$$
$$+ \frac{\gamma}{2} \int_{Q_\tau} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{x_i x_j}, u_{x_k x_l} e^{-\gamma t} \Psi(x) dxdt -$$
$$- \frac{1}{2} \int_{Q_\tau} \sum_{i,j,k,l=1}^{n} a_{ijt}^{kl}(x,t) u_{x_i x_j}, u_{x_k x_l} e^{-\gamma t} \Psi(x) dxdt,$$

with $\Omega_\tau = Q_\tau \cap \{t = \tau\}$. From $(\mathbf{A_1})$ there exists a constant a_3 such that the following inequality

$$\sum_{i,j,k,l=1}^{n} a_{ijt}^{kl} \xi_{ij} \xi_{kl} \le a_3 \sum_{i,j=1}^{n} \xi_{ij}^2$$

is satisfied for almost all $(x,t) \in Q_T$ and for all $\xi \in \mathbb{R}^{\frac{n(n+1)}{2}}$. Then from $(\mathbf{A_1})$

$$I_2^1 \ge \frac{1}{2} a_2 \int_{\Omega_\tau} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 e^{-\gamma\tau} \Psi(x) dx +$$
$$+ \frac{\gamma}{2} a_2 \int_{Q_\tau} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 e^{-\gamma t} \Psi(x) dxdt - \frac{1}{2} a_3 \int_{Q_\tau} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 e^{-\gamma t} \Psi(x) dxdt.$$

Next

$$I_2^3 := \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{x_i x_j}, u_t e^{-\gamma t} \Psi_{x_k x_l} dx dt \le$$

$$\le \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^{n} \left[\delta_2 (a_{ij}^{kl})^2 |u_{x_i x_j}|^2 \Psi(x) + + \frac{1}{\delta_2} |u_t|^2 \frac{(\Psi_{x_k x_l})^2}{\Psi} \right] e^{-\gamma t} dx dt \le$$

$$\le \frac{\delta_2 a_2^0 n^2}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 e^{-\gamma t} \Psi(x) dx dt + + \frac{n^4 \mu_0}{2\delta_2} \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dx dt,$$

where $\Psi(x) = [\phi_R(x)]^\beta$,

$$\phi_R(x) = \begin{cases} \frac{1}{R}(R^2 - |x|^2), & \text{for } 0 \le |x| \le R, \\ 0, & \text{for } |x| > R, \end{cases}$$

and μ_0 is a constant such that $|\Psi_{x_i x_j}| \le \mu_0 (\phi_R)^{\beta-2}$;

$$I_2^2 := \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{x_i x_j} \Big(u_{t x_k} \Psi_{x_l} + u_{t x_l} \Psi_{x_k} \Big) e^{-\gamma t} dx dt \le$$

$$\le \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^{n} \left[\frac{a_2^0}{2\delta_3} |u_{x_i x_j}|^2 \left(\frac{\Psi_{x_l}^2}{\Psi} + \frac{\Psi_{x_k}^2}{\Psi} \right) + \frac{\delta_3}{2} (|u_{t x_k}|^2 + |u_{t x_l}|^2) \Psi \right] e^{-\gamma t} dx dt \le$$

$$\le n^3 \delta_3 \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{t x_i}|^2 e^{-\gamma t} \Psi(x) dx dt + \frac{4\beta^2 a_2^0}{\delta_3} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} (u_{x_i x_j})^2 (\phi_R)^{\beta-2} e^{-\gamma t} dx dt.$$

From the assumption ($\mathbf{A_2}$) and the initial condition we obtain

$$I_3 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} (u_t \Psi(x))_{x_j} e^{-\gamma t} dx dt = I_3^1 + I_3^2,$$

where

$$I_3^1 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u_{x_i}, u_{t x_j} e^{-\gamma t} \Psi(x) dx dt =$$

$$= \frac{1}{2} \int\limits_{\Omega_\tau} \sum_{i,j=1}^{n} (a_{ij} u_{x_i} u_{x_j} e^{-\gamma t} \Psi(x))_t dx + \frac{\gamma}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} e^{-\gamma t} \Psi(x) dx -$$

$$- \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ijt} u_{x_i} u_{x_j} e^{-\gamma t} \Psi(x) dx = \int\limits_{\Omega_\tau} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} e^{-\gamma \tau} \Psi(x) dx +$$

$$+ \frac{\gamma}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} e^{-\gamma t} \Psi(x) dx - \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ijt} u_{x_i} u_{x_j} e^{-\gamma t} \Psi(x) dx \ge 0$$

and

$$I_3^2 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u_{x_i}, u_t e^{-\gamma t} \Psi_{x_j} dxdt =$$

$$= \int\limits_0^\tau \sum_{i,j=1}^{n} (a_{ij} u, u_t e^{-\gamma t} \Psi_{x_j})_{x_i} dt - \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ijx_i} u u_t e^{-\gamma t} \Psi_{x_j} dxdt -$$

$$- \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u u_{tx_i} e^{-\gamma t} \Psi_{x_j} dxdt - \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u u_t e^{-\gamma t} \Psi_{x_i x_j} dxdt \leq$$

$$\leq \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[\delta_4 |u|^2 \Psi + \frac{a_{ijx_i}}{\delta_4} |u_t|^2 \frac{(\Psi_{x_j})^2}{\Psi} \right] e^{-\gamma t} dxdt +$$

$$+ \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[\delta_5 |u_{tx_i}|^2 \Psi + \frac{(a_{ij})^2}{\delta_5} |u|^2 \frac{(\Psi_{x_j})^2}{\Psi} \right] e^{-\gamma t} dxdt +$$

$$+ \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[(a_{ij})^2 |u_t|^2 |\Psi_{x_i x_j}| + |u|^2 |\Psi_{x_i x_j}|^2 \right] e^{-\gamma t} dxdt \leq$$

$$\leq \frac{\beta^2 a_1^1 n^2}{\delta_4} \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta - 2} e^{-\gamma t} dxdt + \frac{1}{2} n^2 \delta_4 \int\limits_{Q_\tau} |u|^2 \Psi e^{-\gamma t} dxdt +$$

$$+ \frac{\beta^2 a_1^0 n^2}{\delta_5} \int\limits_{Q_\tau} |u|^2 (\phi_R)^{\beta - 2} e^{-\gamma t} dxdt + \frac{1}{2} n \delta_5 \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{tx_i}|^2 \Psi e^{-\gamma t} dxdt +$$

$$+ \frac{a_1^0 n^2 \mu_0}{2} \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta - 2} e^{-\gamma t} dxdt + \frac{n^2 \mu_0}{2} \int\limits_{Q_\tau} |u|^2 (\phi_R)^{\beta - 2} e^{-\gamma t} dxdt.$$

Next, from $(\mathbf{A_3})$ and the initial condition we have

$$I_4 := \int\limits_{Q_\tau} a(x,t) u u_t e^{-\gamma t} \Psi dxdt = \frac{1}{2} \int\limits_{\Omega_\tau} (a(u)^2 \Psi e^{-\gamma t})_t dx +$$

$$+ \frac{\gamma}{2} \int\limits_{Q_\tau} a(x,t) |u|^2 \Psi e^{-\gamma t} dxdt - \frac{1}{2} \int\limits_{Q_\tau} a_t(x,t) |u|^2 \Psi e^{-\gamma t} dxdt \geq$$

$$\geq \frac{1}{2} a_0 \int\limits_{\Omega_\tau} |u|^2 \Psi e^{-\gamma \tau} dx + \frac{\gamma}{2} a_0 \int\limits_{Q_\tau} |u|^2 \Psi e^{-\gamma t} dxdt - \frac{1}{2} a_1 \int\limits_{Q_\tau} |u|^2 \Psi e^{-\gamma t} dxdt,$$

where $a_1 = \operatorname{ess\,sup}_{Q_T} a_t(x,t)$. From **(B)** we have

$$I_5 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} b_{ij}(x,t) u_{tx_i} (u_t \Psi(x))_{x_j} dxdt = I_5^1 + I_5^2,$$

where

$$I_5^1 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} u_{tx_j} e^{-\gamma t} \Psi(x) dxdt \geq b_0 \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{tx_i}|^2 e^{-\gamma t} \Psi(x) dx$$

and

$$I_5^2 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} u_t e^{-\gamma t} \Psi_{x_j} dxdt \leq$$

$$\leq \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[\delta_0 (b_{ij})^2 |u_{tx_i}|^2 \Psi + \frac{1}{\delta_0} |u_t|^2 \frac{(\Psi_{x_j})^2}{\Psi} \right] e^{-\gamma t} dxdt \leq$$

$$\leq \frac{\delta_0 b^0 n}{2} \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{tx_i}|^2 e^{-\gamma t} \Psi(x) dxdt + \frac{n\beta^2 2}{\delta_0} \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt.$$

Moreover, from the condition **(G)** we obtain

$$I_6 = \int\limits_{Q_\tau} (g(x,t,u_t^2) - g(x,t,u_t^1) u_t \Psi e^{-\gamma t} dxdt \geq 0.$$

From the estimates of the integrals I_1–I_6 and (5) we have

$$\frac{1}{2} \int\limits_{\Omega_\tau} |u_t|^2 \Psi e^{-\gamma \tau} dx + \frac{a_2}{2} \int\limits_{\Omega_\tau} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 \Psi e^{-\gamma \tau} dx +$$

$$+ \frac{a_0}{2} \int\limits_{\Omega_\tau} |u|^2 \Psi e^{-\gamma \tau} dx + \frac{\gamma}{2} \int\limits_{Q_\tau} |u_t|^2 \Psi e^{-\gamma t} dxdt +$$

$$+ \left[\frac{\gamma a_2}{2} - \frac{a_3}{2} - \frac{\delta_2 a_2^0 n^2}{2} \right] \int\limits_{Q_\tau} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 \Psi e^{-\gamma t} dxdt +$$

$$+ \left[b_0 - n^3 \delta_3 - \frac{b^0 \delta_0 n}{2} - \frac{\delta_5 n}{2} \right] \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{tx_i}|^2 \Psi e^{-\gamma t} dxdt + \tag{7}$$

$$+ \left[\frac{\gamma a_0}{2} - \frac{a_1}{2} - \frac{\delta_4 n^2}{2} \right] \int\limits_{Q_\tau} |u|^2 \Psi e^{-\gamma t} dxdt \leq$$

$$\leq \left[\frac{n^4 \mu_0}{2\delta_2} + \frac{2\beta^2 n}{\delta_0} + \frac{2n a_1^1 \beta^2}{\delta_4} + \frac{n^2 a_1^0 \mu_0}{2} \right] \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt +$$

$$+ \left[\frac{n^2 \mu_0}{2} + \frac{2 a_1^0 n \beta^2}{\delta_5} \right] \int\limits_{Q_\tau} |u|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt + \frac{4 a_2^0 \beta^2}{\delta_3} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt.$$

Let R_1 be fixed and $R_1 < R$. Now we choose

$$\gamma = \gamma_0 + \gamma_1,\ \delta_2 = a_3/(a_2^0 n^2),\ \delta_4 = a_1 n^{-2},\ \gamma_1 = \max\{2a_1/a_0, 2a_3/a_2\},$$

$$\delta_0 = 2b_0/(3b^0 n),\ \delta_3 = b_0/(3n^3),\ \delta_5 = 2b_0/(3n).$$

From (7) we obtain the following inequality

$$\begin{aligned} \gamma_0 \int\limits_{Q_\tau^{R_1}} \sum_{i,j=1}^{n} \Big[|u_t|^2 + |u_{x_i x_j}|^2 + |u|^2 \Big] \Psi e^{-\gamma_0 t} dx dt \leq \\ \leq K R^{\beta-2} \int\limits_{Q_\tau^{R}} \sum_{i,j=1}^{n} \Big[|u_t|^2 + |u_{x_i x_j}|^2 + |u|^2 \Big] e^{-\gamma_0 t} dx dt. \end{aligned} \tag{8}$$

Then from (8) we have

$$\gamma_0 \int\limits_{Q_\tau^{R_1}} w e^{-\gamma_0 t} \Psi dx dt \leq K R^{\beta-2} \int\limits_{Q_\tau^{R}} w e^{-\gamma_0 t} dx dt, \tag{9}$$

where

$$w = \sum_{ij=1}^{n} \big[|u_t|^2 + |u_{x_i x_j}|^2 + |u|^2 \big].$$

In Ω^{R_1}

$$\Psi(x) = [\phi_R(x)]^\beta = \left[\frac{1}{R}(R - |x|)(R + |x|) \right]^\beta \geq (R - R_1)^\beta. \tag{10}$$

Using (10) we obtain from (9)

$$\gamma_0 e^{-\gamma_0 \tau} (R - R_1)^\beta \int\limits_{Q_\tau^{R_1}} w dx dt \leq K R^{\beta-2} \int\limits_{Q_\tau^{R}} w dx dt$$

or

$$\int\limits_{Q_\tau^{R_1}} w dx dt \leq \frac{1}{\gamma_0} \left(\frac{R}{R - R_1} \right)^\beta \frac{e^{\gamma_0 \tau}}{R^2} \int\limits_{Q_\tau^{R}} w dx dt. \tag{11}$$

Since $\lim\limits_{R\to\infty} \frac{R}{R-R_1} = 1$, we get from (11)

$$\int\limits_{Q_\tau^{R_1}} w dx dt \leq \frac{K}{\gamma_0} \frac{1}{R^2} e^{\gamma_0 \tau} \int\limits_{Q_\tau^{R}} w dx dt. \tag{12}$$

Let $R = R_2$. Then

$$\int\limits_{Q_\tau^{R_1}} w dx dt \leq \frac{K}{\gamma_0} \frac{1}{(R_2 - R_1)^2} e^{\gamma_0 \tau} \int\limits_{Q_\tau^{R_2}} w dx dt. \tag{13}$$

Moreover, let

$$\rho_k = (R_2 - R_1)/k,\ k \in \mathbb{N},\ R_1(s) = R_1 + s\rho_k,\ R_2(s) = R_1(s) + \rho_k,\ s \in \{0, 1, ..., k-1\}.$$

Then inequality (13) for $R_1(s)$, $R_2(s)$ and ρ_k will be in the form

$$\int\limits_{Q_\tau^{R_1(s)}} w dx dt \le \frac{K}{\gamma_0}\frac{1}{\rho_k^2} e^{\gamma_0 \tau} \int\limits_{Q_\tau^{R_2(s)}} w dx dt.1 \tag{14}$$

From the form of $R_1(s)$, $R_2(s)$, ρ_k and (14) we have

$$\int\limits_{Q_\tau^{R_1}} w dx dt \le \left(\frac{K}{\gamma_0 \rho_k^2}\right)^k e^{\gamma_0 \tau} \int\limits_{Q_\tau^{R_2}} w dx dt. \tag{15}$$

Choosing K and γ_0 such that $\left(\frac{K}{\gamma_0 \rho_k^2}\right)^k < e^{-1}$, putting the constants κ, λ, a, b_0 such that

$$R_1 = 2^m,\ R_2 = 2^{m+1},\ m \in N, \quad \kappa = \lambda 2^{m+1},\ \lambda = 2 + [a],\ \gamma = b_0 \lambda^2 2^{m+1},\ b_0 = 8K \cdot e,$$

and assuming that

$$\int\limits_{Q_\tau^{R_2}} w dx dt \le e^{a R_2^2},$$

we obtain from (15)

$$\int\limits_{Q_\tau^{R_1}} w dx dt \le e^{(-\kappa + \gamma_0 \tau_0)} \int\limits_{Q_\tau^{R_2}} w dx dt \le e^{(-\kappa + \gamma_0 \tau_0 + a R_2^2)}.$$

Since

$$-\kappa + \gamma_0 \tau_0 + a R_2^2 = (-2 + a - [a] + b_0 \lambda^2 \tau_0) 2^{m+1} \le (-1 + b_0 \lambda^2 \tau_0) 2^{m+1},$$

for $\tau_0 = \min\{T, \frac{1}{2b_0}\lambda^2\}$ it follows that

$$\int\limits_{Q_{\tau_0}^{R_1}} w dx dt \le e^{-2^{m+1}}.$$

Hence for $m \to +\infty$, $w(x,t) = 0$ almost everywhere in Q_{τ_0}. If $\tau_0 < T$, then by analogy we can prove that $w(x,t) = 0$ almost everywhere in $Q_{\tau_0, 2\tau_0}$ e.t.c.
Hence $u(x,t) = u^1(x,t) - u^2(x,t) = 0$. □

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