УДК 517.5

I. I. Marchenko, A. Szkibiel

ON STRONG TRACTS OF SUBHARMONIC FUNCTIONS OF FINITE LOWER ORDER

I. I. Marchenko, A. Szkibiel. *On strong tracts of subharmonic functions of finite lower order*, Matematychni Studii, **22** (2004) 35–44.

We define the notion of a strong asymptotic tract for subharmonic function of finite lower order λ. We estimate the number of strong tracts using Petrenko's magnitude of the deviation from ∞ of a subharmonic function $u(z)$. The estimates in the paper are exact.

И. И. Марченко, А. Шкибель. *О строгих трактах субгармонических функций конечного нижнего порядка* // Математичні Студії. – 2004. – Т.22, №1. – С.35–44.

Введено понятие строгого асимптотического тракта для субгармонических функций конечного нижнего порядка λ и оценено число строгих трактов, используя величину отклонения Петренко от ∞ субгармонической функции $u(z)$. Оценки, полученные в статье, являются точными.

Let $u(z)$ be a subharmonic function in the plane of finite lower order λ, where

$$\lambda = \liminf_{r\to\infty} \frac{\log \max\{u(z) : \ |z| = r\}}{\log r}.$$

Consider the sets $\mathrm{E}(n) = \{z \in \mathbb{C} : \ u(z) \geqslant n\}$ for $n \in \mathbb{N}$. Let $\mathrm{C}(n)$ be a thick component of $\mathrm{E}(n)$, which means that $u(z) \not\equiv n$ on $\mathrm{C}(n)$ (see [1]). There is $k \in \mathbb{N}$ such that, for every $n \geqslant k$, the function $u(z)$ is unbounded on all thick components $\mathrm{C}(n)$. Let $n_2 > n_1 > k$. Then every thick component $\mathrm{C}(n_1)$ contains at least one thick component $\mathrm{C}(n_2)$. If $\mathrm{P}(n)$ is the number of different sets $\mathrm{C}(n)$, then $\mathrm{P}(n_2) \geqslant \mathrm{P}(n_1)$. We define

$$p = \lim_{n\to\infty} \mathrm{P}(n)$$

and call it the *number of tracts of the function* $u(z)$. If $\lambda < \infty$, then $p < \infty$ (see [1]). Hence, there is n_0, such that, for all $n \geqslant n_0$, $\mathrm{P}(n) = p$. We consider the set $\mathrm{E}(n_0)$ and its components $\mathrm{C}_j(n_0)$ for $j \in \{1, 2, \ldots, p\}$.

If $\mathrm{C}_j(n_0)$ is a tract of $u(z)$, then there exists (see [2]) a continuous curve $\Gamma_j \subset \mathrm{C}_j(n_0)$ given by the equation $z = z(t)$, where $0 \leqslant t < \infty$ and $z(t) \to \infty$ as $t \to \infty$, such that

$$\lim_{\substack{z\to\infty \\ z\in\Gamma_j}} u(z) = \lim_{t\to\infty} u(z(t)) = \infty.$$

2000 *Mathematics Subject Classification:* 31A05, 30D20.

Definition 1. We call the tract $\mathrm{C}(n_0)$ of the subharmonic function $u(z)$ *a strong tract* if there is a continuous curve $\Gamma \subset \mathrm{C}(n_0): \ z = z(t)$, $0 \leqslant t < \infty$, such that $z(t) \to \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{u(z(t))}{\max\{u(z): \ |z| = |z(t)|\}} = 1.$$

We are going to estimate the number of strong tracts of a subharmonic function of finite lower order λ.

In order to state the main result of the paper, we use *Petrenko's magnitude of the deviation from* ∞ of a subharmonic function $u(z)$ as

$$\beta(\infty, u) = \liminf_{r\to\infty} \frac{\max\{u(z): \ |z| = r\}}{T(r,u)},$$

where $T(r,u)$ is the Nevanlinna characteristics of the subharmonic function $u(z)$ (see [1]).

Theorem A [3]. *For a subharmonic function of finite lower order* λ *we have*

$$\beta(\infty, u) \leqslant \begin{cases} \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } \lambda \leqslant 0.5, \\ \pi\lambda & \text{if } \lambda > 0.5. \end{cases}$$

The exact estimate of $\beta(\infty, u)$ for meromorphic functions was obtained by Petrenko in 1969 ([4]).

The paper presents the proof of the following result.

Theorem. *Let* $u(z)$ *be a subharmonic function of finite lower order* λ *and* p *strong tracts. Then*

$$p \leqslant \begin{cases} \max\left\{1, \left[\frac{\pi\lambda\sin\pi\lambda}{\beta(\infty,u)}\right]\right\} & \text{if } \lambda \leqslant 0.5, \\ \left[\frac{\pi\lambda}{\beta(\infty,u)}\right] & \text{if } \lambda > 0.5. \end{cases}$$

In the case $u(z) = \log|f(z)|$, where $f(z)$ is an entire function, the theorem was proved by one of the authors ([5]). The estimates of Theorem are exact, see [5].

Corollary. *For a subharmonic function* $u(z)$ *of finite lower order* λ *we have*

$$\beta(\infty, u) \leqslant \begin{cases} \frac{\pi\lambda}{p} & \text{if } \lambda > 0.5, \\ \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } \lambda \leqslant 0.5, \ p = 1, \\ \frac{\pi\lambda\sin\pi\lambda}{p} & \text{if } \lambda \leqslant 0.5, \ p > 1, \end{cases}$$

where p *is the number of strong tracts of* $u(z)$.

1. Auxiliary results. Let $u(z)$ be a subharmonic function and let p be the number of tracts of $u(z)$. Let n_0 be such that, for all $n \geqslant n_0$, $\mathrm{E}(n)$ has exactly p connected components. Consider the functions (see [1])

$$u_j(z) = \begin{cases} u(z) & \text{if } z \in C_j(n_0) \\ n_0 & \text{if } z \notin C_j(n_0). \end{cases}$$

for $j \in \{1, 2, \ldots, p\}$. The functions $u_j(z)$ are subharmonic in $\mathbb{C}$. Also, for every $j \in \{1, 2, \ldots, p\}$, $u_j(z)$ is the pointwise limit of a nonincreasing sequence $\{v_j^k(z)\}$ of subharmonic functions with continuous second partial derivatives [1].

Now, we define the functions (see [6])

$$m^*(r, \theta, u_j) = \frac{1}{2\pi} \sup_{|E|=2\theta} \int\limits_E u_j(re^{i\varphi}) \mathrm{d}\varphi,$$

where $|E|$ is the Lebesgue measure of the set E and

$$m_0^*(r, \theta, u) = \sum_{j=1}^{p} m^*(r, \theta, u_j).$$

According to Baernstein's Theorem (see [6]), the functions $m^*(r, \theta, u_j)$ are subharmonic in $K = \{re^{i\theta} : 0 < r < \infty,\ 0 < \theta < \pi\}$, continuous on $K \cup (-\infty, 0) \cup (0, +\infty)$ and convex in $\log r$ for any fixed $\theta \in [0, \pi]$. Also the functions defined above are nondecreasing with respect to r for any fixed θ. Hence the function $m_0^*(r, \theta, u)$, as the sum of $m^*(r, \theta, u_j)$'s, has the same properties. Moreover

$$m^*(r, \theta, u_j) = \frac{1}{\pi} \int\limits_0^\theta \widetilde{u_j}(re^{i\varphi}) \mathrm{d}\varphi, \qquad \frac{\partial m^*}{\partial \theta}(r, \theta, u_j) = \frac{1}{\pi} \widetilde{u_j}(re^{i\theta}), \quad 0 < \theta < \pi,$$

where $\widetilde{u_j}(re^{i\theta})$ is the circular rearrangement of the function $u_j(re^{i\theta})$ [7].

Lemma 1. *The sequence* $\{m^*(r, \theta, v_j^k)\}$ *converges to* $m^*(r, \theta, u_j)$ *uniformly on the set* $\{re^{i\theta} : 1 \leqslant r \leqslant R,\ 0 \leqslant \theta \leqslant \pi\}$ *for every* $R > 1$ $(j \in \{1, 2, \ldots, p\})$.

Proof. We have $u_j(z) \leqslant v_j^k(z)$ for all $j \in \{1, 2, \ldots, p\}$. Then for any measurable set E

$$\int\limits_E u_j(re^{i\varphi}) \mathrm{d}\varphi \leqslant \int\limits_E v_j^k(re^{i\varphi}) \mathrm{d}\varphi \quad \text{and}$$

$$\sup_{|E|=2\theta} \int\limits_E u_j(re^{i\varphi}) \mathrm{d}\varphi \leqslant \sup_{|E|=2\theta} \int\limits_E v_j^k(re^{i\varphi}) \mathrm{d}\varphi \quad \text{for every} \quad k.$$

Hence

$$\sup_{|E|=2\theta} \int\limits_E u_j(re^{i\varphi}) \mathrm{d}\varphi \leqslant \lim_{k\to\infty} \sup_{|E|=2\theta} \int\limits_E v_j^k(re^{i\varphi}) \mathrm{d}\varphi \quad \text{and}$$

$$m^*(r, \theta, u_j) \leqslant \lim_{k\to\infty} m^*(r, \theta, v_j^k) \quad \text{for} \quad j \in \{1, 2, \ldots, p\}. \tag{1}$$

Let $\varepsilon > 0$ be given. Then, by the Egorov Theorem, for any E with $|E| = 2\theta$ there exists a set E_ε with $|E_\varepsilon| < \varepsilon$ such that $v_j^k(re^{i\varphi}) \to u_j(re^{i\varphi})$ uniformly on $E \setminus E_\varepsilon$, where r is fixed.

Then there exists k_0 such that, for $k \geqslant k_0$, $v_j^k(re^{i\varphi}) < u_j(re^{i\varphi}) + \varepsilon$ on $E \setminus E_\varepsilon$. Hence

$$\begin{aligned}\int_E v_j^k(re^{i\varphi})\mathrm{d}\varphi &= \int_{E\setminus E_\varepsilon} v_j^k(re^{i\varphi})\mathrm{d}\varphi + \int_{E_\varepsilon} v_j^k(re^{i\varphi})\mathrm{d}\varphi \leq \\ &\leqslant \int_{E\setminus E_\varepsilon} \left(u_j(re^{i\varphi}) + \varepsilon\right)\mathrm{d}\varphi + \int_{E_\varepsilon} v_j^1(re^{i\varphi})\mathrm{d}\varphi \leq \\ &\leqslant \int_E \left(u_j(re^{i\varphi}) + \varepsilon\right)\mathrm{d}\varphi + \max_{|z|=r} v_j^1(z)\varepsilon = \\ &= \int_E u_j(re^{i\varphi})\mathrm{d}\varphi + \left(2\theta + \max_{|z|=r} v_j^1(z)\right)\varepsilon,\end{aligned}$$

so

$$\sup_{|E|=2\theta} \int_E v_j^k(re^{i\varphi})\mathrm{d}\varphi \leqslant \sup_{|E|=2\theta} \int_E u_j(re^{i\varphi})\mathrm{d}\varphi + \left(2\theta + \max_{|z|=r} v_j^1(z)\right)\varepsilon$$

for $k \geqslant k_0$ and finally

$$\lim_{k\to\infty} m^*(r,\theta,v_j^k) \leqslant m^*(r,\theta,u_j) \quad \text{for} \quad j \in \{1,2,\ldots,p\}. \tag{2}$$

Using (1) and (2) we get

$$\lim_{k\to\infty} m^*(r,\theta,v_j^k) = m^*(r,\theta,u_j) \quad \text{for} \quad j \in \{1,2,\ldots,p\}.$$

Hence $m^*(r,\theta,u_j)$ is the pointwise limit of $\{m^*(r,\theta,v_j^k)\}$. Moreover the sequence $\{m^*(r,\theta,v_j^k)\}$ is nonincreasig and the functions $m^*(r,\theta,u_j)$ and $m^*(r,\theta,v_j^k)$ for all $k \in \mathbb{N}$ are continuous on the set $\{re^{i\theta} : \ r > 0, \ 0 \leqslant \theta \leqslant \pi\}$, so we can use the Dini Theorem. This completes the proof of Lemma 1. □

For a real function $\varphi(r)$, we consider the operator

$$\mathrm{L}\varphi(r) = \liminf_{h\to 0} \frac{\varphi(re^h) + \varphi(re^{-h}) - 2\varphi(r)}{h^2}.$$

If the function $\varphi(r)$ is twice differentiable in r, then

$$\mathrm{L}\varphi(r) = r\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r).$$

Since $m^*(r,\theta,u_j)$ is convex in $\log r$, for all $r > 0$ and $\theta \in [0,\pi]$ we have

$$\mathrm{L}m^*(r,\theta,u_j) \geqslant 0.$$

Lemma 2. *Let $u(z)$ be a subharmonic function with continuous second partial derivatives. For all $r > 0$ and for almost all $\theta \in (0,\pi)$ we have*

$$\mathrm{L}m^*(r,\theta,u_j) \geqslant -\frac{1}{\pi}\frac{\partial \widetilde{u}_j(re^{i\varphi})}{\partial\varphi}\bigg|_{\varphi=\theta}.$$

The proof of this lemma is similar to that of Lemma 1 in [5], also see [8].

Lemma A [9]. *Let the function $f(x)$ be nondecreasing on an interval $[a,b]$ and let $\varphi(x)$ be a nonnegative function having a bounded derivative on $[a,b]$. Then*

$$\int_a^b f'(x)\varphi(x)\mathrm{d}x \leqslant f(x)\varphi(x)\Big|_a^b - \int_a^b \varphi'(x)f(x)\mathrm{d}x.$$

From now on we assume that $u(z)$ is a subharmonic function with p strong tracts. For every α with $0<\alpha<\min\{\pi,\pi/2\lambda\}$ and for every k we put (see [10])

$$\sigma_k(r) = \int_0^\alpha m_0^*(r,\theta)\cos\lambda\theta\mathrm{d}\theta,$$

where $m_0^*(r,\theta)=\sum_{j=1}^p m^*(r,\theta,v_j^k)$.

Lemma B [11]. *Let (S_i) and (R_i) be sequences such that $\lim_{i\to\infty}S_i=\infty$, $\lim_{i\to\infty}R_i=\infty$, $\lim_{i\to\infty}S_i/R_i=0$ and for each i the numbers $2S_i$ and $2R_i$ are Pólya peaks of the function $T(r,u)$. Then for every $\varepsilon>0$ there exists $i_0(\varepsilon)$ such that for $i>i_0$*

$$\frac{T(2S_i,u)}{S_i^\lambda}+\frac{T(2R_i,u)}{R_i^\lambda}<\varepsilon\int_{S_i}^{R_i}\frac{T(r,u)}{r^{\lambda+1}}\mathrm{d}r.$$

Lemma 3. *There exist sequences (S_i) and (R_i) with $\lim_{i\to\infty}S_i=\infty$, $\lim_{i\to\infty}R_i=\infty$ and $\lim_{i\to\infty}S_i/R_i=0$, such that for every $\varepsilon>0$ and for $i>i_0(\varepsilon)$, and $k>k_0(\varepsilon,i)$ we have*

$$\frac{(\sigma_k)'_-(R_i)}{R_i^{\lambda-1}}+\frac{(\sigma_k)'_-(S_i)}{S_i^{\lambda-1}}+\frac{\lambda\sigma_k(R_i)}{R_i^\lambda}+\frac{\lambda\sigma_k(S_i)}{S_i^\lambda}\leqslant\varepsilon\int_{S_i}^{R_i}\frac{T(r,u)}{r^{\lambda+1}}\mathrm{d}r,$$

where $(\sigma_k)'_-(r)$ is the left derivative of $\sigma_k(r)$.

Proof. Let $\varepsilon>0$ be given. Let (S_i), (R_i) be the sequences defined in Lemma B. Let i_0 be such that

$$\frac{T(2S_i,u)}{S_i^\lambda}+\frac{T(2R_i,u)}{R_i^\lambda}<\frac{\varepsilon}{9}\int_{S_i}^{R_i}\frac{T(r,u)}{r^{\lambda+1}}\mathrm{d}r$$

for all $i\geqslant i_0$. Using Lemma 1, for $k>k_0$, we get

$$m_0^*(r,\theta)\leqslant m_0^*(r,\theta,u)+\varepsilon\leqslant T(r,u)+pn_0+\varepsilon$$

on the set $\{re^{i\theta}:\ 1\leqslant r\leqslant 2R_i,\ 0\leqslant\theta\leqslant\pi\}$. Hence, for $\alpha=\min\{\pi,\pi/2\lambda\}$,

$$\sigma_k(r)=\int_0^\alpha m_0^*(r,\theta)\cos\lambda\theta\mathrm{d}\theta\leqslant\int_0^\alpha T(r,u)\cos\lambda\theta\mathrm{d}\theta+\int_0^\alpha(pn_0+\varepsilon)\cos\lambda\theta\mathrm{d}\theta\leq$$
$$\leqslant\pi T(r,u)+\pi(pn_0+\varepsilon).$$

Therefore

$$\frac{\lambda\sigma_k(S_i)}{S_i^\lambda} \leqslant \frac{\pi T(S_i,u)}{S_i^\lambda} + \frac{\pi(pn_0+\varepsilon)}{S_i^\lambda} \leqslant 5\frac{T(S_i,u)}{S_i^\lambda} \tag{3}$$

and

$$\frac{\lambda\sigma_k(R_i)}{R_i^\lambda} \leqslant 5\frac{T(R_i,u)}{R_i^\lambda}. \tag{4}$$

Now, since $m_0^*(r,\theta)$ is a monotone function of r,

$$(\sigma_k)'_-(r) = \int\limits_0^\alpha \frac{\partial m_0^*(r,\theta)}{\partial r}\cos\lambda\theta \mathrm{d}\theta,$$

but $m_0^*(r,\theta)$ is convex in $\log r$ if θ is fixed, so $\frac{r\partial m_0^*(r,\theta)}{\partial r}$ is nondecreasing and for $x > r$, using Lemma A, we have

$$m_0^*(x,\theta) \geqslant m_0^*(x,\theta) - m_0^*(r,\theta) \geqslant \int\limits_r^x \frac{\partial m_0^*(t,\theta)}{\partial t}\mathrm{d}t = \int\limits_r^x \frac{t\frac{\partial}{\partial t}m_0^*(t,\theta)}{t}\mathrm{d}t$$

$$= \int\limits_r^x t\frac{\partial}{\partial t}m_0^*(t,\theta)\mathrm{d}(\log t) \geqslant r\frac{\partial}{\partial r}m_0^*(r,\theta)\log\frac{x}{r},$$

where, on the right side of the inequality, for fixed θ we have the left derivative of $m_0^*(r,\theta)$. Hence

$$\frac{\partial m_0^*(r,\theta)}{\partial r} \leqslant \frac{1}{r\log\frac{x}{r}}m_0^*(x,\theta) \leqslant \frac{1}{r\log\frac{x}{r}}\left(T(r,u)+pn_0+\varepsilon\right) \quad \text{as } k \geqslant k_0,$$

so

$$\left.\frac{\partial m_0^*(r,\theta)}{\partial r}\right|_{r=S_i} \leqslant \frac{1}{S_i\log 2}\left(T(2S_i,u)+pn_0+\varepsilon\right)$$

and

$$\left.\frac{\partial m_0^*(r,\theta)}{\partial r}\right|_{r=R_i} \leqslant \frac{1}{R_i\log 2}\left(T(2R_i,u)+pn_0+\varepsilon\right).$$

Thus

$$\frac{(\sigma_k)'_-(R_i)}{R_i^{\lambda-1}} = \frac{1}{R_i^{\lambda-1}}\int\limits_0^\alpha \left.\frac{\partial m_0^*(r,\theta)}{\partial r}\right|_{r=R_i}\cos\lambda\theta\mathrm{d}\theta$$

$$\leqslant \frac{1}{R_i^{\lambda-1}}\frac{1}{R_i\log 2}\left(T(2R_i,u)+pn_0+\varepsilon\right)\int\limits_0^\alpha\cos\lambda\theta\mathrm{d}\theta \leqslant 4\frac{1}{R_i^\lambda}T(2R_i,u) \tag{5}$$

and

$$\frac{(\sigma_k)'_-(S_i)}{S_i^{\lambda-1}} \leqslant 4\frac{1}{S_i^\lambda}T(2S_i,u). \tag{6}$$

Finally, using (3), (4), (5), (6) and Lemma B, we get

$$\frac{(\sigma_k)'_-(R_i)}{R_i^{\lambda-1}} + \frac{(\sigma_k)'_-(S_i)}{S_i^{\lambda-1}} + \frac{\lambda\sigma_k(R_i)}{R_i^{\lambda}} + \frac{\lambda\sigma_k(S_i)}{S_i^{\lambda}} \leqslant$$

$$\leqslant 9\frac{1}{R_i^{\lambda}}T(2R_i,u) + 9\frac{1}{S_i^{\lambda}}T(2S_i,u) \leqslant 9\left(\frac{1}{R_i^{\lambda}}T(2R_i,u) + \frac{1}{S_i^{\lambda}}T(2S_i,u)\right) \leqslant \varepsilon \int\limits_{S_i}^{R_i} \frac{T(r,u)}{r^{\lambda+1}}\mathrm{d}r.$$

The lemma is proved. □

2. Proof of Theorem. For $p = 1$ the proof is not necessary, as the conclusion follows easily from Theorem A. Hence, we assume that the number of strong tracts $p > 1$. Using the definition of $\sigma_k(r)$ and Lemma 2, we get

$$\mathrm{L}\sigma_k(r) \geqslant -\frac{1}{\pi}\sum_{j=1}^{p}\int\limits_0^{\alpha} \frac{\partial \widetilde{v_j^k}(re^{i\theta})}{\partial\theta}\cos\lambda\theta\mathrm{d}\theta.$$

Integrating by parts we obtain

$$\mathrm{L}\sigma_k(r) \geqslant -\frac{1}{\pi}\sum_{j=1}^{p}\widetilde{v_j^k}(re^{i\theta})\cos\lambda\theta\Big|_0^{\alpha} - \frac{\lambda}{\pi}\sum_{j=1}^{p}\int\limits_0^{\alpha}\widetilde{v_j^k}(re^{i\theta})\sin\lambda\theta\mathrm{d}\theta =$$

$$= -\frac{1}{\pi}\sum_{j=1}^{p}\widetilde{v_j^k}(re^{i\theta})\cos\lambda\theta\Big|_0^{\alpha} - \lambda\sum_{j=1}^{p} m^*(r,\alpha,v_j^k)\sin\lambda\alpha +$$

$$+\lambda^2\sum_{j=1}^{p}\int\limits_0^{\alpha} m^*(r,\theta,v_j^k)\cos\lambda\theta\mathrm{d}\theta =$$

$$= -\frac{1}{\pi}\sum_{j=1}^{p}\widetilde{v_j^k}(re^{i\theta})\cos\lambda\theta\Big|_0^{\alpha} - \lambda m_0^*(r,\theta)\sin\lambda\alpha + \lambda^2\sigma_k(r) \equiv h_k(r) + \lambda^2\sigma_k(r). \qquad (7)$$

Since $\mathrm{L}m^*(r,\theta,v_j^k) \geqslant 0$, using Fatou Lemma, we get

$$\mathrm{L}\sigma_k(r) \geqslant \int\limits_0^{\alpha}\mathrm{L}m_0^*(r,\theta)\cos\lambda\theta\mathrm{d}\theta \geqslant \sum_{j=1}^{p}\int\limits_0^{\alpha}\mathrm{L}m^*(r,\theta,v_j^k)\cos\lambda\theta\mathrm{d}\theta \geqslant 0$$

and $\sigma_k(r)$ is convex in $\log r$. It follows that $r(\sigma_k)'_-(r)$ is an increasing function on $[0,\infty)$. Thus, for almost all $r \geqslant 0$ we have

$$\mathrm{L}\sigma_k(r) = r\frac{\mathrm{d}}{\mathrm{d}r}r(\sigma_k)'_-(r).$$

We now divide inequality (7) by $r^{\lambda+1}$ and integrate it over the interval $[S_i, R_i]$,

$$\int\limits_{S_i}^{R_i}\frac{\frac{\mathrm{d}}{\mathrm{d}r}r(\sigma_k)'_-(r)}{r^{\lambda}}\mathrm{d}r \geqslant \int\limits_{S_i}^{R_i}\frac{h_k(r)}{r^{\lambda+1}}\mathrm{d}r + \lambda^2\int\limits_{S_i}^{R_i}\frac{\sigma_k(r)}{r^{\lambda+1}}\mathrm{d}r,$$

where the numbers S_i and R_i are defined in Lemma 3. Since the function $r(\sigma_k)'_-(r)$ is increasing, by Lemma A, we have

$$\int_{S_i}^{R_i} \frac{\frac{\mathrm{d}}{\mathrm{d}r} r(\sigma_k)'_-(r)}{r^\lambda}\mathrm{d}r \leqslant \left(\frac{r(\sigma_k)'_-(r)}{r^\lambda} + \lambda\frac{\sigma_k(r)}{r^\lambda}\right)\Bigg|_{S_i}^{R_i} + \lambda^2 \int_{S_i}^{R_i} \frac{\sigma_k(r)}{r^{\lambda+1}}\mathrm{d}r.$$

Therefore

$$\int_{S_i}^{R_i} \frac{h_k(r)}{r^{\lambda+1}}\mathrm{d}r \leqslant \left(\frac{(\sigma_k)'_-(r)}{r^{\lambda-1}} + \lambda\frac{\sigma_k(r)}{r^\lambda}\right)\Bigg|_{S_i}^{R_i}. \tag{8}$$

Let $\varepsilon > 0$ be given. By Lemma 3, for $i > i_0$ and $k > k_0$, we have

$$\int_{S_i}^{R_i} \frac{h_k(r)}{r^{\lambda+1}}\mathrm{d}r < \varepsilon \int_{S_i}^{R_i} \frac{T(r,u)}{r^{\lambda+1}}\mathrm{d}r.$$

Now, if $v_j^k(z)$ is a nonincreasing sequence tending to $u_j(z)$, then also $\widetilde{v_j^k}(z)$ is a nonincreasing sequence tending to $\widetilde{u_j}(z)$. Hence, using Lemma 1, by Monotone Convergence Theorem, we get

$$\int_{S_i}^{R_i} \frac{h(r)}{r^{\lambda+1}}\mathrm{d}r < \varepsilon \int_{S_i}^{R_i} \frac{T(r,u)}{r^{\lambda+1}}\mathrm{d}r,$$

where

$$h(r) = -\frac{1}{\pi}\sum_{j=1}^{p} \widetilde{u_j}(re^{i\theta})\cos\lambda\theta\Big|_0^\alpha - \lambda m_0^*(r,\alpha,u)\sin\lambda\alpha.$$

Thus, for every $i > i_0$, there exists $r_i \in [S_i, R_i]$ such that $h(r_i) < \varepsilon T(r_i,u)$. Also $m_0^*(r,\theta,u) \leqslant T(r,u) + pn_0$. Hence, using the definition of $h(r)$, we get

$$\sum_{j=1}^{p} \widetilde{u_j}(r_i) - \sum_{j=1}^{p} \widetilde{u_j}(r_i e^{i\alpha})\cos\lambda\alpha - \pi\lambda T(r_i,u)\sin\lambda\alpha \leqslant 2\pi\varepsilon T(r_i,u), \tag{9}$$

for big enough i's.

Now

$$\widetilde{u_j}(r_i) = \max_{|z|=r_i} u_j(z) \geqslant \max_{\substack{|z|=r_i\\ z\in\Gamma_j}} u_j(z) = \max_{\substack{|z|=r_i\\ z\in\Gamma_j}} u(z) > (1-\varepsilon)\max_{|z|=r_i} u(z),$$

as $\mathrm{C}_j(n_0)$ is a strong tract. Since

$$\beta(\infty,u) = \liminf_{r\to\infty} \frac{\max\{u(z):\ |z|=r\}}{T(r,u)},$$

we have

$$\widetilde{u_j}(r_i) > (1-\varepsilon)^2\beta(\infty,u)T(r_i,u).$$

Inequality (9) shows that

$$p(1-\varepsilon)^2\beta(\infty,u)T(r_i,u) - \pi\lambda T(r_i,u)\sin\lambda\alpha - \sum_{j=1}^{p} \widetilde{u_j}(r_i e^{i\alpha})\cos\lambda\alpha < 2\pi\varepsilon T(r_i,u) \tag{10}$$

(i) First we consider the case $\lambda \geqslant 0.5$. Putting $\alpha = \frac{\pi}{2\lambda}$ in (10) we get

$$p(1-\varepsilon)^2\beta(\infty,u)T(r_i,u) - \pi\lambda T(r_i,u) < 2\pi\varepsilon T(r_i,u)$$

Thus

$$p < \frac{\pi\lambda + 2\pi\varepsilon}{(1-\varepsilon)^2\beta(\infty,u)}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$p \leqslant \frac{\pi\lambda}{\beta(\infty,u)}.$$

(ii) Now consider the case $0 < \lambda < 0.5$. We put $\alpha = \pi$. We have

$$\widetilde{u_j}(-r_i) = \min_{|z|=r_i} u_j(z) = n_0$$

for $j \in \{1,2,\ldots,p\}$. Using (10) we get

$$p(1-\varepsilon)^2\beta(\infty,u)T(r_i,u) - \pi\lambda T(r_i,u)\sin\pi\lambda - pn_0\cos\pi\lambda < 2\pi\varepsilon T(r_i,u).$$

Thus

$$p(1-\varepsilon)^2\beta(\infty,u) - \pi\lambda\sin\pi\lambda < 3\pi\varepsilon.$$

Hence

$$p < \frac{\pi\lambda\sin\pi\lambda + 3\pi\varepsilon}{(1-\varepsilon)^2\beta(\infty,u)}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$p \leqslant \frac{\pi\lambda\sin\pi\lambda}{\beta(\infty,u)}.$$

(iii) Finally we consider $\lambda = 0$. Then $p \leqslant 0$, which contradicts to the assumption that $p > 1$. Hence $p \leqslant 1$.

The theorem is proved.

REFERENCES

1. Hayman W.K., Kennedy P.B. Subharmonic functions. Vol. I, Academic Press, 1976.
2. Talpur M.N.M. *A subharmonic analogue of Iversen's theorem.*, Proc. London Math. Soc.(3) **32** (1976), 181–192.
3. Hayman W.K. Subharmonic functions. Vol. II, Academic Press, 1989.
4. Петренко В.П. *Рост мероморфных функций конечного нижнего порядка*, Изв. Акад. наук СССР, **33** (1969), №2, 414–454.
5. Марченко И.И. *Об асимптотических значениях целых функций*, Изв. РАН, сер. матем. **63** (1999), №3, 133–146.
6. Baernstein A. *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), no. 3–4, 139–169.

7. Hayman W.K. Multivalent Functions, Cambridge Univ. Press, Cambridge 1958.

8. Gariepy R., Lewis J.L. *Space Analogues of some theorems for subharmonic and meromorphic functions*, Ark. Mat. **13** (1975), 91–105.

9. Marchenko I.I. *On the magnitudes of deviations and spreads of meromorphic functions of finite lower order*, Mat.Sb. **186** (1995), 391–408.

10. Essen M. and Shea D.F. *Applications of Denjoy integral inequalities and differential inequalities to growth problems for subharmonic and meromorphic functions*, Proc. Roy. Irish Acad. **A82** (1982), 201–216.

11. Петренко В.П. Рост мероморфных функций.– Харьков: Вища шк., 1978.– 136 с.

Kharkiv State University
Institute of Mathematics, University of Szczecin
marchenko@wmf.univ.szczecin.pl
olaszkibiel@poczta.onet.pl

Received 28.04.2004