УДК 515.12+512.58

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ALGEBRAS OF LAWSON MONADS

T. M. Radul. *Algebras of Lawson monads*, Matematychni Studii, **22** (2004) 21–26.

We investigate algebras of Lawson monads and show that not all of them can be represented as subalgebras of products of intervals.

Т. Н. Радул. *Алгебры монад Лоусона* // Математичні Студії. – 2004. – Т.22, №1. – C.21–26.

Изучаются монады Лоусона. Доказано, что не все они могут быть представлены в виде подалгебр произведений интервалов.

0. The algebraic aspect of the theory of functors in categories of topological spaces and continuous maps was investigated rather recently. It is based, mainly, on the existence of monad (or triple) structure in the sense of S. Eilenberg and J. Moore [2].

Many classical constructions lead to monads: hyperspaces, spaces of probability measures, superextensions etc. There were many investigations of monads in categories of topological spaces and continuous maps (see for example [3] or [6]). But it seems that the main difficulty to obtain general results in the theory of monads is the different nature of functors.

Some functional representations of the hyperspace functor were found in [4] and [5]. In [1] we introduced a class of Lawson monads, which contains sufficiently wide class of monads. The Lawson monads have a functional representation, i.e., their functorial part FX can be naturally imbedded in $\mathbb{R}^{CX}$.

The main goal of this paper is to investigate the algebras of Lawson monads.

The paper is arranged in the following manner. In Section 2 we give some characterization of the category of algebras of the monad $\mathbb{V}$ which contains all the Lawson monads as submonads. More exactly, we prove that this category is equivalent to the category whose objects are products of closed intervals and morphisms preserve the structure of products. In Section 3 we give an example of a Lawson monad with algebras which cannot be embedded in products of intervals.

1. By $\mathcal{C}omp$ we denote the category of compact Haussdorff spaces (compacta) and continuous maps.

We denote by I the segment $[0,1]$. Let $X \in \mathcal{C}omp$. We denote by CX the Banach space of all continuous functions $\varphi\colon X \to \mathbb{R}$ with the usual sup-norm: $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$. In what follows, all spaces and maps are assumed to be in $\mathcal{C}omp$ except for $\mathbb{R}$ and maps in the sets CX with X compact Hausdorff.

2000 *Mathematics Subject Classification:* 18C15, 54B30.

We need some definitions concerning monads and algebras. A *monad* $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathcal{E}$ consists of an endofunctor $T\colon \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta\colon \mathrm{Id}_{\mathcal{E}} \to T$ (unity), $\mu\colon T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$. (By $\mathrm{Id}_{\mathcal{E}}$ we denote the identity functor on the category $\mathcal{E}$ and T^2 is the superposition $T \circ T$ of T.)

A natural transformation $\psi\colon T \to T'$ is called a *morphism* from a monad $\mathbb{T} = (T, \eta, \mu)$ into a monad $\mathbb{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \eta T' \circ T\psi$. If all the components of ψ are monomorphisms then the monad $\mathbb{T}$ is called a *submonad* of $\mathbb{T}'$.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in a category $\mathcal{E}$. The pair (X, ξ) is called a $\mathbb{T}$-*algebra* if $\xi \circ \eta X = \mathrm{id}_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let (X, ξ), (Y, ξ') be two $\mathbb{T}$-algebras. A map $f\colon X \to Y$ is called a $\mathbb{T}$-algebras morphism if $\xi' \circ Tf = f \circ \xi$. The category of $\mathbb{T}$-algebras and their morphisms is denoted by $\mathcal{E}^{\mathbb{T}}$.

For any real $t \geq 0$, we denote by I_t the segment $[-t, t]$. If t_1, t_2 are real numbers with $0 \leq t_1 \leq t_2$, by $j_{t_1}^{t_2}$ we denote the natural embedding $j_{t_1}^{t_2}\colon I_{t_1} \to I_{t_2}$. We shall also use the notation $j_t\colon I_t \to \mathbb{R}$ for the inclusion map and φ_t for the codomain restriction onto I_t of a map $\varphi\colon X \to \mathbb{R}$, provided that $t \geq \|\varphi\|$. Then we have $\varphi = j_t \circ \varphi_t$.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in the category $\mathcal{C}omp$. A family $\{\xi_t\colon TI_t \to I_t \mid t \geq 0\}$ of $\mathbb{T}$-algebras is called *coherent* if for each t_1, $t_2 \in \mathbb{R}$ with $0 \leq t_1 \leq t_2$ the embedding $j_{t_1}^{t_2}$ is a $\mathbb{T}$-algebras morphism. A monad $\mathbb{T} = (T, \eta, \mu)$ is called *Lawson* if there exists a coherent family of $\mathbb{T}$-algebras $\{\xi_t\colon TI_t \to I_t \mid t \geq 0\}$ such that for each $X \in \mathcal{C}omp$ there exists a point-separating family of $\mathbb{T}$-algebras morphisms $\{f_\alpha\colon (TX, \mu X) \to (I_{t(\alpha)}, \xi_{t(\alpha)}) \mid \alpha \in A\}$ [1].

By VX we denote the product $\prod_{\varphi \in CX} I_{\|\varphi\|}$, i.e. the set of all mappings (not necessarily continuous) $\nu\colon CX \to \mathbb{R}$ which satisfy the condition $-\|\varphi\| \leq \nu(\varphi) \leq \|\varphi\|$ for each $\varphi \in CX$.

For a map $\varphi \in CX$ we denote by π_φ or $\pi(\varphi)$ the corresponding projection $\pi_\varphi\colon VX \to \mathbb{R}$. Then any map $f\colon Z \to VX$ in $\mathcal{C}omp$ is uniquely determined by its projections $f_\varphi = \pi_\varphi \circ f$ in CZ for every $\varphi \in CX$, provided only that $\|f_\varphi\| \leq \|\varphi\|$ for all φ.

Now, for each map $f\colon X \to Y$ define a map $Vf\colon VX \to VY$ by the condition $\pi_\varphi \circ Vf = \pi_{\varphi \circ f}$ for $\varphi \in CY$. Since $\|\pi(\varphi \circ f)\| = \|\varphi \circ f\| \leq \|\varphi\|$, the map Vf is well-defined. One can check that V is a covariant functor on the category $\mathcal{C}omp$.

Now we shall construct natural transformations $h\colon I_{\mathcal{C}omp} \to V$ and $m\colon V^2 \to V$ of unit and multiplication which complete the functor V to a monad $\mathbb{V} = (V, h, m)$.

For a compactum X we define the components hX and mX by $\pi_\varphi \circ hX = \varphi$ and $\pi_\varphi \circ mX = \pi(\pi_\varphi)$ for all $\varphi \in C(X)$. The map mX is well-defined because $\|\pi_\varphi\| = \|\varphi\|$.

It was shown in [1] that the triple $\mathbb{V} = (V, h, m)$ forms a monad on the category $\mathcal{C}omp$. It was also shown in [1] that any monad $\mathbb{F}$ is Lawson iff there is a monad embedding $l\colon \mathbb{F} \to \mathbb{V}$.

2. In this section we give some characterization of the category of $\mathbb{V}$-algebras. We will need some definitions from the category theory; see [7] for more details. Consider any two categories $\mathcal{C}$ and $\mathcal{D}$. A quadruple $\langle F, G, \eta, \varepsilon\rangle$ is an adjuncion if $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ are functors and $\eta\colon \mathrm{Id}_{\mathcal{C}} \to GF$, $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$ natural transformations such that $G\varepsilon X \circ \eta GX = \mathrm{id}_X$ for each $X \in \mathcal{C}$ and $\varepsilon FY \circ F\eta Y = id_Y$ for each $Y \in \mathcal{D}$.

The triple $\mathbb{T} = (GF, \eta, G\varepsilon F)$ forms a monad in the category $\mathcal{C}$ which is called the monad defined by the adjuncion $\langle F, G, \eta, \varepsilon\rangle$. There exists a functor $K\colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ which is called the comparison functor. For any $f\colon X \to Y$ in $\mathcal{D}$ we define K by $KX = (GX, G\varepsilon X)$ and $Kf = Gf\colon (GX, G\varepsilon X) \to (GY, G\varepsilon Y)$.

Let us define a category $\mathcal{P}rod$ which will be equivalent to the category of $\mathbb{V}$-algebras. The objects of this category are products of closed intervals $\prod_{\alpha \in A} I_{t(\alpha)}$. Let us describe the morphisms. Consider any $t \geq 0$ and $\prod_{\alpha\in A} I_{t(\alpha)} \in \mathcal{P}rod$. A function $f\colon \prod_{\alpha\in A} I_{t(\alpha)} \to I_t$ is called *product-preserving* if $f(x) = 0$ for each $x \in \prod_{\alpha\in A} I_{t(\alpha)}$ or there exists $\alpha \in A$ such that $t(\alpha) \leq t$ and $f = j^t_{t(\alpha)} \circ p_\alpha$ where $p_\alpha\colon \prod_{\beta\in A} I_{t(\beta)} \to I_{t(\alpha)}$ is the natural projection.

Consider any two objects $\prod_{\alpha\in A} I_{t(\alpha)}$ and $\prod_{\beta\in B} I_{t(\beta)}$ in $\mathcal{P}rod$. A function $f\colon \prod_{\alpha\in A} I_{t(\alpha)} \to \prod_{\beta\in B} I_{t(\beta)}$ is a morphism in the category $\mathcal{P}rod$ iff for each $\beta \in B$ the function $p_\beta \circ f\colon \prod_{\alpha\in A} I_{t(\alpha)} \to I_{t(\beta)}$ is product-preserving. It is easy to see that for each $X \in \mathcal{C}omp$ the space VX can be regarded as an object of $\mathcal{P}rod$. We see also that Vf is a morphism in $\mathcal{P}rod$ for each morphism f of the category $\mathcal{C}omp$. So, we can define the functor $F\colon \mathcal{C}omp \to \mathcal{P}rod$ by changing the codomain of the functor $V\colon \mathcal{C}omp \to \mathcal{C}omp$. By $U\colon \mathcal{P}rod \to \mathcal{C}omp$ we denote the natural forgetful functor. Hence $U \circ F = V$. Thus we have the natural transformation $h\colon \mathrm{Id}_{\mathcal{C}omp} \to U \circ F$.

For each $P = \prod_{\alpha\in A} I_{t(\alpha)} \in \mathcal{P}rod$ let us consider the map $\varepsilon P\colon FU(P) \to P$ defined by the equalities $p_\alpha \circ \varepsilon P = \pi(j_{t(\alpha)} \circ p_\alpha)_{t(\alpha)}$ for each $\alpha \in A$. One can check that the quadruple $\langle F, U, h, \varepsilon\rangle$ is an adjuncion and $\mu = U\varepsilon F$. Hence the monad $\mathbb{V}$ is defined by the adjuncion $\langle F, U, h, \varepsilon\rangle$. There exists a comparison functor $K\colon \mathcal{P}rod \to \mathcal{C}omp^{\mathbb{V}}$.

Our aim is to prove that K is an equivalence of categories. We need some definitions and a version of the Beck theorem (see [7] for more details).

Let $f, g\colon X \to Y$ be a parallel pair in a category $\mathcal{C}$. A morphism $e\colon Y \to Z$ in $\mathcal{C}$ is called a coequalizer of the pair f, g if $e \circ f = e \circ g$ and for each morphism $h\colon Y \to T$ with $h \circ f = h \circ g$ there exists a unique morphism $d\colon Z \to T$ with $h = d \circ e$. We say that e is a split coequalizer of the pair f, g if there exist morphisms $t\colon Y \to X$ and $s\colon Z \to Y$ such that $e \circ s = \mathrm{id}_Z$, $f \circ t = \mathrm{id}_Y$ and $g \circ t = s \circ e$.

A parallel pair $f, g\colon X \to Y$ is said to be contractible if there is a morphism $t\colon Y \to X$ such that $g \circ t \circ f = g \circ t \circ g$ and $f \circ t = \mathrm{id}_Y$. If the pair f, g has a split coequalizer, then f, g is contractible. Let us remark that for any contractible pair f, g the morphism f is necessarilly epi. The class of epimorphisms coincides with the class of surjective maps in the categories $\mathcal{C}omp$ and $\mathcal{P}rod$.

We say that a functor $U\colon \mathcal{D} \to \mathcal{C}$ reflects coequalizers for a pair $f, g\colon X \to Y$ in $\mathcal{D}$ if any morfism $e\colon Y \to Z$ such that Ue becomes a coequalizer of Uf, Ug is already a coequalizer in $\mathcal{D}$; U preserves coequalizers for f, g if for each $h\colon Y \to Z$ which is a coequalizer for f, g the morphism Uh is a coequalizer of Uf, Ug.

Beck's Theorem. *The following assertions are equivalent:*

1. *The comparison functor $K\colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is an equivalence of categories.*
2. *If f, g is any parallel pair in $\mathcal{D}$ for which Uf, Ug has a split coequalizer then $\mathcal{D}$ has a coequalizer for f, g and U preserves and reflects coequalizers for these pairs.*

Let $f, g\colon P_1 = \prod_{\alpha\in A} I_{t(\alpha)} \to P_2 = \prod_{\beta\in B} I_{t(\beta)}$ be a parallel pair in $\mathcal{P}rod$ such that f is surjective. We assume that $t(\beta) > 0$ for each $\beta \in B$ and $t(\alpha) > 0$ for each $\alpha \in A$. Let us denote $E(B) = \{\beta \in B \mid p_\beta \circ f = p_\beta \circ g\}$. Define the map $\tilde{g}\colon B \to A \cup \{0\}$ as follows. Since the map g is a morphism in $\mathcal{P}rod$, we see that for each $\beta \in B$ either $p_\beta \circ g(P_1) \subset \{0\}$ or there exists $\alpha \in A$ such that $p_\beta \circ g = j^{t(\beta)}_{t(\alpha)} \circ p_\alpha$. Put $\tilde{g}(\beta) = 0$ if $p_\beta \circ g(P_1) \subset \{0\}$ and $\tilde{g}(\beta) = \alpha$ otherwise. Since f is an epimorphism in $\mathcal{P}rod$, analogously we can define a map $\tilde{f}\colon B \to A$. It is easy to see that $\tilde{g}|E(B) = \tilde{f}|E(B)$.

Lemma 1. *If the pair Uf, Ug is contractible then $\tilde{f}(B \setminus E(B)) \cap \tilde{g}(B \setminus E(B)) = \varnothing$.*

Proof. Let $t\colon UP_2 \to UP_1$ be the map such that $Uf \circ t = \mathrm{id}_{UP_2}$ and $Ug \circ t \circ Uf = Ug \circ t \circ Ug$. Let us remark that $Uf \circ t = \mathrm{id}_{UP_2}$ implies $p_{\tilde{f}(\beta)} \circ t = p_\beta$ for each $\beta \in B$.

Assume the contrary: there exist β_1, $\beta_2 \in B \setminus E(B)$ such that $\tilde{f}(\beta_1) = \tilde{g}(\beta_2) = \alpha \in A$. Obviously, $\beta_1 \neq \beta_2$. Put $\alpha_1 = \tilde{g}(\beta_1)$. Since $\beta_1 \in B \setminus E(B)$, we see that $\alpha_1 \neq \alpha$. Consider the case $\alpha_1 \neq 0$. Choose $x \in P_1$ such that $p_\alpha(x) \neq p_{\alpha_1}(x)$. Then we have $p_{\beta_2} \circ Ug \circ t \circ Uf(x) = p_\alpha \circ t \circ Uf(x) = p_{\beta_1} \circ Uf(x) = p_\alpha(x) \neq p_{\alpha_1}(x) = p_{\beta_1} \circ Ug(x) = p_\alpha \circ t \circ Ug(x) = p_{\beta_2} \circ Ug \circ t \circ Ug(x)$. If $\alpha_1 = 0$ we can choose x such that $p_\alpha(x) \neq 0$ and use the same arguments. We obtain the contradiction and the lemma is proved. □

Theorem 1. *The functor K is an equivalence of the categories $\mathcal{P}rod$ and $\mathcal{C}omp^{\mathbb{V}}$.*

Proof. Let $f, g\colon P_1 = \prod_{\alpha \in A} I_{t(\alpha)} \to P_2 = \prod_{\beta \in B} I_{t(\beta)}$ be a parallel pair in $\mathcal{P}rod$ such that there exists a split coequalizer $e\colon UP_2 \to Z$ of the pair Uf, Ug in $\mathcal{C}omp$. It is easy to check that the natural projection $h\colon \prod_{\beta \in B} I_{t(\beta)} \to \prod_{\gamma \in E(B)} I_{t(\gamma)} = P_3$ is a coequalizer of the pair f, g in $\mathcal{P}rod$. (In the case $E(B) = \varnothing$ we consider $\prod_{\gamma \in E(B)} I_{t(\gamma)}$ as the one-point set.)

Let us show that Uh is a coequalizer of Uf and Ug in $\mathcal{C}omp$. Since $Uh \circ Uf = Uh \circ Ug$, there exists a map $s\colon Z \to UP_3$ such that $s \circ e = Uh$. Since Uh is surjective, s is surjective too. Let us show that s is injective. Let $t\colon UP_2 \to UP_1$ and $i\colon UP_3 \to UP_2$ be the maps such that $e \circ i = \mathrm{id}_{UP_3}$, $Uf \circ t = \mathrm{id}_{UP_2}$ and $Ug \circ t = i \circ e$. Consider any two points z_1, $z_2 \in Z$ such that $s(z_1) = s(z_2)$. Then we have $h \circ i(z_1) = s \circ e \circ i(z_1) = s(z_1) = s(z_2) = h \circ i(z_2)$. Since e is a split coequalizer, the pair f, g is contractible and we can choose $x \in P_1$ such that $f(x) = i(z_1)$ and $g(x) = i(z_2)$ by Lemma 1. Then we have $z_1 = e \circ f(x) = e \circ g(x) = z_2$. Hence, the map s is a homeomorphism and Uh is a coequalizer in $\mathcal{C}omp$. Thus we see that U preserves coequalizers.

One can check that U reflects coequalizers. The theorem is proved. □

3. Let $\mathbb{F} = (F, \eta, \mu)$ be a Lawson monad and $\{(I_t, \xi_t) \mid t \geq 0\}$ is a coherent family from the definition. Since $\mathbb{F}$ is a submonad of $\mathbb{V}$, the category of $\mathbb{V}$-algebras (which is equivalent to $\mathcal{P}rod$) is a subcategory of the category of $\mathbb{F}$-algebras. Each free $\mathbb{F}$-algebra $(FX, \mu X)$ can be embedded as a subalgebra in a product of intervals by the definition of Lawson monad. Moreover, for many known monads such as hyperspaces, probability measures, hyperspaces of inclusion, superextension etc., all their algebras can be represented as subalgebras of products of intervals (see for example [3] or [6]).

Thus, the following question arises naturally. For each product $P = \prod_{\alpha \in A} I_{t(\alpha)}$ we define a sructure of $\mathbb{F}$-algebra $\xi_P\colon FP \to P$ by the formula $p_\alpha \circ \xi = \xi_{t(\alpha)} \circ F(p_\alpha)$. Can any $\mathbb{F}$-algebra be embedded by an $\mathbb{F}$-algebras morphism in (P, ξ_P), for some product $P = \prod_{\alpha \in A} I_{t(\alpha)}$?

We give the negative answer to this question using the monad $\mathcal{S}$ defined in [8].

By $\mathcal{T}ych$ we denote the category of Tychonov spaces and continuous maps. Let us consider any $X \in \mathcal{T}ych$. For each $n \in \mathbb{N} \cup \{0\}$ we consider the product X^{2^n} as a subset of $X^{2^{n+1}}$ with the natural inclusion $X^{2^n} = \{(x, y) \in X^{2^{n+1}} \mid x = y\}$. Put $X_\infty = \bigcup_{i=0}^{\infty} X^{2^i}$.

Let us define a function $f\colon \mathbb{R}_\infty \to \mathbb{R}$ by the induction. Put $f_0 = \mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$. Assume that we have defined $f_i\colon \mathbb{R}^{2^i} \to \mathbb{R}$ for each $i < n \geq 1$. Let us define $f_n\colon \mathbb{R}^{2^n} \to \mathbb{R}$ by the formula $f_n(a_1, a_2) = 1/3 \min\{1, |f_{n-1}(a_1) - f_{n-1}(a_2)|\} f_{n-1}(a_1) + (1 - (1/3) \min\{1, |f_{n-1}(a_1) - f_{n-1}(a_2)|\}) f_{n-1}(a_2)$ for $(a_1, a_2) \in \mathbb{R}^{2^n}$. We can define the function $f\colon \mathbb{R}_\infty \to \mathbb{R}$ by the condition $f|\mathbb{R}^{2^i} = f_i$ for each $i \in \mathbb{N} \cup \{0\}$.

Let $g\colon X \to Y$ be any morphism in $\mathcal{T}ych$. Let us define a function $g_\infty\colon X_\infty \to Y_\infty$ as follows. If the function $g_\infty|X^{2^i}\colon X^{2^i} \to Y^{2^i}$ is defined, define $g_\infty|X^{2^{i+1}}\colon X^{2^{i+1}} \to Y^{2^{i+1}}$ by the formula $g_\infty|X^{2^{i+1}}(x, y) = (g_\infty|X^{2^i}(x), g_\infty|X^{2^i}(y))$ for each $(x, y) \in X^{2^{i+1}}$.

Consider any $X \in \mathcal{T}ych$. By $BC(X)$ we denote the set of all continuous bounded functions from X to $\mathbb{R}$. Consider an equivalence relation ρ in the set X_∞ defined by $x\rho y$ iff $f \circ \varphi_\infty(x) = f \circ \varphi_\infty(y)$ for each $\varphi \in BC(X)$. Denote by $S_\infty X$ the quotient set X_∞/ρ. The equivalence class with a representative $x \in X_\infty$ will be denoted by $[x]$.

We are going to define a topology of $S_\infty X$. For each $\varphi \in BC(X)$ define a function $\varphi_f \colon S_\infty X \to \mathbb{R}$ by the formula $\varphi_f([x]) = f \circ \varphi_\infty(x)$. Define the map $p_i \colon X^{2^i} \to X$ by the formula $p_i(x_1, \dots, x_{2^i}) = x_{2^i}$. The condition $p|X^{2^i} = p_i$ defines the map $p\colon X_\infty \to X$. Let us define for each $\varphi \in BC(X)$ the function $\varphi_p \colon S_\infty X \to \mathbb{R}$ by the formula $\varphi_p([x]) = \varphi(p(x))$. It was shown in [8] that the map φ_p is well-defined. We will consider $S_\infty X = \bigcup_{n=0}^\infty S_n X$ where $S_n X = \{[x] \in S_\infty X \mid \text{there exists } y \in X^{2^i} \text{ such that } y \in [x]\}$.

Consider a family $\mathbb{F}(X) \subset BC(S_\infty X)$ defined as $\mathbb{F}(X) = \{\varphi_f \mid \varphi \in BC(X)\} \cup \{\varphi_p \mid \varphi \in BC(X)\}$. For $\varphi_1, \dots, \varphi_n \in \mathbb{F}(X)$, define the function $d_{\varphi_1,\dots,\varphi_n} \colon S_\infty \times S_\infty \to \mathbb{R}$ by the formula $d_{\varphi_1,\dots,\varphi_n}(x, y) = \max\{|\varphi_i(x) - \varphi_i(y)| i \in \{1, \dots, n\}\}$. It is easy to check that $d_{\varphi_1,\dots,\varphi_n}$ is a pseudometric on $S_\infty X$. The family of pseudometrics $\{d_{\varphi_1,\dots,\varphi_n} \mid n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n \in \mathbb{F}(X)\}$ defines a uniformity $\mathcal{U}_{\mathbb{F}(\mathcal{X})}$ on $S_\infty X$. We consider $S_\infty X$ with the topology generated by this uniformity.

By $\eta X \colon X \to S_\infty X$ we denote the continuos map defined by the formula $\eta X(x) = [(x)]$ for $x \in X$. Let us remark that $\eta X(X) = S_0 X$.

For each continuous map $g \colon X \to Y$ define a function $S_\infty g \colon S_\infty X \to S_\infty Y$ by the formula $S_\infty g([x]) = [g_\infty(x)]$ for $x \in X_\infty$.

For each $a, b \in X^{2^n}$ we can consider the element $(a, b) \in X^{2^{n+1}}$. Define the map $s\colon S_\infty X \times S_\infty X \to S_\infty X$ by the formula $s([a], [b]) = [(a, b)]$ for $[a], [b] \in S_\infty X$. It is easy to check that the map s is well-defined.

By S^2_∞ we denote the iteration $S_\infty \circ S_\infty$ of the functor S_∞. Now we are going to define a map $\mu x \colon S^2_\infty X \to S_\infty X$ for each $X \in \mathcal{T}ych$. Put $\mu X|S_0(S_\infty X) = (\eta S_\infty X)^{-1}$. Assume that we have defined $\mu X|S_i(S_\infty X)$ for each $i < n \geq 1$. Consider any $[x] \in S_n(S_\infty X)$. Then $[x] = [(x_1, x_2)]$ where $x_1, x_2 \in (S_\infty X)^{2^{n-1}}$. Put $\mu X([x]) = s(\mu X([x_1]), \mu X([x_2]))$.

It was shown in [8] that the maps $S_\infty g \colon (S_\infty X, \mathcal{U}_{\mathbb{F}(\mathcal{X})}) \to (\mathcal{S}_\infty \mathcal{Y}, \mathcal{U}_{\mathbb{F}(\mathcal{Y})})$ and $\mu X \colon (S^2_\infty X, \mathcal{U}_{\mathbb{F}(\mathcal{S}_\infty \mathcal{X})}) \to (\mathcal{S}_\infty \mathcal{X}, \mathcal{U}_{\mathbb{F}(\mathcal{X})})$ are uniformly continuous.

It is easy to check that the uniformity $\mathcal{U}_{\mathbb{F}(\mathcal{X})}$ defined on $S_\infty X$ is totally bounded. For each $X \in \mathcal{C}omp$ define a uniform space $(SX, \mathcal{V}_{\mathbb{F}(X)})$ being a completion of the uniform space $(S_\infty X, \mathcal{U}_{\mathbb{F}(\mathcal{X})})$. Then we have $SX \in \mathcal{C}omp$. We consider $S_\infty X$ as a subset of SX which is certainly dense in SX.

Consider any morphism $g \colon X \to Y$. The map $S_\infty g \colon S_\infty X \to S_\infty Y$ is uniformly continuous. Thus, there exists a unique extension $Sg \colon SX \to SY$. We have defined a functor $S \colon \mathcal{C}omp \to \mathcal{C}omp$.

For each $X \in \mathcal{C}omp$ define the map $hX \colon X \to SX$ by $hX(x) = \eta X(x) \in S_\infty X \subset SX$. Now, the set $S^2_\infty X$ is dense in $S^2 X$. The map $\mu X \colon S^2_\infty X \to S_\infty X$ is uniformly continuous. Hence there exists a unique extension $mX \colon S^2 X \to SX$. We have defined the natural transformations $h \colon \mathrm{Id}_{\mathcal{C}omp} \to S$ and $m \colon S^2 \to S$.

It was shown in [8] that the triple $\mathcal{S} = (S, h, m)$ forms a monad in $\mathcal{C}omp$.

For $t \geq 0$ define the map $k_t \colon S_\infty I_t \to I_t$ by the formula $k_t = (j_t)_f$, where $j_t \colon I_t \to \mathbb{R}$ is the natural embedding. It follows from the definition of the uniformity $\mathcal{U}_{\mathbb{F}(\mathcal{X})}$ that the map k_t is uniformly continuous (we consider I_t with the uniformity generated by the natural metric). Hence there exists a unique extension $\xi_t \colon SI_t \to I_t$. It was shown in [8] that the family $\{\xi_t \mid t \geq 0\}$ is a coherent family of $\mathcal{S}$-algebras and the monad $\mathcal{S}$ is Lawson.

We are going to give an example of $\mathcal{S}$-algebra (X, ξ) such that any $\mathcal{S}$-algebras morphism

$f\colon (X,\xi)\to (I_t,\xi_t)$ is constant. Denote $I=[0,1]$. Define the map $l\colon S_\infty I\to I$ by the formula $l([x])=(j)_p$, where $j\colon I\to\mathbb{R}$ is the natural embedding. It follows from the definition of the uniformity $\mathcal{U}_{\mathbb{F}(\mathcal{X})}$ that the map l is uniformly continuous (we consider I with the uniformity generated by the natural metric). Hence there exists a unique extension $\xi\colon SI\to I$. It is easy to check that the pair (I,ξ) is an $\mathcal{S}$-algebra.

Let $g\colon (I,\xi)\to (I_t,\xi_t)$ be any $\mathcal{S}$-algebras morphism. Suppose that there are points t_1, $t_2\in I$ such that $g(t_1)\neq g(t_2)$. Consider the elements $x_1=[(t_1)]$, $x_2=[(t_2,t_1)]\in S_\infty I\subset SI$. We have $\xi(x_1)=\xi(x_2)=t_1$. On the other hand, we obtain that $\xi_t\circ S(g)(x_1)=g(t_1)\neq (1/3)\min\{1,|g(t_2)-g(t_1)|\}g(t_2)+(1-(1/3)\min\{1,|g(t_2)-g(t_1)|\})g(t_1)=\xi_t\circ S(g)(x_2)$. We obtain a contradiction. Hence, $g(t_1)\neq g(t_2)$ and we can not represent the algebra (I,ξ) as a subalgebra of some product of algebras (I_t,ξ_t).

REFERENCES

1. Radul T. *On functional representations of Lawson monads,* Applied Categorical Structures **9** (2001), 457–463 .
2. Eilenberg S. Moore J. *Adjoint functors and triples,* Ill. J.Math. **9** (1965), 381–389.
3. Заричный М.М., Радул Т.Н. *Монады в категории компактов,* Усп. мат. наук, **50** (1995), Вып. 3, 83–108.
4. Шапиро Л.Б. *Об операторах продолжения функций и нормальных функторах*, Вестник моск. универ. сер.1 (1992) вып.1, 35–42.
5. Radul T. *A functional representation of the hyperspace monad*, Comment. Math. Univ. Carolin. **38** (1997), 165–168.
6. Teleiko A., Zarichnyi M. Categorical Topology of Compact Hausdorff Spaces, VNTL Publishers, Lviv, 1999.
7. MacLane S. Categories for Working Matematicians, Springer Verlag, 1976.
8. Radul T. *Categorical properties of iterated power*, Revista Colombiana de Matematicas **35** (2001), 13–20.

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Received 21.12.2003