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SKETCH OF GROUP BALLEANS

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A ballean is a set X endowed with some family of subset of X which are called the balls. We postulate the properties of the family of balls in such a way that the balleans with the appropriate morphisms can be considered as the asymptotic counterparts of the uniform topological spaces. The purpose of the paper is to find the asymptotic analogues for topological groups.

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Боллеан — это множество X, снабжённое неким семейством подмножеств, которые называются шарами. Свойства системы шаров постулируются таким образом, чтобы боллеаны можно было рассматривать как асимптотические двойники равномерных топологических пространств. Цель статьи — найти подходящие асимптотические аналоги для топологических групп.

A *ball structure* is a triplet (X, P, B), where X, P are non-empty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called the *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of (X, P, B), P is called the *set of radii.* Given any $x \in X$, $A \subseteq X$, $\alpha \in P$ we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure (X, P, B) is called *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B^*(x, \alpha') \subseteq B(x, \alpha), \quad B(x, \beta') \subseteq B^*(x, \beta).$$

A ball structure (X, P, B) is called *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta').$$

A ball structure (X, P, B) is called *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta).$$

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A ball structure (X, P, B) is called *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

Let (X, P, B) be a lower symmetric, lower multiplicative ball structure. Then the family

$$\left\{ \bigcup_{x \in X} B(x,\alpha) \times B(x,\alpha) : \alpha \in P \right\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

We say that a ball structure is a *ballean* if it is upper symmetric and upper multiplicative. The balleans arose independently in asymptotic topology [8], under the name coarse structure, and in combinatorics [12]. For good motivation to study the balleans related to metric spaces see the survey [2].

Let (X_1, P_1, B_1), (X_2, P_2, B_2) be balleans. A mapping $f\colon X_1 \to X_2$ is called a $\prec$-mapping if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta).$$

If $f\colon X_1 \to X_2$ is a bijection such that f and f^{-1} are $\prec$-mappings, we say that the balleans (X_1, P_1, B_1) and (X_2, P_2, B_2) are isomorphic. By the definition, $\prec$-mappings can be considered as the asymptotic counterparts of the uniformly continuous mappings between the uniform topological spaces. The approach to study balleans with $\prec$-morphisms is reflected in the papers [3, 14–20].

Now assume that the support X of the ballean (X, P, B) has some algebraic structure. Continuing the analogy between uniform topology and asymptotology we run into the question: does the ballean structure of X respect its algebraic structure? Perhaps, one of the first step to agree the asymptotic and algebraic structures was done in [4], where X was a semilattice. In this paper we present one possible approach to make an agreement between the algebraic and asymptotic structure of groups, and sketch some results obtained keeping this agreement.

1. Balleans on groups. Let G be a group with the unit e, (G, P, B) be a ballean. We say that (G, P, B) is

- *left* (resp. *right*) *invariant* if all the shifts $x \mapsto gx$ (resp. $x \mapsto xg$) are the $\prec$-mappings;
- *uniformly left* (resp. *right*) *invariant* if, for every $\alpha \in P$, there exists $\beta \in P$ such that $gB(x,\alpha) \subseteq B(gx,\alpha)$ (resp. $B(x,\alpha)g \subseteq B(xg,\beta)$) for all $x, g \in G$;
- a *group ballean* if it is uniformly left and right invariant.

If (G, P, B) is uniformly left invariant then it is left invariant, but the converse statement does not hold. Note also that a uniformly left invariant ballean needs not to be a group ballean.

A ballean (G, P, B) is uniformly left invariant if and only if, for every $\alpha \in P$, there exists $\beta \in P$ such that

$$gB(e,\alpha) \subseteq B(g,\beta), \quad g^{-1}B(g,\alpha) \subseteq B(e,\beta)$$

for every $g \in G$.

A family $\mathcal{I}$ of subsets of a set X is called an *ideal* if, for any subsets $A, B \in \mathcal{I}$ and $A' \subseteq A$, $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. A subset $\mathcal{I}' \subseteq \mathcal{I}$ is called a base for $\mathcal{I}$ if, for every $A \in \mathcal{I}$, there exists $A' \in \mathcal{I}'$ such that $A \subseteq A'$.

We say that an ideal $\mathcal{I}$ on a group G is a *group ideal* if, for any subsets $A, B \in \mathcal{I}$, $AB \in \mathcal{I}$ and $A^{-1} \in \mathcal{I}$.

Given a ballean (G, P, B), we say that a subset $A \subseteq G$ is *bounded from the unit* if there exists $\alpha \in P$ such that $A \subseteq B(e, \alpha)$.

For every uniformly left invariant ballean (G, P, B), the family $\mathcal{I}$ of all subsets of G bounded from the unit is a group ideal on G. Moreover, the identity mapping $\mathrm{id}\colon G \to G$ is an isomorphism between (G, P, B) and the ballean $(G, \mathcal{I}, B_l)$, where $B_l(g, A) = gA \cup \{g\}$ for all $g \in G, A \in \mathcal{I}$. On the other hand, for every group ideal $\mathcal{I}$ on G, $(G, \mathcal{I}, B_l)$ is a uniformly left invariant ballean. Thus, we get the natural correspondence between the family of all uniformly left invariant balleans having the support G and the family of all group ideals on G. Following this correspondence, given an arbitrary group ideal $\mathcal{I}$ on G, we write $(G, \mathcal{I})$ instead of $(G, \mathcal{I}, B_l)$.

We say that a group ideal $\mathcal{I}$ on G is *uniformly invariant* if, for every $A \in \mathcal{I}$, $\bigcup_{x \in G} x^{-1}Ax \in \mathcal{I}$. A uniformly left invariant ballean $(G, \mathcal{I})$ is a group ballean if and only if the ideal $\mathcal{I}$ is uniformly invariant. Thus, we obtain an identification between the family of all group balleans with the support G and the family of all uniformly invariant group ideals on G.

Let us observe that a left invariant ballean $(G, \mathcal{I})$ is right invariant if and only if the multiplication $(G, \mathcal{I}) \times (G, \mathcal{I}) \to (G, \mathcal{I})$ is a $\prec$-mapping. Automatically, in this case the inversion $(G, \mathcal{I}) \to (G, \mathcal{I})$ also is a $\prec$-mapping. Hence, a ballean (G, P, B) is a group ballean if and only if the multiplication on G is a $\prec$-mapping.

Let (X, P, B) is a ballean, $x, y \in X$. We say that x, y are *connected* if there exists $\alpha \in P$ such that $y \in X(x, \alpha)$. A subset $Y \subseteq X$ is called connected if any two points of Y are connected. The connectedness is an equivalence relation on X, so X disintegrates into connected components. A ballean is called connected if its support is connected.

For every group ideal $\mathcal{I}$ on G, the subset $H = \bigcup \mathcal{I}$ is a subgroup of G and every coset gH is a connected component of the ballean $(G, \mathcal{I})$. Hence, $(G, \mathcal{I})$ is connected if and only if $\bigcup \mathcal{I} = G$.

Given an arbitrary group G, we denote by $\mathcal{F}(G)$ the family of all finite subset of G. The group ideal $\mathcal{F}(G)$ is called *finitary*. If $\mathcal{I}$ is a group ideal, the ballean $(G, \mathcal{I})$ is connected if and only if $\mathcal{F}(G) \subseteq \mathcal{I}$.

We say that a group ideal $\mathcal{I}$ is *proper* if $\mathcal{F}(G) \subseteq \mathcal{I}$ and $G \notin \mathcal{I}$. If $G \in \mathcal{I}$ the ballean $(G, \mathcal{I})$ is called bounded. Thus, $\mathcal{I}$ is proper if and only if $(G, \mathcal{I})$ is connected and unbounded.

Let (X, d) be a metric space, $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$. Given any $x \in X, r \in \mathbb{R}^+$, we put

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

The ballean $(X, \mathbb{R}^+, B_d)$ is called a *metric ballean*. We say that a ballean is *metrizable* if it is isomorphic to some metric ballean. For metrizable balleans see [14].

Let $\mathcal{I}$ be a group ideal on a group G. By [14, Theorem 1], the ballean $(G, \mathcal{I})$ is metrizable if and only if $\mathcal{I}$ has a countable base and $\bigcup \mathcal{I} = G$.

A function $\|\cdot\|\colon G \to \mathbb{R}^+$ is called a norm if

$$\|xy\| \leq \|x\| + \|y\|, \|x^{-1}\| = \|x\|$$

for all $x, y \in G$. For every $n \in \omega$, let $B_n = \{x \in G : |x\| \leq n\}$. Then the family $\{B_n : n \in \omega\}$ is the base for some group ideal on G. On the other hand, if $\mathcal{I}$ is a group ideal on G with a countable base and $\bigcup \mathcal{I} = G$, then $\mathcal{I}$ can be determined by some norm on G taking the integer values. In this case the ballean $(G, \mathcal{I})$ can be metrizable by the left invariant metric $d(x, y) = \|x^{-1}y\|$. If $\mathcal{I}$ is uniformly invariant, then the norm $\|x\|$ can be chosen so that $d(x, y)$ is left and right invariant.

Let $\mathcal{I}_1, \mathcal{I}_2$ be group ideals on a group G. We put $\mathcal{I}_1 \wedge \mathcal{I}_2 = \{A_1 \cap A_2 : A_1 \in \mathcal{I}_1, A_2 \in \mathcal{I}_2\}$ and note that $\mathcal{I}_1 \wedge \mathcal{I}_2$ is a group ideal. If $\mathcal{I}_1, \mathcal{I}_2$ are proper, then $\mathcal{I}_1 \wedge \mathcal{I}_2$ is proper. We denote by $\mathcal{I}_1 \vee \mathcal{I}_2$ the smallest group ideal on G containing $\mathcal{I}_1$ and $\mathcal{I}_2$.

Let $\mathcal{I}$ be a group ideal on a group G and let φ be a filter on G. We say that φ is going to infinity with respect $\mathcal{I}$ if $G \backslash A \in \varphi$ for every $A \in \mathcal{I}$.

2. Group ideals on countable groups. Let G be an arbitrary group, $A \subseteq G$. We denote by $\mathcal{I}(A)$ the smallest group ideal on G such that $A \in \mathcal{I}(A), \mathcal{F}(G) \subseteq \mathcal{I}(A)$ and say that $\mathcal{I}(A)$ is *single-generated*. Now we describe some base for $\mathcal{I}(A)$.

Let $X = \{x_n : n \in \omega\}$ be an alphabet, $X_n = x_k : \; k \leq n$. For every finite subset $F \subseteq G$ and every $n \in \omega$, let $W_{n,F}$ be the set of all group words in the alphabet $F \cup X_n$. Given $v(x_0, \ldots, x_n) \in W_{n,F}$, we put $v(A) = v(A, \ldots, A)$ and

$$W_{n,F}(A) = \{v(A) : v \in W_{n,F}\}.$$

Then the family $\{W_{n,F}(A) : n \in \omega, F \in \mathcal{F}(G)\}$ is a base for the group ideal $\mathcal{I}(A)$. It follows that $\mathcal{I}(A)$ is proper if and only if $G \backslash W_{n,F}(A) \neq \varnothing$ for all $n \in \omega, F \in \mathcal{F}(G)$. If G is countable, $\mathcal{I}(A)$ has a countable base.

We are going to show that, for every countable group G, there exists an infinite subset A such that the ideal $\mathcal{I}(A)$ is proper. In particular, every countable group admits a non-finitary proper group ideal. Our construction is based on the following auxiliary statement which can be proved by Ramsey argument.

Lemma 1. *Let X, Y be infinite sets, Z be an infinite subset of $Y, n \in \omega$ and $f \colon X^n \to Y$ be an arbitrary mapping. Then there exists an infinite subset A of X such that $Z \backslash f(A^n)$ is infinite.*

Lemma 2. *Let G be a countable group, B, C be infinite subsets of G. Then there exists an infinite subset $A \subseteq B$ such that $C \notin \mathcal{I}(A)$.*

Proof. Let $G = \{g_n : n \in \omega\}, X = \{x_n : n \in \omega\}$ be an alphabet, $X \cap G = \varnothing$. We can enumerate $\{v_n : n \in \omega\}$ the set W of all group words in the alphabet $G \cup X$. By above description of the base for a single-generated ideal, it suffices to find $A \subseteq B$ such that $C \backslash \bigcup_{k \leq n} v_k(A)$ is infinite for every $n \in \omega$. We shall construct $A = \{a_n : n \in \omega\}$ inductively. By Lemma 1, there exist infinite subsets $A_0 \subseteq B, C_0 \subseteq C$ such that $v_0(A_0) \cap C_0 = \varnothing$. Fix $a_0 \in A_0$ and assume that we have chosen elements $a_0, a_1, \ldots, a_n$ of G and infinite subsets

$$A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n, \quad C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n.$$

Let $v_{n+1} = v_{n+1}(x_0, \ldots, x_m)$. We consider the set W' of all group words obtained by substitutions in v_{n+1} some of the elements $a_0, a_1, \ldots, a_n$ instead of some variables $x_0, \ldots, x_m$. Clearly, W' is finite. Applying Lemma 1, we get the infinite subset $A_{n+1} \subseteq A_n$ and $C_{n+1} \subseteq C_n$ such that $v(A_{n+1}) \cap C_{n+1} = \varnothing$ for every $w \in W'$. Fix an arbitrary element $a_{n+1} \in A_{n+1}$. After ω steps we get the required subset A. $\square$

Lemma 3. *Let G be a countable group and let $\mathcal{H}$ be a family of proper ideals on G such that $|\mathcal{H}| \leq \aleph_0$ and every ideal from $\mathcal{H}$ has a countable base. Then there exists an infinite subset A of G such that the ideal $\mathcal{I}(A)$ is proper and $\mathcal{I}(A) \notin \mathcal{H}$.*

Proof. For every ideal $\mathcal{I} \in \mathcal{H}$, we choose some countable base for $\mathcal{I}$ and enumerate $\{B_n : n \in \omega\}$ all the obtained subsets of G. For every $n \in \omega$, pick $b_n \in G \setminus B_n$ and put $B' = \{b_n : n \in \omega\}$. Clearly, for every $\mathcal{I} \in \mathcal{H}$ and every infinite subset D of B', we have $D \notin \mathcal{I}$. Then we partition B' into two infinite subset B, C and apply Lemma 2. □

Theorem 1. *For every countable group G, there exist at least $\aleph_1$ proper single-generated group ideals on G.*

Proof. Apply Lemma 2 and Lemma 3. □

Question 1. *Do there exist $2^{\aleph_0}$ proper single-generated group ideals on every countable group?*

Every uncountable group G also has a non-finitary proper group ideal but by the trivial reason: it suffices to take an arbitrary infinite subset $A \subset G$ such that $|A| < |G|$ and look at the ideal $\mathcal{I}(A)$.

Question 2. *Let G be an uncountable group. Does there exist a subset $A \subset G$ such that $|A| = |G|$ and the ideal $\mathcal{I}(A)$ is proper?*

In the next section we give the positive answers to Question 1 and Question 2 in the case of Abelian groups.

Theorem 2. *A group G admits a proper uniformly invariant group ideal if and only if, for any $n \in \omega$ and $g_1, g_2, \dots, g_n \in G$,*

$$g_1^G g_2^G \dots g_n^G \neq G,$$

where $g^G = \{x^{-1}gx : x \in G\}$.

Proof. Clearly, the family of all subsets of the form $g_1^G g_2^G \dots g_n^G$ is the base for some uniformly invariant group ideal on G. On the other hand, if $\mathcal{I}$ is a connected uniformly invariant group ideal on G and $g \in G$, then $g^G \in \mathcal{I}$. □

By [5], there exists an infinite group with only two conjugated classes. By Theorem 2, this group does not admit proper uniformly invariant group ideals.

3. Group ideals on Abelian groups.

Lemma 4. *Let G be an Abelian group, $\mathcal{I}$ be a group ideal on G with a countable base. Let $X = \{x_n : n \in \omega\}$ be a countable subset of G such that the sequence $(x_n)_{n\in\omega}$ tends to infinity with respect to $\mathcal{I}$. Then there exists an infinite subset Y of X such that, for every partition of Y into two infinite cells Y_1 and Y_2,*

$$(\mathcal{I}(Y_1) \vee \mathcal{I}) \wedge (\mathcal{I}(Y_2) \vee \mathcal{I}) = \mathcal{I}.$$

Proof. Let $\{B_n : n \in \omega\}$ be a base for $\mathcal{I}$. We consider two cases.

Case 1. For every natural number k the sequence $\{kx_n : n \in \omega\}$ tends to infinity with respect to $\mathcal{I}$. We can choose inductively a subsequence $(y_n)_{n\in\omega}$ of $(x_n)_{n\in\omega}$ such that

$$\left(B_n + \sum_{i<n}\{\pm y_i, \pm 2y_i, ..., \pm ny_i\}\right) \cap (B_n + \{\pm y_n, \pm 2y_n, ..., \pm ny_n\}) = \varnothing$$

for every $n \in \omega$. Put $Y = \{y_n : n \in \omega\}$.

Case 2. There exist a subsequence $(z_n)_{n\in\omega}$ of $(x_n)_{n\in\omega}$ and a natural number $k > 1$ such that the subsequence $\{kz_n : n \in \omega\}$ is bounded with respect to $\mathcal{I}$. We may suppose that the sequence $\{mz_n : n \in \omega\}$ tends to infinity for every $m \in \{1, \dots, k-1\}$. Then we choose inductively a subsequence $(y_n)_{n\in\omega}$ of $(z_n)_{n\in\omega}$ such that

$$\left(B_n + \sum_{i<n}\{\pm y_i, \pm 2y_i, ..., \pm (k-1)y_i\}\right) \cap (B_n + \{\pm y_n, \pm 2y_n, ..., \pm (k-1)y_n\}) = \varnothing$$

for every $n \in \omega$. Put $Y = \{y_n : n \in \omega\}$. □

In the following lemma we identify the cardinal k with the set of ordinals $< k$ and omit its easy proof.

Lemma 5. *Let G be an infinite group, k be an infinite cardinal. Assume that G has a subgroup H which is the direct sum $\oplus_{\alpha\in k} H_\alpha$ of non-zero groups. Then, for every partition $k = k' \cup k''$,*

$$\mathcal{I}\Big(\bigoplus_{\alpha\in k'} H_\alpha\Big) \wedge \mathcal{I}\Big(\bigoplus_{\alpha\in k''}, H_\alpha\Big) = \mathcal{F}(G).$$

Theorem 3. *Let G be an infinite Abelian group of cardinality k. Then there exist 2^k proper single-generated group ideals and 2^{2^k} proper group ideals on G.*

Proof. Assume that G has a subgroup H which is a direct sum $\oplus_{\alpha\in k} H_\alpha$ of non-zero groups. The first statement follows directly from Lemma 5. To prove the second statement, for every free ultrafilter φ on k, we denote by $\mathcal{I}_\varphi$ the group ideal on G with the base of subsets of the form $F + \oplus_{\alpha\in k\setminus\Phi} H_\alpha$, where $F \in \mathcal{F}(G)$ and Φ runs over φ. Let ψ be a free ultrafilter on k and $\varphi \neq \psi$. Partition $k = \Phi + \Psi$ so that $\Phi \in \varphi, \Psi \in \psi$. Applying Lemma 5, we conclude that $\mathcal{I}_\varphi \neq \mathcal{I}_\psi$ and notice that the set of all free ultrafilters on k is of cardinality 2^{2^k}.

Suppose that G has no subgroups which are the direct sums of k-many non-zero groups. Then G is countable and we can choose an injective sequence $(x_n)_{n\in\omega}$ tending to infinity with respect to the finitary ideal $\mathcal{F}(G)$. Put $\mathcal{I} = \mathcal{F}(G)$ and let Y be the subset of $X = \{x_n : n \in \omega\}$ given by Lemma 4. The first statement follows directly from Lemma 4. For every free ultrafilter φ on Y, we denote by $\mathcal{I}_\varphi$ the smallest group ideal containing the ideal $\{Y\setminus\Phi : \Phi \in \varphi\}$. If ψ is a free ultrafilter on Y and $\varphi \neq \psi$, we apply Lemma 4 to show that $\mathcal{I}_\varphi \neq \mathcal{I}_\psi$, so there are $2^{2^{\aleph_0}}$ proper group ideals on G. □

Question 3. *Is the family of all proper group ideals on every countable group of cardinality $2^{2^{\aleph_0}}$?*

Now we describe the constructive family of proper group ideals on $\mathbb{Z}$ using the arithmetic to negative bases. It is easy to check that every integer can be uniquely represented in the form

$$\sum d_n(-2)^n, \quad d_n \in \{0,1\}$$

with $d_n = 0$ for all except finitely many n.

We consider the following more general construction. Let $(b_n)_{n\in\omega}$ be a sequence of positive integer with $b_0 = 1$ and $b_n \geq 2$ for each $n > 0$. We put $f_n = b_0 b_1 \cdots b_n$ for each $n \in \omega$. By [1, Theorem 4.1], every integer m has a unique representation in the form

$$m = \sum d_n(-1)^n f_n, \quad d_n \in \{0, 1, ..., b_{n+1}\}$$

with $d_n = 0$ for all except finitely many n. We define the supp m as the set of all $n \in \omega$ such that $d_n \neq 0$ in this representation and put

$$B_k = \{m \in \mathbb{Z} : |\operatorname{supp} m| \leq k\}, \quad k \in \omega.$$

Then the family $\{B_k : k \in \omega\}$ is a base for some proper group ideal determined by the sequence $(b_n)_{n\in\omega}$. This ideal is single-generated by the set B_1. It can be show that varying $(b_n)_{n\in\omega}$ we get $2^{\aleph_0}$ distinct ideals on $\mathbb{Z}$. In the case $b_n = (2)_{n\in\omega}$ the corresponding group ballean can be considered as the counterpart of 2-adic topology on $\mathbb{Z}$.

4. Lattice of group ideals. The set of all group ideals on a group G is a complete lattice with the operations $\wedge$ and $\vee$ defined in Section 1. The set of all proper group ideals is a complete $\wedge$-semilattice with the smallest element $\mathcal{F}(G)$. The set of all proper group ideals with countable base is a $\wedge$-semilattice. The union $\bigcup\mathcal{C}$ of every chain $\mathcal{C}$ of proper group ideals is a proper group ideal. By Zorn Lemma, every proper group ideal is contained in some maximal proper group ideal.

Theorem 4. *Let G be an infinite group and let $\mathcal{I}$ be a non-finitary proper group ideal on G. If G is either countable or Abelian, then there exists a single-generated ideal $\mathcal{I}'$ on G such that $\mathcal{F}(G) \subset \mathcal{I}' \subset \mathcal{I}$.*

Proof. If G is countable, we take an infinite subset $B' \in \mathcal{I}$, partition B' into two infinite subset B, C, apply Lemma 2 and put $\mathcal{I}' = \mathcal{I}(A)$.

Assume that G is Abelian. Since $\mathcal{I} \neq \mathcal{F}(G)$, there exists a countable subset $Y \in \mathcal{I}$. Let S be a subgroup of G generated by Y. By the above paragraph, there exists a subset $A \subset S$ such that $\mathcal{F}(S) \subset \mathcal{I}(A) \wedge \mathcal{F}(S) \subset \mathcal{I} \wedge \mathcal{F}(S)$. Put $\mathcal{I}' = \mathcal{I}(A)$. □

Question 4. *Is Theorem 4 true for every infinite group?*

Theorem 5. *Let G be an Abelian group, $\mathcal{I}_1, \mathcal{I}_2$ be proper group ideals on G. If $\mathcal{I}_1$ has a countable base and $\mathcal{I}_1 \subset \mathcal{I}_2$, then there exists a single-generated group ideal $\mathcal{I}$ on G such that $\mathcal{I}_1 \subset \mathcal{I} \subset \mathcal{I}_2$.*

Proof. Pick an arbitrary subset $A \in \mathcal{I}_2 \backslash \mathcal{I}_1$. Since $\mathcal{I}_1$ has a countable base, we can choose an injective sequence $(x_n)_{n\in\omega}$ in A tending to infinity with respect to $\mathcal{I}_1$. Put $X = \{x_n : n \in \omega\}$ and choose the subset Y of X given by Lemma 4. Partition Y into two infinite subsets Y_1, Y_2 and put $\mathcal{I} = \mathcal{I}(Y_1) \vee \mathcal{I}_1$. □

Question 5. *Let G be a countable Abelian group, $\mathcal{I}_1, \mathcal{I}_2$ be proper group ideals on G such that $\mathcal{I}_1 \subset \mathcal{I}_2$. Does there exist a group ideal $\mathcal{I}$ on G such that $\mathcal{I}_1 \subset \mathcal{I} \subset \mathcal{I}_2$?*

Let G be a group, $\mathcal{I}_1$ and $\mathcal{I}_2$ be proper group ideals on G. We say that $\mathcal{I}_2$ is a $\wedge$-*complement* to $\mathcal{I}_1$ if $\mathcal{I}_1 \wedge \mathcal{I}_2 = \mathcal{F}(G)$ and $\mathcal{I}_2 \neq \mathcal{F}(G)$. If $\mathcal{I}$ has at least one $\wedge$-complement, we say that $\mathcal{I}$ is $\wedge$-*complementable*. The following statement resembles the criterion of complementability of group topologies on an Abelian group [11, Theorem 2.4.4].

Theorem 6. *Let G be an Abelian group and let $\mathcal{I}$ be a proper group ideal on G with a countable base. Then $\mathcal{I}$ is $\wedge$-complementable if and only if at least one of the following statements holds:*

- (i) *for every prime p, the subgroup $pG = \{pg : g \in G\}$ is unbounded in $(G, \mathcal{I})$;*
- (ii) *there exists a prime p such that the subgroup $S_p = \{g \in G : pg = 0\}$ is unbounded in $(G, \mathcal{I})$.*

Proof. Assume that $\mathcal{I}'$ is a $\wedge$-complement to $\mathcal{I}$. Choose a subset $A \in \mathcal{I}'$ such that A is unbounded in $(G, \mathcal{I})$. Since $\mathcal{I}$ has a countable base, we can choose an injective sequence $(a_n)_{n\in\omega}$ tending to infinity with respect to $\mathcal{I}$. If the sequence $(ka_n)_{n\in\omega}$ tends to infinity with respect to $\mathcal{I}$ for every natural number k, then (i) holds. Otherwise, we can choose an injective sequence $(b_n)_{n\in\omega}$ and a prime number p such that $\{b_n : n \in \omega\} \in \mathcal{I}'$, $(b_n)_{n\in\omega}$ tends to infinity with respect to $\mathcal{I}$ and $\{pb_n : n \in \omega\}$ is bounded in $(G, \mathcal{I})$. Since $\mathcal{I}'$ is a $\wedge$-complement to $\mathcal{I}$, $\{pb_n : n \in \omega\}$ is finite. Then we pick a subsequence $(c_n)_{n\in\omega}$ of $(b_n)_{n\in\omega}$ such that $pc_n = pc_m$ for all n, m. Clearly, $\{c_n - c_0 : n \in \omega\}$ is bounded with respect to $\mathcal{I}$ and $c_n - c_0 \in S_p$ for each $n \in \omega$. Hence, (ii) holds.

Now let (i) be satisfied. Then we can choose an injective sequence $(x_n)_{n\in\omega}$ such that the sequence $(kx_n)_{n\in\omega}$ tends to infinity with respect to $\mathcal{I}$ for every natural number k. Let Y be a subset given by case 1 of Lemma 4. It is easy to check that $\mathcal{I}(A)$ is a $\wedge$-complement to $\mathcal{I}$, where A is an infinite subset of Y such that $Y \setminus A$ is infinite. If (ii) is satisfied, we can choose an injective sequence $(x_n)_{n\in\omega}$ in S_p tending to infinity with respect to $\mathcal{I}$. Put $X = \{x_n : n \in \omega\}$ and note that $\mathcal{I}(X)$ is a $\wedge$-complement to $\mathcal{I}$. □

Question 6. *Is Theorem 6 true for an arbitrary proper group ideal on an Abelian group?*

Let $\mathcal{I}_1, \mathcal{I}_2$ be proper group ideals on a group G. We say that $\mathcal{I}_2$ is a $\vee$-complement to $\mathcal{I}$ if $\mathcal{I}_1 \vee \mathcal{I}_2$ is the ideal of all subsets of G. If $\mathcal{I}_2$ is a $\vee$-complement and a $\wedge$-*complement* to $\mathcal{I}_1$, we say that $\mathcal{I}_2$ is a *complement* to $\mathcal{I}_1$.

Question 7. *Find criteria of $\vee$-complementability and complementability of a proper group ideal on an Abelian group.*

5. Ultrafilters and group ideals. Let G be a discrete group and let βG be the Stone-Čech compactification of G. We take the points of βG to be the ultrafilters on G with the points of G identified with the principal ultrafilters. For every subset $A \subseteq G$, we put $\bar{A} = \{q \in \beta G : A \in q\}$. The topology of βG can be defined by stating that the family $\{\bar{A} : A \subseteq G\}$ is a base for the open sets. Using the universal property of the Stone-Čech compactification, the multiplication on G can be extended to βG is such a way that, for every $r \in \beta G$, the right shift $x \mapsto xr$ is continuous, and, for every $g \in G$, the left shift

$x \mapsto gx$ is continuous. Formally, the product rq of the ultrafilters $r, q \in \beta G$ is defined by the rule: given an arbitrary subset A of G,

$$A \in rq \Leftrightarrow \{g \in G : g^{-1}A \in q\} \in r.$$

For more information about compact right topological semigroup βG and its combinatorial applications see [6] and [10].

Given an infinite group G and a proper group ideal $\mathcal{I}$ on G, we denote by $(G,\mathcal{I})^\sharp$ the set of all ultrafilters on G tending to infinity with respect to $\mathcal{I}$, and put $(G,\mathcal{I})^\flat = \beta G \setminus (G,\mathcal{I})^\sharp$. We omit the direct verification of the following statement.

Theorem 7. *For every infinite group G and every proper group ideal $\mathcal{I}$ on G, $(G,\mathcal{I})^\sharp$ is a closed two-sided ideal of the semigroup βG, $(G,\mathcal{I})^\flat$ is a subsemigroup of βG.*

It is well-known [6, Chapter 2] that every closed ideal of βG contains the minimal ideal. Hence, by Theorem 7, the set

$$G^\sharp = \bigcap \{(G,\mathcal{I})^\sharp : \mathcal{I} \text{ is a proper group ideal on } G\}$$

of all ultrafilters on G tending to infinity with respect to each proper group ideal on G is non-empty.

Question 8. *Give an intrinsic characterization of the ultrafilters from $G^\sharp$.*

Theorem 8. *Let G be an infinite Abelian group, φ be a free ultrafilter on G. Then there exists a countable subset $A \subset G$ such that the group ideal $\mathcal{I}(A)$ is proper and φ tends to infinity with respect to $\mathcal{I}(A)$.*

Proof. If every element of φ is an uncountable subset of G, we choose an arbitrary countable subset A of G. Otherwise, we may suppose that G is countable. Put $\mathcal{I} = \mathcal{F}(G)$ and take the subset Y given by Lemma 4. Partition Y into two infinite subsets Y_1 and Y_2. Then either $\mathcal{I}(Y_1) \cap \varphi = \varnothing$ or $\mathcal{I}(Y_2) \cap \varphi = \varnothing$. If follows that φ tends to infinity either with respect to $\mathcal{I}(Y_1)$ or with respect to $\mathcal{I}(Y_2)$. □

Question 9. *Let G be a countable group and let φ be a free ultrafilter on G. Does there exist a proper group ideal $\mathcal{I}$ on G such that φ tends to infinity with respect to $\mathcal{I}$?*

A topological space without isolated points is called *maximal* if it has an isolated point in every stronger topology. By [13], every infinite group admits a maximal topology in which every left shift is continuous. In this topology the only one free ultrafilter converges to the unit. Moreover, under Martin Axiom there exist maximal topological groups but such a group cannot be constructed in ZFC without additional set-theoretic assumptions.

A connected ballean (X, P, B) is called *prebounded* [15] if it is unbounded but every stronger ballean with the support X is bounded. By Zorn Lemma, every connected unbounded ballean can be strengthened to some prebounded ballean.

Question 10. *Does there exist a countable group and a proper group ideal $\mathcal{I}$ on G such that the uniformly left invariant ballean $(G,\mathcal{I})$ is prebounded?*

6. Uniform groups and group balleans. Let G be a group with the unit e, $\mathcal{U}$ be a uniformity on G, φ_e be the filter of neighborhoods of e of the uniform topological space $(G,\mathcal{U})$. We say that $(G,\mathcal{U})$ is

• *left* (resp. *right) invariant* if all the shifts $x \mapsto gx$ (resp. $x \mapsto xg$) are uniformly continuous;

• *uniformly left* (resp. *right) invariant* if the family $\{gU \times gU : g \in G, U \in \varphi_e\}$ (resp. $\{Ug \times Ug : g \in G, U \in \varphi_e\}$) forms a base for the uniformity $\mathcal{U}$;

• *uniform* if it is uniformly left and right invariant.

We say that a filter ψ on a group G is a *group filter* if, for any $A, B \in \psi, AB \in \psi$ and $A^{-1} \in \psi$. If $(G, \mathcal{U})$ is uniformly left invariant, then φ_e is a group filter. On the other hand, if ψ is a group filter, then there is only one uniformity $\mathcal{U}$ on G such that $(G, \mathcal{U})$ is uniformly left invariant and $\psi = \varphi_e$. Thus, we can identify $(G, \mathcal{U})$ with the pair (G, φ_e).

Let us say that a group filter ψ on a group G is *proper* if $\bigcap \psi = \{e\}$ and $\{e\} \notin \psi$. Clearly, for every proper group filter on G, the topology on G determined by the corresponding uniformity is Hausdorff and non-discrete. Among the exotic groups constructed by Ol'shanskii [9] one can find an infinite torsion group in which any two non-identity subgroups have non-identity intersection. It is easy to see that this group does not admit the proper group filters. On the other hand, every non-torsion group has a proper group filter.

Question 11. *Which countable torsion groups have at least one proper group filter?*

A uniformly left invariant group (G, φ_e) is uniform if and only if φ_e is invariant, i.e. for every $U \in \varphi_e$ there exists a subset $V \in \varphi_e$ such that $x^{-1}Vx \subseteq U$ for every $x \in G$. In this case the multiplication $(G, \varphi_e) \times (G, \varphi_e) \to (G, \varphi_e)$ and the inversion $(G, \varphi_e) \to (G, \varphi_e)$ are uniformly continuous. In particular, every uniform group is a topological group with equal left and right group uniformities, i.e. a SIN-group. On the other hand, if G is a SIN-group and φ_e is a filter of its neighborhoods of e, then (G, φ_e) is a uniform group.

Question 12. *Which countable groups have at least one proper invariant group filter? In other words, which countable groups admit a non-discrete Hausdorff SIN-topology?*

As a possible answer to this question we expect some modification of Markov criterion of topologizability of a countable group [7].

The only glance at the above definitions calls some feelings of duality between group ideals and group filters, group balleans and uniform groups. It would be interesting to formalize these feelings in the frame of some duality theory.

REFERENCES

1. Budak T., Isik N., Pym J. *Subsemigroup of Stone-Čech compactification*, Math. Proc.Camb. Phil. Soc., **116** (1994), 99–118.
2. Dranishnikov A. *Asymptotic topology*, Russian Math. Surveys, **55** (2000), 71–116.
3. Fillali M., Protasov I. *Slowly oscillating function on locally compact groups*, Appl. Gener. Topology (submitted).
4. Frider V., Zarichnyi M. *On coarse anti-Lawson semilattices*, Mat. Stud. **21** (2004), no.1, 3–12.
5. Higman G., Neumann B., Heumann H. *Embedding theorems for groups*, J. London Math. Soc., **24** (1949), 247–254.
6. Hindman N., Strauss D. Algebra in the Stone-Čech Compactification, de Grueter Exposition in Math., V.27, 1998.

7. Markov A.A. *On unconditionally closed sets*, Topology and Topol. Algebra. Translations Series 1, **8** (1962), 273–304.

8. Mitchener P.D. *Coarse Homology Theories*, Algebr. Ceom. Topology, **1** (2001), 271-294.

9. Ольшанский А.Ю. Геометрия определяющих соотношений, М.: Наука, 1984.

10. Protasov I.V. Combinatorics of Numbers, Math. Stud. Monogr. Ser., V.2, VNTL, Lviv, 1997.

11. Protasov I.V., Zelenyuk E.G. *Topologies on Groups Determined by Sequences*, Math. Stud. Monogr. Ser., V.4, VNTL, Lviv, 1999.

12. Protasov I., Banakh T. *Ball Structures and Colorings of Groups and Graphs*, Mat. Stud. Monogr. Ser., V.11, VNTL, Lviv, 2003.

13. Protasov I.V. *Maximal topologies on groups*, Sib. Math. J., **39** (1998), 1368-1381.

14. Protasov I.V. *Metrizable ball structures*, Algebra and Discrete Math. (2002), no.1, 129–141.

15. Protasov I.V. *Uniform ball structures*, Algebra and Discrete Math. (2003), no.1, 93–102.

16. Protasov I.V. *Normal ball structures*, Math. Stud. **20** (2003), no.1, 3–16.

17. Protasov I.V. *Morphisms of ball structures of groups and graphs*, Ukr. Math. J. **54** (2002), 847-855.

18. Protasov I.V. *Binary coronas of balleans*, Algebra and Discrete Math. (2003), no.4, 50–65.

19. Protasov I.V. *Coronas of balleans*, Topol. Appl. (to appear).

20. Protasov I.V. *Resolvability of ball structures*, Appl. Gener. Topology (to appear).

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