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O. V. Makhney, R. M. Tatsiy

THE STRUCTURE OF CAUCHY FUNCTION OF A VECTOR QUASIDIFFERENTIAL EQUATION

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The representation of a Cauchy matrix-function of a vector quasidifferential equation and its mixed quasiderivatives in the sense of the initial equation and the adjoint equation is constructed using a fundamental system of solutions and their quasiderivatives. Every element of such a matrix-function may be represented in the form of a ratio of two determinants, where the second one is a quasiwronskian and the first one differs from it by only one row.

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Построено изображение матрицы-функции Коши векторного квазидифференциального уравнения и её смешанных квазипроизводных в смысле исходного и сопряжённого уравнений через фундаментальную систему решений и их квазипроизводные. Каждый элемент такой матричной функции может быть представлен в виде отношения двух определителей, причём второй является квазивронскианом, а первый отличается от него только одной строкой.

1. Preliminary remarks. Differential expressions and equations which contain summands of the form $\left(p_{ij}y^{(s-i)}\right)^{(m-j)}$ are frequently occurred in applied problems; it is accepted to call them quasidifferential ones. Probably, D. Shin [1] was the first one who had suggested this name and the method of the introduction of quasiderivatives for their investigation (the paper [1] had been published also in French).

We shall consider the initial problem

$$\sum_{i=0}^{s}\sum_{j=0}^{m}(-1)^{m-j}\left(A_{ij}(x)Y^{(s-i)}\right)^{(m-j)}=0, \tag{1}$$

$$Y^{[\nu]}(x_0)=C_\nu, \quad x_0\in I,\ \ \nu\in\{0,\dots,n-1\},\ \ n=s+m, \tag{2}$$

where s, m are natural numbers, $A_{ij}(x)$ ($i\in\{0,\dots,s\}$, $j\in\{0,\dots,m\}$) are square complex-valued matrices-functions of the l-th order such that $A_{00}^{-1}(x)$ is bounded and measurable one on an open interval I, all elements of the matrices $A_{i0}(x)$ and $A_{0j}(x)$ ($i\in\{1,\dots,s\}$, $j\in\{1,\dots,m\}$) are square-integrable functions on I, $A_{ij}(x)=B'_{ij}(x)$, all elements of the

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matrices $B_{ij}(x)$ have locally bounded variation on I and are continuous on the right there, and the quasiderivatives of the matrix solution $Y(x)$ are defined by the formulas:

$$Y^{[k]} = Y^{(k)}, \quad k \in \{0, \ldots, s-1\}; \;\; Y^{[s]} = \sum_{i=0}^{s} A_{i0}(x) Y^{(s-i)};$$
$$Y^{[s+k]} = -\left(Y^{[s+k-1]}\right)' + \sum_{i=0}^{s} A_{ik}(x) Y^{(s-i)}, \quad k \in \{1, \ldots, m\}.$$

The quasiderivatives in the sense of the adjoint of equation (1)

$$\sum_{j=0}^{m} \sum_{i=0}^{s} (-1)^{s-i} \left(A_{ij}^*(x) Y^{(m-j)}\right)^{(s-i)} = 0, \tag{3}$$

where $A_{ij}^*(x)$ are Hermitian adjoint matrices, are defined [2, p. 39] by the formulas:

$$Y^{\{k\}} = Y^{(k)}, \; k \in \{0, \ldots, m-1\}; \;\; Y^{\{m\}} = -\sum_{j=0}^{m} A_{0j}^*(x) Y^{(m-j)};$$
$$Y^{\{m+k\}} = -\left(Y^{\{m+k-1\}}\right)' - \sum_{j=0}^{m} A_{kj}^*(x) Y^{(m-j)}, \; k \in \{1, \ldots s\}.$$

With the help of the transformation of equation (1) to the well-posed system of quasidifferential equations of the first order by the method of the introduction of quasiderivatives we can prove [2, p. 41] that the matrix solution of the initial problem (1), (2) exists and is a unique one in the class of absolutely continuous on I matrices-functions, its quasiderivatives to the $(s-1)$-th order are absolutely continuous on I matrices-functions and all elements of the rest quasiderivatives up to the order $n-1$ have locally bounded on I variation and they are continuous on the right there. Analogously, there exists a matrix solution of equation (3) with the initial conditions $Y^{\{\nu\}}(x_0) = \tilde{C}_\nu$, $x_0 \in I$, $\nu \in \{0, \ldots, n-1\}$, which together with its quasiderivatives to the $(m-1)$-th order is an absolutely continuous one and the rest of its quasiderivatives up to the order $n-1$ are continuous on the right matrices-functions of locally bounded on I variation.

By a Cauchy matrix-function of equation (1) one understands a matrix function $K(x,t)$ of the order $l \times l$ which satisfies equation (1) by the first variable and, moreover, $K^{[i]}(t,t) = 0$, $i \in \{0, \ldots, n-2\}$, $K^{[n-1]}(t,t) = E$.

We shall use the symbol $K^{*\{j\}*[i]}(x,t)$ for the mixed quasiderivatives of the matrix-function $K(x,t)$. This symbol means that at first the i-th quasiderivative by x in the sense of the initial equation is taken, then the Hermitian adjoining, then the j-th quasiderivative by t in the sense of the adjoint equation and, after all, the Hermitian adjoining again. According to [2, p. 46], $K^{*\{j\}*[i]}(x,t) = K^{[i]*\{j\}*}(x,t)$, $i,j \in \{0, \ldots, n-1\}$.

2. Main results. In problems of not only applied but also theoretical nature one can run into a problem of constructing a Cauchy function and its mixed quasiderivatives in the sense of the initial and adjoint equations via a certain fundamental system of solutions and its quasiderivatives.

Theorem. *Let $Y_1(x)$, $Y_2(x)$, ..., $Y_n(x)$ be a fundamental system of solutions of the matrix equation (1). By $y_{ipqj}(x)$ we shall denote the element which lies in the intersection of p-th row and q-th column of the matrix $Y_j^{[i-1]}(x)$, that is $Y_j^{[i-1]}(x) = (y_{ipqj}(x))(i,j \in \{1, \ldots, n\}$, $p,q \in \{1, \ldots, l\})$. We shall also designate the determinant $W(t) = \det\left(Y_j^{[i-1]}(t)\right)_{i,j=1}^{n} =$*

$\det\left(\left(y_{ipqj}(t)\right)_{p,q=1}^{l}\right)_{i,j=1}^{n}$ *by quasiwronskian of quasidifferential equation (1). Then a Cauchy matrix-function of equation (1) and its mixed quasiderivatives are square matrices of l-th order, every element of which is represented by a ratio of two determinants:*

$$K_{pq}^{*\{j\}*[i]}(x,t) = \frac{M_{ipqj}(x,t)}{W(t)}, \quad i,j \in \{0,\dots,n-1\}, \quad p,q \in \{1,\dots,l\}, \tag{4}$$

where each of determinants $M_{ipqj}(x,t)$ differs from the quasiwronskian $W(t)$ by only one "non-standard" row $(y_{i+1,p11}(x),\dots,y_{i+1,pl1}(x),\dots,y_{i+1,p1n}(x),\dots,y_{i+1,pln}(x))$ which lies in the $((n-j-1)l+q)$-th position from above and the element $K_{pq}^{*\{j\}*[i]}(x,t)$ is situated in the intersection of p-th row and q-th column of the matrix $K^{*\{j\}*[i]}(x,t)$.

Proof. Let $R(x) = \left(Y_j^{[i-1]}(t)\right)_{i,j=1}^{n}$ be an integral matrix of equation (1). Then $B(x,t) = R(x)R^{-1}(t)$ is the evolutionary operator which has the structure [3]

$$B(x,t) = \begin{pmatrix} K^{*\{n-1\}*}(x,t) & \cdots & K(x,t) \\ \vdots & \ddots & \vdots \\ K^{*\{n-1\}*[n-1]}(x,t) & \cdots & K^{[n-1]}(x,t) \end{pmatrix}, \tag{5}$$

where $K(x,t)$ is a Cauchy matrix-function of equation (1).

By virtue of (5), we can take

$$K^{*\{j\}*[i]}(x,t) = \frac{1}{W(t)} \sum_{k=1}^{n} Y_k^{[i]}(x) V_{n-j,k}(t), \quad i,j \in \{0,\dots,n-1\},$$

where $V_{ij}(t)$ is the transpose of the matrix formed out of the algebraic adjoints of the elements of the matrix $Y_j^{[i-1]}(t)$ in the determinant $W(t)$, that is

$$V_{ij}(t) = \begin{pmatrix} A_{i11j}(t) & \cdots & A_{il1j}(t) \\ \vdots & \ddots & \vdots \\ A_{i1lj}(t) & \cdots & A_{illj}(t) \end{pmatrix}, \quad i,j \in \{1,\dots,n\}.$$

Therefore

$$K_{pq}^{*\{j\}*[i]}(x,t) = \frac{1}{W(t)} \sum_{k=1}^{n} \sum_{g=1}^{l} y_{i+1,pqk}(x) A_{n-j,qgk}(t) = \frac{M_{ipqj}(x,t)}{W(t)},$$

so far as $A_{n-j,qgk}(t)$ is the algebraic adjoint of the element $y_{n-j,qgk}(t)$ in the determinant $W(t)$. □

The row dependent on x is situated at the q-th position in the $(n-j)$-th zone from above in the determinant $M_{ipqj}(x,t)$ though it was taken from the p-th position in the $(i+1)$-th zone from above in the determinant $W(x)$. In fact, it means that even if $j \neq 0$ the matrix $K^{*\{j\}*[i]}(x,t)$ depends only on the elements of the matrices $Y_1(x), \dots, Y_n(x), Y_1(t), \dots, Y_n(t)$ and their quasiderivatives in the sense of the initial equation.

Remark. In the scalar case ($l = 1$) formulas (4) take up the form [4]

$$K^{\{j\}[i]}(x,t) = \frac{1}{W(t)} \begin{vmatrix} \varphi_1(t) & \cdots & \varphi_n(t) \\ \cdots & \cdots & \cdots \\ \varphi_1^{[n-j-2]}(t) & \cdots & \varphi_n^{[n-j-2]}(t) \\ \varphi_1^{[i]}(x) & \cdots & \varphi_n^{[i]}(x) \\ \varphi_1^{[n-j]}(t) & \cdots & \varphi_n^{[n-j]}(t) \\ \cdots & \cdots & \cdots \\ \varphi_1^{[n-1]}(t) & \cdots & \varphi_n^{[n-1]}(t) \end{vmatrix}, \quad i, j \in \{0, \ldots, n-1\},$$

where $\varphi_1(t)$, $\varphi_2(t)$, $\ldots$, $\varphi_n(t)$ form a fundamental system of solutions of equation (1), and $W(t)$ is a quasiwronskian formed out of them.

The suggested method of the construction of a Cauchy matrix-function and its mixed quasiderivatives in the sense of the initial equation and the adjoint equation via a fundamental system of solutions and their quasiderivatives allows us to overcome some difficulties which are occurred in problems of not only applied but also theoretical nature.

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National University "Lvivs'ka Polytechnika"
st. S. Bandery 12, 79013, Lviv, Ukraine
makhney1@yahoo.com

Akademia Bydgoska im. Kazimierza Wielkego, Institute of Mathematics,
pl. Weyssenhoffa 11, PL-85-072, Bydgoszcz, Poland

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