УДК 517.95

T. Winiarska

QUASILINEAR EVOLUTION EQUATIONS WITH OPERATORS DEPENDENT ON t

T. Winiarska. *Quasilinear evolution equations with operators dependent on t*, Matematychni Studii, **21** (2004) 170–178.

The purpose of this paper is to present some theorems on existence and uniqueness of solutions for some semilinear Cauchy problems of second order with operators $A(t)$ not densely defined in a given Banach space X. To this end, we begin with reduction of our problem to a problem in which the operators have the same (independent of t) domain D.

Т. Винярска. *Квазилинейные эволюционные уравнения с операторами, зависящими от t* // Математичні Студії. – 2004. – Т.21, №2. – C.170–178.

Целью этой работы является представление некоторых теорем о существовании и единственности решений некоторых полулинейных задач Коши второго порядка с операторами $A(t)$, не плотно определенными в заданном банаховом пространстве X. Для этого мы начинаем с редукции нашей задачи к задаче, в которой операторы имеют одну и ту же (независимую от t) область D.

1. Introduction. Our main objective is to investigate the abstract semilinear second order initial value problem

$$\begin{cases} \dfrac{d^2u}{dt^2} = A(t)u + f\left(t, u, \dfrac{du}{dt}\right), & t \in (0, T], \\ u(0) = u_0, \quad \dfrac{du}{dt}(0) = u_1, & u_0, u_1 \in X \end{cases} \tag{1}$$

where $(X, \|\,.\,\|)$ is a Banach space, u a mapping from $\mathbb{R}$ to X, f a nonlinear mapping from $[0, T] \times X \times X$ into X and $\{A(t)\}$, $t \in [0, T]$ a family of linear closed operators

$$A(t)\colon X \supset D_t \longrightarrow X$$

with domains $D_t \subset X$ depending on t unnecessarily dense in X.

Most of the results concerning problem (1) have been obtained under the assumption that the operators $\{A(t)\}$, $t \in [0, T]$, of a given family have domains independent of t (see e. g. [4, 5, 8]).

The case of densely defined operators $A(t)$ with domains dependent on t has been considered in [11].

2000 *Mathematics Subject Classification:* 47A05, 35J25, 42C99.

The main our goal is to present a result on existence and uniqueness of the mild solution for problem (1). To this end, we use some extrapolation spaces associated to the family $\{A(t)\}$, $t \in [0,T]$, in order to reduce problem (1) to a second order initial value problem with operators densely defined.

2. Preliminaries. We make the following assumptions on the family $\{A(t)\}$, $t \in [0,T]$.

(Z_1) There exists a closed subspace $Y \subset X$ such that $Y = \bar{D}_t$ for each $t \in [0,T]$ and $Y \neq X$.

(Z_2) For each $t \in [0,T]$ the resolvent set $\varrho(A(t)) = \varrho(A)$ of $A(t)$ is independent of $t \in [0,T]$ and $[0,\infty) \subset \varrho(A)$.

(Z_3) The family $A(t), t \in [0,T]$, is stable in the sense that there exists a real number $M \geq 1$ such that

$$\|(\lambda - A(t_k))^{-1}(\lambda - A(t_{k-1}))^{-1} \dots (\lambda - A(t_1))^{-1}\| \leq M\lambda^{-k} \tag{2}$$

and

$$\|(\lambda - A(t_1))^{-1}(\lambda - A(t_2))^{-1} \dots (\lambda - A(t_k))^{-1}\| \leq M\lambda^{-k}, \tag{3}$$

for $\lambda > 0$ and every finite sequence $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$, $k \in \mathbb{N}$.

From the assumption (Z_3) for $k = 1$ it follows that

$$\|(\lambda - A(t))^{-1}\| \leq \frac{M}{\lambda} \quad \text{for } \lambda > 0,\ t \in [0,T]. \tag{4}$$

It follows from (4) that it is possible to define for each $t \in [0,T])$ the operator

$$B(t)x = [A(t)]^{\frac{1}{2}}x := \frac{i}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda, A(t))(-A(t))x\,d\lambda,$$

where $x \in D_t$ (cf. [6]).

(Z_4) For each $t \in [0,T]$, $\lambda > 0$ and $n \in N$

$$\left\| \frac{d^n}{d\lambda^n}(\lambda(\lambda^2 - A(t))^{-1}) \right\| \leq \frac{Mn!}{\lambda^{n+1}},$$

and

$$\left\| \frac{d^n}{d\lambda^n}(B(t)(\lambda^2 - A(t))^{-1}) \right\| \leq \frac{Mn!}{\lambda^{n+1}},$$

where $M \geq 1$ is from (2) and (3).

(Z_5) The mapping

$$[0,T] \ni t \longrightarrow R(\lambda, A(t))x \quad \text{is of class } C^1 \text{ for } x \in X.$$

(Z_6) For each t, $s \in [0,T]$ the operator $A^{-1}(t)A(s)$ is closable and the mapping

$$[0,T] \ni t \to \overline{A^{-1}(t)A(s)}$$

is continuous in $t = s$, i.e.

$$\lim_{t \to s} \|\overline{A^{-1}(t)A(s)} - I\| = 0$$

3. Existence and uniqueness of solution of problem (1). Following [11] we shall use some extrapolation spaces in order to reduce problem (1) to a similar problem with an operator $\hat{A}_0(t)$ densely defined in an extrapolation space $\hat{X}_0$.

To this purpose we define for each $t \in [0,T]$ the operator $A_0(t)$,

$$A_0(t) := A(t)|_{D_t^0}, \quad D_t^0 := \{x \in D_t : A(t)x \in Y\},$$

i.e. $A_0(t)$ is the part of $A(t)$ in Y. By the definition of $A_0(t)$ we see that the operator $A_0(t)\colon Y \supset D_t^0 \to Y$ is densely defined. In the space Y, for each $t \in [0,T]$, we define a weaker norm

$$|x|_t^{A_0} := \|R(0, A_0(t))x\|, \quad \text{for } x \in Y,\ t \in [0,T]$$

From (Z_6) it follows that for each $t \in [0,T]$ the norms $|\,.\,|_t^{A_0}$ defined by () are equivalent (cf. [12]). Taking in the space Y the norm

$$|x|_0^{A_0} := \|R(0, A_0(0))x\| = \|A_0^{-1}(0)x\|, \quad \text{for } x \in Y \tag{5}$$

we denote by $\hat{Y}^{A_0}$ the space which is the completion of the space Y with norm (5) to a Banach space. Since the operator

$$A_0(t)\colon Y \supset D_t^0 \longrightarrow \hat{Y}^{A_0}$$

is bounded for each $t \in [0,T]$, we can extend it to the closure of its domain, i.e. to $\bar{D}_t^0 = Y$. Then we define

$$\hat{A}_0(t)\colon \hat{Y}^{A_0} \supset Y \longrightarrow \hat{Y}^{A_0} \quad \text{for } t \in [0,T]$$

to be the extension of $A_0(t)$. In this way we get a family of linear densely defined operators $\{\hat{A}_0(t)\}$, $t \in [0,T]$, for which $D(\hat{A}_0(t)) = Y$ for $t \in [0,T]$, where Y is dense in $\hat{Y}^{A_0}$.

Our main objective of this paper is to consider problem (1) in the case when f is the mapping

$$f\colon [0,T] \times X \times X \longrightarrow X.$$

To the purpose we define $\hat{X}_0^A$ as the completion of the space X with the norm

$$|x|_0^A = \|R(0, A(0))x\|, \quad x \in X$$

to a Banach space. By virtue of ([7], Th. 3.1.10) we may identify $\hat{Y}^{A_0}$ with $\hat{X}_0^A$ and $\hat{A}_0(t)$ with $\hat{A}(t)$, where all of the mappings

$$\hat{A}(t) : \hat{X}_0^A \supset Y \longrightarrow \hat{X}_0^A, \quad t \in [0,T]$$

are defined on the same subspace Y which is dense in $\hat{X}_0^A$.

From now on we will consider the problem

$$\begin{cases} \dfrac{d^2u}{dt^2} = \hat{A}(t)u + f(t,u,u') \\ u(0) = u_0 \in X \\ u'(0) = u_1 \in X. \end{cases} \tag{6}$$

Under assumptions (Z_1)–(Z_6) we get that the family $\{\hat{A}(t)\}$, $t \in [0,T]$, has the following properties:

1^0. $D(\hat{A}(t)) = Y$, for each $t \in [0,T]$.

2^0. $D(\hat{A}(t))$ is dense in $\hat{X}_0^A$.

3^0. $\hat{A}(t)x = \hat{B}^2(t)x$, for each $t \in [0,T]$ and $x \in Y$, where

$$\hat{B}(t)x = \frac{i}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda, \hat{A}(t))(-\hat{A}(t))x d\lambda \quad \text{for } x \in Y. \tag{7}$$

The operator $\hat{B}(t)$ may not be closed as an operator defined on $\hat{X}_0^A$, but it is closable because

$$\hat{B}^{-1}(t) = \frac{1}{\pi i}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda, \hat{A}(t)) d\lambda$$

is bounded in $\hat{X}_0^A$.

In the sequel the closure of operator (7), i.e. the operator

$$\hat{B}(t)\colon \hat{X}_0^A \supset D(\hat{B}(t)) \longrightarrow \hat{X}_0^A$$

will be denoted by the same symbol $\hat{B}(t)$.

Lemma 1. *Under assumptions (Z_1)–(Z_6) the family of operators*

$$\hat{B}(t)\hat{B}^{-1}(0)\colon \hat{X}_0^A \longrightarrow \hat{X}_0^A$$

is uniformly bounded, i.e. there exists a constant $K \geq 0$ such that

$$|\hat{B}(t)\hat{B}^{-1}(0)x|_0^A \leq K|x|_0^A \quad \textit{for } x \in \hat{X}_0^A,\ t \in [0,T]\,.$$

Proof. For $x \in (Y, |\cdot|_0^A) \subset \hat{X}_0^A$ we have

$$\begin{aligned}|\hat{B}(t)\hat{B}^{-1}(0)x|_0^A &= \|A_0^{-1}(0)\hat{B}(t)\hat{B}^{-1}(0)x\| \leq \\ &\leq \|A_0^{-1}(0)A_0(t)\|\,\|A_0^{-1}(t)\hat{B}(t)\hat{B}^{-1}(0)x\|.\end{aligned}$$

By (Z_6), we have

$$\begin{aligned}|\hat{B}(t)\hat{B}^{-1}(0)x|_0^A &\leq C\|A_0^{-1}(t)\hat{B}(t)\hat{B}^{-1}(0)x\| = C\|A_0^{-1}(t)B(t)B^{-1}(0)x\| = \\ &= C\|B^{-1}(t)B^{-1}(0)x\| = C\|B^{-1}(t)B^{-1}(0)A_0(0)A_0^{-1}(0)x\| \leq \\ &\leq C\|B^{-1}(t)B^{-1}(0)A_0(0)\|\,|x|_0^A = C\|B^{-1}(t)B(0)\|\,|x|_0^A\,,\end{aligned}$$

where $C = \sup\{\|A_0^{-1}(t)A_0(t)\| : t \in [0,T]\}$. By virtue of ([11], Lemma 2), there exists $C_0 > 0$ such that

$$\|B^{-1}(t)B(0)\| \leq C_0 \quad \text{for } t \in [0,T]$$

which ends the proof (with $K = CC_0$). □

Lemma 2. *Under assumptions (Z_1)–(Z_6) the norms*

$$|||x|||_t := |\hat{B}(t)x|_0^A \quad \textit{for } x \in Y$$

corresponding to $t \in [0,T]$ are equivalent.

Proof. Let $x \in (Y, |\,.\,|_0^A)$ be arbitrary. By Lemma 1, $|\hat{B}(t)\hat{B}^{-1}(0)|_0^A \le K$, for $t \in [0,T]$. Thus

$$|||x|||_t = |\hat{B}(t)x|_0^A = |\hat{B}(t)\hat{B}^{-1}(0)\hat{B}(0)x|_0^A \le K|\hat{B}(0)x|_0^A = K|||x|||_0\,,$$

Similarly we get

$$|||x|||_0 = |\hat{B}(0)x|_0^A \le L|\hat{B}(t)x|_0^A = L|||x|||_t\,,$$

where $L = \sup\{|\hat{B}(0)\hat{B}^{-1}(t)|_0^t : t \in [0,T]\}$. □

Lemma 3. *Under the assumptions of Lemma 2 the family* $\{\hat{B}(t)\}$, $t \in [0,T]$, *has a constant domain, i.e.* $D(\hat{B}(t)) = D(\hat{B}(0))$ *for each* $t \in [0,T]$.

Proof. Let $x \in D(\hat{B}(0))$ and let $\{x_n\} \subset Y$ be such that $x_n \to x$ and $\hat{B}(0)x_n \longrightarrow \hat{B}(0)x$ in $\hat{X}_0^A$. Since, for each $t \in [0,T]$, $Y \subset D(\hat{B}(t))$,

$$|\hat{B}(t)x_p - \hat{B}(t)x_q|_0^A = |\hat{B}(t)(x_p - x_q)|_0^A \le M|\hat{B}(0)(x_p - x_q)|_0^A \to 0$$

when $p, q \longrightarrow \infty$. This means that $\{\hat{B}(t)x_n\}$ satisfies the Cauchy condition. This implies the convergence of $\{\hat{B}(t)x_n\}$ in $\hat{X}_0^A$. Let $y \in \hat{X}_0^A$ be a limit of $\{\hat{B}(t)x_n\}$. Hence we have

$$x_n \underset{n\to\infty}{\longrightarrow} x \quad \text{and } \hat{B}(t)x_n \longrightarrow y \text{ when } n \to \infty \tag{8}$$

From (8) and closeness of the operator $\hat{B}(t)$ it follows that $x \in D(\hat{B}(t))$ and $\hat{B}(t)x = y$. Thus $D(\hat{B}(0)) \subset D(\hat{B}(t))$, for each $t \in [0,T]$. Analogously we obtain the inverse inclusion, i.e. $D(\hat{B}(t)) \subset D(\hat{B}(0))$ for each $t \in [0,T]$.

We denote

$$D_0^B := D(\hat{B}(0)) = D(\hat{B}(t)), \quad t \in (0,T].$$

Since $Y \subset D_0^B$, D_0^B is a dense subspace of $\hat{X}_0^A$. We denote by $[D_0^B]$ the space D_0^B equipped with the graph norm of the operator $\hat{B}(0)$, i.e.

$$|x| := |x|_0^A + |\hat{B}(0)x|_0^A. \tag{9}$$

The space $[D_0^B]$ may be defined in another way. Since $0 \in \varrho(\hat{B}(t))$, for each $t \in [0,T]$, the norms $|x|_0^A + |\hat{B}(0)x|_0^A$ and $|\hat{B}(0)x|_0^A$ are equivalent for each $x \in D_0^B$. On the other hand we have

$$|\hat{B}(0)x|_0^A = \|R(0,\hat{A}(0))\hat{B}(0)x\| = \|\hat{A}^{-1}(0)\hat{B}(0)x\| = \|\hat{B}^{-1}(0)x\| = \|R(0,\hat{B}(0))x\|\,.$$

Denoting

$$|x|_0^B := \|R(0,\hat{B}(0))x\| \quad \text{for } x \in D_0^B, \tag{10}$$

we see that norms (9) and (10) are equivalent. From this we get

$$[D_0^B] = \hat{Y}^B\,,$$

where $\hat{Y}^B$ is the space which is the completion of the space Y with norm (10) to a Banach space.

Using the extrapolation space $\hat{X}_0^A$ and $\hat{Y}^B$ we are able to reduce Cauchy problem (6) to the following first order problem in the space $\hat{Y}^B \times \hat{X}_0^A$:

$$\begin{cases} \dfrac{d\mathcal{U}}{dt} = \hat{\mathcal{A}}(t)\mathcal{U} + F(t,\mathcal{U}), \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{11}$$

where

$$\mathcal{U} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \hat{\mathcal{A}}(t) = \begin{bmatrix} 0 & I \\ \hat{A}(t) & 0 \end{bmatrix}, \quad F(t,\mathcal{U}) = \begin{bmatrix} 0 \\ f(t,u,v) \end{bmatrix},$$
$$\mathcal{U}_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \qquad v = u', \qquad D(\hat{\mathcal{A}}(t)) = Y \times D_0^B .$$

□

Theorem 1. *Under assumptions* (Z_1)–(Z_6) *if*

$$f\colon [0,T] \times X \times X \ni (t,x,y) \longrightarrow f(t,x,y) \in \hat{X}_0^A \quad \textit{is of class } C^1$$

and

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le L(|x_1 - x_2|_0^B + |y_1 - y_2|_0^A), \tag{12}$$

$(u_0,v_0) \in D(\hat{\mathcal{A}}(t))$, *then problem* (11) *has a unique classical solution* $\mathcal{U} \in C^1([0,T], \hat{Y}^B \times \hat{X}_0^A)$, *which is the unique solution of the integral equation*

$$\mathcal{U}(t) = \mathcal{V}(t,0)\mathcal{U}_0 + \int_0^t \mathcal{V}(t,s)F(s,\mathcal{U}(s))ds,$$

where $\mathcal{V}(t,s)$, $t,s \in [0,T]$ *is the fundamental solution of problem* (11).

Proof. From (Z_3) by ([11], Th. 5] it follows that the family $\{\hat{A}(t)\}$, $t \in [0,T]$, satisfies the following inequalities in the space $\hat{X}_0^A$:

$$\|(\lambda - \hat{A}(t_k))^{-1}(\lambda - \hat{A}(t_{k-1}))^{-1} \dots (\lambda - \hat{A}(t_1))^{-1}\| \le \frac{\bar{M}}{\lambda^k}$$

and

$$\|(\lambda - \hat{A}(t_1))^{-1}(\lambda - \hat{A}(t_2))^{-1} \dots (\lambda - \hat{A}(t_k))^{-1}\| \le \frac{\bar{M}}{\lambda^k}$$

for $\lambda > 0$, $\bar{M} \ge 1$ and every finite sequence $0 \le t_1 \le \dots \le t_k \le T$, $k \in \mathbb{N}$.

From this and (7) using ([1]; Lemma 2) it follows that the family $\{\hat{B}(t)\}$, $t \in [0,T]$, is double stable in $\hat{X}_0^A$, i.e.

$$\|(\mu - \hat{B}(t_k))^{-1}(\mu - \hat{B}(t_{k-1}))^{-1} \dots (\mu - \hat{B}(t_1))^{-1}\| \le \frac{\bar{M}}{|\mu|^k}$$

and

$$\|(\mu - \hat{B}(t_1))^{-1}(\mu - \hat{B}(t_2))^{-1} \dots (\mu - \hat{B}(t_k))^{-1}\| \le \frac{\bar{M}}{|\mu|^k}$$

for $\mu \neq 0$ and every finite sequence $0 \leq t_1 \leq \cdots \leq t_k \leq T$.
From ([5], Lemmas 3.4 and 3.5) we obtain the stability of the family $\{\hat{\mathcal{A}}(t)\}$, $t \in [0,T]$. By assumption (Z_5) we deduce that the mapping

$$[0,T] \ni t \longrightarrow \hat{\mathcal{A}}(t) \begin{bmatrix} x \\ y \end{bmatrix} \in \hat{Y}^B \times \hat{X}_0^A$$

is of class C^1. From this and stability of family $\{\hat{\mathcal{A}}(t)\}$, $t \in [0,T]$, we get existence of the fundamental solution $\mathcal{V}(t,s)$ for problem (11).

On the other hand, from (12), by inequalities

$$|x|_0^B \leq C\|x\|, \quad |x|_0^A \leq C_1\|x\|$$

for every $x \in X$ it follows that

$$\|f(t,x_1,y_1) - f(t_1,x_2,y_2)\| \leq L_1(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

and

$$|f(t,x_1,y_1) - f(t,x_2,y_2)|_0^A \leq L_2(|x_1 - x_2|_0^B + |y_1 - y_2|_0^A).$$

To obtain the function $\mathcal{U}$ we use the following iterative method

$$\begin{cases} \mathcal{U}_1'(t) = \hat{\mathcal{A}}(t)\mathcal{U}_1(t) + F(t,\mathcal{U}_0), \\ \mathcal{U}_n'(t) = \hat{\mathcal{A}}(t)\mathcal{U}_n(t) + F(t,\mathcal{U}_{n-1}(t)) \quad n \in \mathbb{N} \\ \mathcal{U}_n(0) = \mathcal{U}_0\,. \end{cases} \tag{13}$$

From this we get

$$\mathcal{U}_n \in C([0,T], \hat{Y}^B \times \hat{X}_0^A) \cap C^1((0,T], \hat{Y}^B \times \hat{X}_0^A), \quad n \in \mathbb{N}$$

and

$$\mathcal{U}_n(t) = \mathcal{V}(t,0)\mathcal{U}_0 + \int_0^t \mathcal{V}(t,s)F(s,\mathcal{U}_{n-1}(s))ds, \quad t \in [0,T].$$

Since $Y \subset X$ is not dense in $(X, |.|_0^B)$, we see that $\hat{Y}^B \subset X$. From this ([9], Th. 4.17) it follows that

$$\mathcal{U}(t) = \lim_{n\to\infty} \mathcal{U}_n(t) \tag{14}$$

is continuous in $[0,T]$, because $\mathcal{U}_n(t)$ converges uniformly in $[0,T]$.

We shall prove that $\mathcal{U} \in C^1((0,T], \hat{Y}^B \times \hat{X}_0^A)$. Indeed, let

$$\mathcal{U}_n(t) = \begin{bmatrix} u_n(t) \\ v_n(t) \end{bmatrix}, \quad \text{for } n \in \mathbb{N},\ t \in [0,T]\,.$$

Hence by (13) we have for $t \in (0,T]$

$$u_n'(t) = v_n(t)\,, \quad v_n'(t) = \hat{A}(t)u_n(t) + f(t,u_{n-1}(t),v_{n-1}(t))\,.$$

From this we get for $p,q \in N$

$$|u_p'(t) - u_q'(t)|_0^B = |v_p(t) - v_q(t)|_0^B\,,$$
$$|v_p'(t) - v_q'(t)|_0^A \leq |\hat{A}(t)[u_p(t) - u_q(t)]|_0^A + |f(t,u_{p-1}(t),v_{p-1}(t)) - f(t,u_{q-1}(t),v_{q-1}(t))|_0^A\,.$$

Thus,

$$\begin{aligned}
|u_p'(t)-u_q'(t)|_0^B &\le C\|v_p(t)-v_q(t)\|\,,\\
|v_p'(t)-v_q'(t)|_0^A &\le \|\hat{A}^{-1}(0)\hat{A}(t)[u_p(t)-u_q(t)]\| + \\
&+ C_1\|f(t,u_{p-1}(t),v_{p-1}(t))-f(t,u_{q-1}(t),v_{q-1}(t))\| \le \\
&\le K\|u_p(t)-u_q(t)\| + C_1L_1(\|u_{p-1}(t)-u_{q-1}(t)\|+\|v_{p-1}(t)-v_{q-1}(t)\|).
\end{aligned}$$

From (14) it follows that for $\varepsilon>0$ there exists $n_0\in N$ such that for $p,q>n_0$ we have

$$\sup\{|u_p'(t)-u_q'(t)|_0^B : t\in[\tau,T]\}<\varepsilon$$

and

$$\sup\{|v_p'(t)-v_q'(t)|_0^B : t\in[\tau,T]\}<\varepsilon$$

for every $\tau>0$. This means that the mapping

$$(0,T]\ni t\longrightarrow \mathcal{U}(t):=\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}\in \hat{Y}^B\times\hat{X}_0^A$$

is of class C^1 and is the unique classical solution of problem (11), where uniqueness is an immediate consequence of Gronwall's Lemma. □

Theorem 2. *Under the assumptions of Theorem 1 the Cauchy problem* (6) *has a unique solution* $u\in C^1([0,T],X)\cap C^2((0,T],\hat{X}_0^A)$, *which is the unique solution of the integral equation*

$$u(t)=-\frac{\partial}{\partial s}S(t,s)|_{s=0}u_0+S(t,0)u_1+\int_0^t S(t,s)f(s,u(s),u'(s))ds\,, \tag{15}$$

where

$$S(t,s):=\Pi_1\mathcal{V}(t,s)\begin{bmatrix}0\\ x\end{bmatrix}\quad \textit{for each } x\in X_0^A\,,$$

and $\Pi_1\begin{bmatrix}y\\ x\end{bmatrix}:=y$ *for* $y\in\hat{Y}^B$, $x\in\hat{X}_0^A$ (cf. [11]).

Proof. The proof is the same as that of Theorem 7 in [11] and is omitted. □

Definition 1. A function $u\in C^1([0,T],X)$ which is a solution of the integral equation (15) and $u(0)=u_0$, $u'(0)=u_1$ is called *"a mild solution"* of problem (1).

Corollary 1. *Under the assumptions of Theorem 1 problem* (1) *has a unique mild solution given by* (15).

REFERENCES

1. Bochenek J. *Existence of the fundamental solution of a second order evolution equation*, Ann. Polon. Math. **66** (1997) 15–35.
2. Fattorini H. O. *Ordinary differential equations in linear topological spaces*, Intern. J. Diff. Equations, **5** (1968), 72–105.
3. Fattorini H. O. *Ordinary differential equations in linear topological spaces*, Intern. J. Diff. Equations, **6** (1969), 50–70.
4. Kozak M. *A fundamental solution of a second order differential equation in a Banach space*, Univ. Iagel. Acta Math. **32** (1995), 275–289.
5. Kozak M. *An abstract linear second order temporally inhomogeneous differential equation II*, Univ. Iagel. Acta Math. **32** (1995), 263–274.
6. Krein S. Linear Differential Equations in Banach Space, Amer. Math. Soc., 1972.
7. von Neerven J. The adjoint of a semigroup of Linear Operators, Springer–Verlag, 1992.
8. Pazy A. Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.
9. Radoń M. Application of the extrapolation spaces in theory of differential equations in Banach space, PhD thesis, 2002 (in Polish).
10. Travis C. C., Webb G. F. *Cosine family and abstract nonlinear second order differential equation*, Acta Math. Hungar. **32** (1978), 75–96.
11. Winiarska T. *Extrapolation Banach spaces and abstract semilinear second order differential equations*, Inter. J. Differ. Equat. and Appl. **6** (2002), No. 4, 449–465.
12. Winiarska T., Winiarski T. *Class of regularity of domains paramaterized families of closed linear operators*, Ann. Polon. Math. **80** (2003), 231–241.

Institute of Mathematics, Cracow University of Technology
Warszawska 24, 31–155 Cracow, Poland
twiniars@usk.pk.edu.pl

Received 2.10.2003