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ONE CRITERION OF γ-TYPE FINITENESS OF A FUNCTION ANALYTIC IN A HALF-PLANE

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Let γ be a growth function and let f be an analytic in the closure of the upper half-plane function such that $|f(t)| \le 1$ for real t. The classes of subharmonic functions of finite γ-type were introduced and studied by K. G. Malyutin. We prove a criterion of γ-type finiteness of $\log|f|$ in the terms of the Fourier coefficients of $\arg f$.

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Пусть γ — функция роста, а функция f аналитическая в замыкании верхней полуплоскости такая, что $|f(t)| \le 1$ для каждого вещественного t. Классы субгармонических функций конечного γ-типа были введены и изучены К. Г. Малютиным. Мы доказываем критерий конечности γ-типа функции $\log|f|$ в терминах коэффициентов Фурье функции $\arg f$.

1. Introduction and main results. The method of Fourier series for entire and meromorphic functions was developed by Rubel and Taylor [1]. Let f be a meromorphic function in the complex plane, $Z(f)$ and $W(f)$ be its sets of zeroes and poles respectively. Let $T(R,f)$ be the Nevanlinna characteristic of the function f and $C_k(R,f)$ be the Fourier coefficients of $\log|f|$,

$$C_k(R,f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \cdot e^{-ik\theta} d\theta, \quad R>0, \quad k\in\mathbb{Z}.$$

Definition 1 ([1]). Let γ be a positive, continuous, unbounded and increasing function on $[0,\infty)$ called *a growth function*. A meromorphic function f is called *a function of finite γ-type* if there exist positive constants A and B such that $T(R,f) \le A\gamma(BR)$ for all $R>0$. We denote the class of such functions by Γ.

The equivalence of the following properties was established in [1] (see also [2, 3]): (1) $f\in\Gamma$; (2) *the sequence $Z(f)$(or the sequence $W(f)$) has finite γ-density and $|C_k(R,f)| \le A_1\gamma(B_1R)$, $k\in\mathbb{Z}$ for some positive constants A_1, B_1, and for all $R>0$.*

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K. G. Malyutin [4] proved the analogue of this statement in the case of complex half-plane. Moreover, he proved his results for subharmonic and δ-subharmonic in the upper half-plane functions. We present this result below as Theorem M.

We deal with the Fourier coefficients of $\arg f$, where f is an analytic in the closure of the upper half-plane function. We give one criterion of γ-type finiteness of such functions in terms of these coefficients.

The notion of *complete measure* corresponding to a subharmonic in $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ function v which has a positive harmonic majorant in each bounded domain in $\mathbb{C}_+$ was introduced by A. F. Grishin [5],

$$\lambda(K) = 2\pi \int\limits_{K\cap\mathbb{C}_+} \operatorname{Im}\zeta d\mu(\zeta) - \nu\left(K \cap \mathbb{R}\right). \tag{1}$$

where μ is the Riesz measure of v. The measure ν is called *the boundary measure.* If $\nu(\{a\}) = \nu(\{b\}) = 0$, then

$$\nu([a;b]) = \lim_{y\to+0} \int\limits_a^b v(x+iy)dx.$$

The measure λ has the following properties:

1) λ is finite for an arbitrary compact set $K \subset \mathbb{C}$;
2) λ is a positive measure outside $\mathbb{R}$;
3) λ is a zero measure on $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

We denote $\lambda(R) = \lambda\left(\overline{\{z : |z| \le R\} \cap \mathbb{C}_+}\right)$.

Definition 2 ([6]). A subharmonic function v in $\mathbb{C}_+$ is said to be *just subharmonic* if

$$\limsup_{z\to t} v(z) \le 0$$

for each $t \in \mathbb{R}$.

The class of just subharmonic functions in $\mathbb{C}_+$ will be denoted by JS.

The notion of a function of finite γ-type in a half-plane was introduced in the paper [4] of K. G. Malyutin. Let

$$m(R,v) = \frac{1}{R}\int_0^\pi v_+(Re^{i\varphi})\sin\varphi d\varphi, \quad N(R,v) = \int_r^R \frac{\lambda_-(t)}{t^3}dt$$

where $\lambda = \lambda_+ - \lambda_-$ is the Jordan decomposition of the complete measure λ corresponding to the function v.

The Nevanlinna characteristic of a just subharmonic function v is defined in [6],

$$T(R,v) = m(R,v) + N(R,v) + m(r,-v), \quad R > r, \quad r > 0.$$

The growth function γ is assumed to satisfy the condition

$$\liminf_{R\to\infty} \frac{\gamma(R)}{R} > 0.$$

Definition 3 ([4]). A function $v \in JS$ is called a function of *finite γ-type in the upper half-plane* if there exist constants A and B, both positive, such that

$$T(R,v) \le \frac{A}{R}\gamma(BR), \quad R > r.$$

The corresponding class of just subharmonic functions of finite γ-type in the upper half-plane will be denoted by $JS(\gamma)$.

Definition 4 ([4]). A positive measure λ has *finite γ-density,* if there exist positive A and B such that

$$N(R,\lambda) := \int_r^R \frac{\lambda(t)}{t^3}dt \le \frac{A}{R}\gamma(BR)$$

for all $R > r$.

Definition 5 ([4]). A positive measure λ is called *a measure of finite γ-type* if there exist positive constants A and B such that for all $R > r$

$$\lambda(R) \le R \cdot A\gamma(BR).$$

If λ is a measure of finite γ-density then λ is a measure of finite γ-type. This statement was proved in [4].

We denote $G_R = \{z : \mathrm{Im} z > 0, r < |z| < R\}$, where $r > 0$ and $R > r$.

We consider a function f which is analytic in the closure of the upper half-plane $\mathbb{C}_+$.

Let $f(z_0) = 1$, $|z_0| = r$, $r > 0$. We define the function $\log f(z)$ as follows. If no zero of $f(z)$ lies on the ray $z(t) = te^{i\theta}$, $t \ge r$, we define $\log f(Re^{i\theta})$ as a value obtained from $\log f(z_0) = 0$ by continuous variation of the argument along the arc $|z| = r$ from z_0 to $re^{i\theta}$ and then along the ray indicated above to $z = Re^{i\theta}$. If the ray contains zeroes we define

$$\log f(Re^{i\theta}) = \lim_{\varepsilon\to 0}\frac{1}{2}\{\log f(Re^{i(\theta+\varepsilon)}) + \log f(Re^{i(\theta-\varepsilon)})\}.$$

We denote $\arg f(Re^{i\theta}) = \mathrm{Im}\log f(Re^{i\theta})$.

We continue the function $\log f$ in lower half-plane $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$ as follows $\log f(Re^{-i\theta}) = -\overline{\log f(Re^{i\theta})}$, $0 < \theta < \pi$.

Since after the continuation in this way the function $\log|f|$ is an odd function with respect to θ and the function $\arg f$ is an even function, the Fourier series of the function $\log f(Re^{i\theta})$ will be the following:

$$\log f(Re^{i\theta}) = \sum_{k=-\infty}^{+\infty} l_k(R,f)e^{ik\theta} = \sum_{k=0}^{\infty} c_k(R,f)\sin k\theta + i\cdot a_k(R,f)\cos k\theta,$$

$$0 \le \theta \le \pi, \quad r < R,$$

where

$$l_k(R,f) = \frac{1}{2\pi}\int_0^{2\pi} \log f(Re^{i\theta})e^{-ik\theta}d\theta,$$

$$c_k(R,f) = \frac{2}{\pi}\int_0^{\pi} \log|f(Re^{i\theta})| \cdot \sin k\theta d\theta, \quad a_k(R,f) = \frac{2}{\pi}\int_0^{\pi} \arg f(Re^{i\theta}) \cdot \cos k\theta d\theta.$$

The main results of this paper are the following

Theorem 1. *Let γ be a growth function and $\liminf\limits_{R\to\infty}\frac{\gamma(R)}{R}>0$. Let f be an analytic in the closure of the upper half-plane function. Then the following properties are equivalent:*

(i) $\log|f|\in JS(\gamma)$;

(ii) $|a_k(R,f)|\le A\gamma(BR)$, *for some positive A, B and for all $R>r, k\in\{0\}\cup\mathbb{N}$.*

Theorem 2. *Let γ be a growth function and $\liminf\limits_{R\to\infty}\frac{\gamma(R)}{R}>0$. Let f be an analytic in the closure of the upper half-plane function. Then the following properties are equivalent:*

(i) $\log|f|\in JS(\gamma)$;

(ii) $|l_k(R,f)|\le A\gamma(BR)$, *for some positive A, B and for all $R>r, k\in\mathbb{Z}$.*

To prove Theorem 1 we need four lemmas. As a corollary of one of them we obtain another proof of the Carleman formula ([7], p. 19) and the analogue of Jensen formula for the upper half-plane.

2. Auxiliary results.

Lemma 1. *Let f be an analytic in $\overline{\mathbb{C}}_+$ function, (z_j) be its sequence of the zeroes. Then the following relations hold:*

$$c_k(R,f)=2\int\limits_r^R\left(\sum_{z_j\in G_t}\left(\frac{|z_j|^k}{t^k}+\frac{t^k}{|z_j|^k}\right)\sin k\theta_j\right)\frac{dt}{t}+$$
$$+\frac{1}{\pi}\int\limits_r^R\left(\frac{x^{k-1}}{R^k}-\frac{R^k}{x^{k+1}}\right)(\log|f(x)|+(-1)^{k+1}\log|f(-x)|)dx+ \tag{2}$$
$$+\frac{1}{\pi}\left(\frac{r^k}{R^k}+\frac{R^k}{r^k}\right)\int\limits_0^\pi\log|f(re^{i\theta})|\cdot\sin k\theta d\theta+\frac{1}{\pi}\left(\frac{r^k}{R^k}-\frac{R^k}{r^k}\right)\int\limits_0^\pi\arg f(re^{i\theta})\cdot\cos k\theta d\theta,$$
$$k\in\mathbb{N},\quad R>r,$$

$$a_k(R,f)=2\int\limits_r^R\left(\sum_{z_j\in G_t}\left(\frac{|z_j|^k}{t^k}-\frac{t^k}{|z_j|^k}\right)\sin k\theta_j\right)\frac{dt}{t}+$$
$$+\frac{1}{\pi}\int\limits_r^R\left(\frac{x^{k-1}}{R^k}+\frac{R^k}{x^{k+1}}\right)(\log|f(x)|+(-1)^{k+1}\log|f(-x)|)dx+ \tag{3}$$
$$+\frac{1}{\pi}\left(\frac{r^k}{R^k}-\frac{R^k}{r^k}\right)\int\limits_0^\pi\log|f(re^{i\theta})|\cdot\sin k\theta d\theta+\frac{1}{\pi}\left(\frac{r^k}{R^k}+\frac{R^k}{r^k}\right)\int\limits_0^\pi\arg f(re^{i\theta})\cdot\cos k\theta d\theta,$$
$$k\in\{0\}\cup\mathbb{N},\quad R>r,$$

where $G_t=\{z:\operatorname{Im}z>0, r<|z|<t\}$.

Proof. Let us consider the domain $G_R=\{z:\operatorname{Im}z>0, r<|z|<R\}$, where $r>0$, $R>r$. If no zero of f lies on ∂G_R we have by the residue theorem applied to the function $(f'(z)/f(z))z^{-k}$, $k\in\mathbb{Z}$, in G_R:

$$\int\limits_{\partial G_R}\frac{f'(z)}{f(z)}z^{-k}dz=2\pi i\sum_{z_j\in G_R}\operatorname*{res}_{z=z_j}\frac{f'(z)}{f(z)}z^{-k},\quad k\in\mathbb{Z}, \tag{4}$$

where $z_j = |z_j|e^{i\theta_j}$ are the zeroes of function f and no zero lies on the real axis. If the function f has zeroes on the real axis and not on $C_R = \{z = Re^{i\theta} : 0 \le \theta \le \pi\}$ then (4) can be rewritten as follows:

$$\int\limits_{\partial G_R} \frac{f'(z)}{f(z)} z^{-k} dz = 2\pi i \sum_{z_j \in G_R, \operatorname{Im} z_j > 0} \operatorname*{res}_{z=z_j} \frac{f'(z)}{f(z)} z^{-k} + \pi i \sum_{z_j \in G_R, \operatorname{Im} z_j = 0} \operatorname*{res}_{z=z_j} \frac{f'(z)}{f(z)} z^{-k}, \quad k \in \mathbb{Z}.$$

But, since the final expressions (2) and (3) are independent of zeroes that lie on the real axis, we can deal only with zeroes in the interior of G_R. More precisely, the last term will disappear. This will be clear from the proof.

It is well-known that an analytic function f in some neighbourhood of its zero $z = a$ with multiplicity m has the representation $f(z) = (z-a)^m \cdot \varphi(z)$, where $\varphi(a) \ne 0$, φ is an analytic function in a neighbourhood of a.

So, the function $f'(z)/f(z)$ in a neighbourhood of a can be presented as follows

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{\varphi'(z)}{\varphi(z)}.$$

Then (4) can be rewritten as

$$\int\limits_{\partial G_R} \frac{f'(z)}{f(z)} z^{-k} dz = 2\pi i \sum_{z_j \in G_R} (z_j)^{-k}, \quad k \in \mathbb{Z}. \tag{5}$$

Further, $\partial G_R = C_R \cup C_r^- \cup [-R; -r] \cup [r; R]$, where C_r^- is C_r taken in the negative direction. Applying this to the left side of (5), we have

$$\begin{aligned} \int\limits_{\partial G_R} \frac{f'(z)}{f(z)} z^{-k} dz = \int\limits_{C_R} \frac{f'(z)}{f(z)} z^{-k} dz + \int\limits_{[-R;-r]} \frac{f'(z)}{f(z)} z^{-k} dz + \\ + \int\limits_{C_r^-} \frac{f'(z)}{f(z)} z^{-k} dz + \int\limits_{[r;R]} \frac{f'(z)}{f(z)} z^{-k} dz, \quad k \in \mathbb{Z}. \end{aligned} \tag{6}$$

Let us consider each term in the right side of (6):

$$\int\limits_{C_R} \frac{f'(z)}{f(z)} z^{-k} dz = \int\limits_0^\pi \frac{f'(Re^{i\theta})}{f(Re^{i\theta})} Rie^{i\theta} R^{-k} e^{-ik\theta} d\theta,$$

$$\int\limits_{C_r^-} \frac{f'(z)}{f(z)} z^{-k} dz = -\int\limits_0^\pi \frac{f'(re^{i\theta})}{f(re^{i\theta})} rie^{i\theta} r^{-k} e^{-ik\theta} d\theta,$$

$$\int\limits_{[r;R]} \frac{f'(z)}{f(z)} z^{-k} dz = \int\limits_r^R \frac{f'(x)}{f(x)} \frac{dx}{x^k},$$

$$\int\limits_{[-R;-r]} \frac{f'(z)}{f(z)} z^{-k} dz = \int\limits_{-R}^{-r} \frac{f'(x)}{f(x)} \frac{dx}{x^k} = (-1)^k \int\limits_r^R \frac{f'(-x)}{f(-x)} \frac{dx}{x^k}.$$

Denoting $F(z)=f'(z)/f(z)$ we can write (6) in the following way:

$$\frac{1}{2\pi}\int_0^\pi F(Re^{i\theta})Re^{i\theta}e^{-ik\theta}d\theta=\sum_{z_j\in G_R}\frac{R^k}{{z_j}^k}+\frac{1}{2\pi}\frac{R^k}{r^k}\int_0^\pi F(re^{i\theta})re^{i\theta}e^{-ik\theta}d\theta-$$
$$-\frac{R^k}{2\pi i}\int_r^R\left(F(x)+(-1)^kF(-x)\right)\frac{dx}{x^k},\quad k\in\mathbb{Z}. \tag{7}$$

According to the definition of the function $\log f$ we have

$$\log f(Re^{i\theta})-\log f(re^{i\theta})=e^{i\theta}\int_r^R\frac{f'(ue^{i\theta})}{f(ue^{i\theta})}du. \tag{8}$$

Relation (7) is proved for all $R>r$ except $R=|z_j|$. We replace R by t in (7). After that we divide (7) by t and integrate it with respect to t from r to R,

$$\frac{1}{2\pi}\int_r^R\left(\int_0^\pi F(te^{i\theta})te^{i\theta}e^{-ik\theta}d\theta\right)\frac{dt}{t}=\int_r^R\left(\sum_{z_j\in G_t}\left(\frac{t}{z_j}\right)^k\right)\frac{dt}{t}+$$
$$+\frac{1}{2\pi}\int_r^R\left(\frac{t}{r}\right)^k\left(\int_0^\pi F(re^{i\theta})re^{i\theta}e^{-ik\theta}d\theta\right)\frac{dt}{t}- \tag{9}$$
$$-\frac{1}{2\pi i}\int_r^R t^k\left(\int_r^t\left(F(x)+(-1)^kF(-x)\right)\frac{dx}{x^k}\right)\frac{dt}{t},\quad k\in\mathbb{Z}.$$

Further, we switch the order of integration with the aid of the Fubini theorem,

$$\int_r^R\left(\int_0^\pi\frac{f'(te^{i\theta})}{f(te^{i\theta})}te^{i\theta}e^{-ik\theta}d\theta\right)\frac{dt}{t}=\int_0^\pi\left(\int_r^R\frac{f'(te^{i\theta})}{f(te^{i\theta})}e^{i\theta}e^{-ik\theta}dt\right)d\theta.$$

Applying this to the left side of (9) and using (8), we have

$$\frac{1}{2\pi}\int_r^R\left(\int_0^\pi F(te^{i\theta})te^{i\theta}e^{-ik\theta}d\theta\right)\frac{dt}{t}=\frac{1}{2\pi}\int_0^\pi\left(\int_r^R F(te^{i\theta})e^{i\theta}e^{-ik\theta}dt\right)d\theta=$$
$$=\frac{1}{2\pi}\int_0^\pi\left(\int_r^R F(te^{i\theta})e^{i\theta}dt\right)e^{-ik\theta}d\theta=\frac{1}{2\pi}\int_0^\pi\left(\log f(Re^{i\theta})-\log f(re^{i\theta})\right)e^{-ik\theta}d\theta= \tag{10}$$
$$=\frac{1}{2\pi}\int_0^\pi\log f(Re^{i\theta})e^{-ik\theta}d\theta-\frac{1}{2\pi}\int_0^\pi\log f(re^{i\theta})e^{-ik\theta}d\theta.$$

At first we consider the case $k=0$. We have

$$\frac{1}{2\pi i}\int_r^R\left(\int_r^t\left(F(x)+F(-x)\right)dx\right)\frac{dt}{t}=\frac{1}{2\pi i}\int_r^R(\log f(t)-\log f(r)+\log f(-r)-$$
$$-\log f(-t))\frac{dt}{t}=\frac{1}{2\pi i}\int_r^R\left(\log f(t)-\log f(-t)\right)\frac{dt}{t}+\frac{1}{2\pi i}(\log f(-r)-\log f(r))\int_r^R\frac{dt}{t} \tag{11}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int\limits_r^R\left(\int\limits_0^\pi F(re^{i\theta})re^{i\theta}d\theta\right)\frac{dt}{t}&=\frac{1}{2\pi i}\int\limits_r^R(\log f(-r)-\log f(r))\frac{dt}{t}=\\&=\frac{1}{2\pi i}(\log f(-r)-\log f(r))\int\limits_r^R\frac{dt}{t}.\end{aligned}\tag{12}$$

Then from (9) and (10), using (11) and (12), we obtain

$$\begin{aligned}\frac{1}{2\pi}\int\limits_0^\pi\log f(Re^{i\theta})d\theta&=\int\limits_r^R\left(\sum_{z_j\in G_t}1\right)\frac{dt}{t}+\frac{1}{2\pi}\int\limits_0^\pi\log f(re^{i\theta})d\theta+\\&+\frac{i}{2\pi}\int\limits_r^R(\log f(t)-\log f(-t))\frac{dt}{t}.\end{aligned}\tag{13}$$

Now we will transform the right side of (9) introducing the following auxiliary notations:

$$I_1(k)=\frac{1}{2\pi}\int\limits_r^R\frac{t^k}{r^k}\left(\int\limits_0^\pi F(re^{i\theta})re^{i\theta}e^{-ik\theta}d\theta\right)\frac{dt}{t},\quad k\in\mathbb{Z}\backslash\{0\},$$

$$I_2(k)=\frac{1}{2\pi i}\int\limits_r^R t^k\left(\int\limits_r^t\big(F(x)+(-1)^kF(-x)\big)\frac{dx}{x^k}\right)\frac{dt}{t},\quad k\in\mathbb{Z}\backslash\{0\}.$$

At first we transform I_1:

$$I_1(k)=\frac{1}{2\pi}\int\limits_r^R\frac{t^{k-1}}{r^k}dt\int\limits_0^\pi F(re^{i\theta})re^{i\theta}e^{-ik\theta}d\theta=\frac{1}{2\pi k}\left(\frac{R^k}{r^k}-1\right)\int\limits_0^\pi F(re^{i\theta})re^{i\theta}e^{-ik\theta}d\theta=$$

$$=\frac{1}{2\pi k}\left(\frac{R^k}{r^k}-1\right)\cdot\int\limits_0^\pi\frac{e^{-ik\theta}}{i}d\log f(re^{i\theta})=\frac{1}{2\pi ik}\left(\frac{R^k}{r^k}-1\right)((-1)^k\log f(-r)-\log f(r))+$$

$$+\frac{1}{2\pi}\left(\frac{R^k}{r^k}-1\right)\int\limits_0^\pi\log f(re^{i\theta})e^{-ik\theta}d\theta,\quad k\in\mathbb{Z}\backslash\{0\}.$$

Then

$$\begin{aligned}I_1(k)&=\frac{i}{2\pi k}\left(\frac{R^k}{r^k}-1\right)\big((-1)^{k+1}\log f(-r)+\log f(r)\big)+\\&+\frac{1}{2\pi}\left(\frac{R^k}{r^k}-1\right)\int\limits_0^\pi\log f(re^{i\theta})e^{-ik\theta}d\theta,\quad k\in\mathbb{Z}\backslash\{0\}.\end{aligned}\tag{14}$$

Before transforming I_2 we denote $g_k(t)=\int\limits_r^t \big(F(x)+(-1)^kF(-x)\big)\frac{dx}{x^k}$. Then integrating by parts we obtain

$$I_2(k)=\frac{1}{2\pi ik}\left(\int\limits_r^R g_k(t)dt^k\right)=\frac{1}{2\pi ik}\left(t^k\cdot g_k(t)|_r^R-\int\limits_r^R t^kg_k'(t)dt\right)=$$

$$=\frac{R^k}{2\pi ik}\int\limits_r^R \big(F(x)+(-1)^kF(-x)\big)\frac{dx}{x^k}-\frac{1}{2\pi ik}\int\limits_r^R\big(F(x)+(-1)^kF(-x)\big)\,dx,\quad k\in\mathbb{Z}\backslash\{0\},$$

Further,

$$\int\limits_r^R\big(F(x)+(-1)^kF(-x)\big)\,dx=\int\limits_r^R d\log f(t)+(-1)^{k+1}\int\limits_r^R d\log f(-t)=$$

$$=\log f(R)-\log f(r)+(-1)^{k+1}(\log f(-R)-\log f(-r)),\quad k\in\mathbb{Z},$$

and

$$R^k\int\limits_r^R\big(F(x)+(-1)^kF(-x)\big)\frac{dx}{x^k}=R^k\int\limits_r^R\frac{d\log f(x)}{x^k}+(-1)^{k+1}R^k\int\limits_r^R\frac{d\log f(-x)}{x^k}=$$

$$=R^k\left(\frac{1}{R^k}\log f(R)-\frac{1}{r^k}\log f(r)+k\int\limits_r^R\log f(x)\frac{dx}{x^{k+1}}\right)+$$

$$+(-1)^{k+1}R^k\left(\frac{1}{R^k}\log f(-R)-\frac{1}{r^k}\log f(-r)+k\int\limits_r^R\log f(-x)\frac{dx}{x^{k+1}}\right)=$$

$$=\log f(R)-\left(\frac{R}{r}\right)^k\log f(r)+(-1)^{k+1}\left(\log f(-R)-\left(\frac{R}{r}\right)^k\log f(-r)\right)+$$

$$+kR^k\int\limits_r^R\big(\log f(x)+(-1)^{k+1}\log f(-x)\big)\frac{dx}{x^{k+1}},\quad k\in\mathbb{Z}\backslash\{0\}.$$

Therefore,

$$I_2(k)=\frac{1}{2\pi ik}(\log f(R)-\frac{R^k}{r^k}\log f(r)+(-1)^{k+1}(\log f(-R)-\frac{R^k}{r^k}\log f(-r)))+$$

$$+\frac{1}{2\pi ik}kR^k\int\limits_r^R\big(\log f(x)+(-1)^{k+1}\log f(-x)\big)\frac{dx}{x^{k+1}}+$$

$$+\frac{1}{2\pi ik}\big(-\log f(R)+\log f(r)+(-1)^k(\log f(-R)-\log f(-r))\big)=$$

$$= \frac{1}{2\pi i k} k R^k \int\limits_r^R \left(\log f(x) + (-1)^{k+1} \log f(-x)\right) \frac{dx}{x^{k+1}} +$$

$$+\frac{1}{2\pi i k}(\log f(r) + (-1)^{k+1} \log f(-r) - \frac{R^k}{r^k}(\log f(r) + (-1)^{k+1} \log f(-r))) =$$

$$= \frac{1}{2\pi i} \int\limits_r^R \frac{R^k}{x^{k+1}} \left(\log f(x) + (-1)^{k+1} \log f(-x)\right) dx +$$

$$+\frac{1}{2\pi i k}\left(1 - \frac{R^k}{r^k}\right)(\log f(r) + (-1)^{k+1} \log f(-r)), \quad k \in \mathbb{Z}\backslash\{0\}.$$

Finally,

$$\begin{aligned} I_2(k) = \frac{1}{2\pi i} \int\limits_r^R \frac{R^k}{x^{k+1}} \left(\log f(x) + (-1)^{k+1} \log f(-x)\right) dx +& \\ +\frac{1}{2\pi i k}\left(1 - \frac{R^k}{r^k}\right)(\log f(r) + (-1)^{k+1} \log f(-r)), \quad k \in \mathbb{Z}\backslash\{0\}.& \end{aligned} \tag{15}$$

Taking into account the transformations, that have been made earlier, (9) can be rewritten as follows:

$$\frac{1}{2\pi} \int\limits_0^\pi \log f(Re^{i\theta}) e^{-ik\theta} d\theta - \frac{1}{2\pi} \int\limits_0^\pi \log f(re^{i\theta}) e^{-ik\theta} d\theta =$$

$$= \int\limits_r^R \left(\sum_{z_j \in G_t} \left(\frac{t}{z_j}\right)^k \right) \frac{dt}{t} + \frac{i}{2\pi k}\left(\left(\frac{R}{r}\right)^k - 1\right)(\log f(r) + (-1)^{k+1} \log f(-r)) +$$

$$+\frac{1}{2\pi}\left(\frac{R^k}{r^k} - 1\right) \int\limits_0^\pi \log f(re^{i\theta}) e^{-ik\theta} d\theta - \frac{1}{2\pi i} \int\limits_r^R \frac{R^k}{x^{k+1}} \left(\log f(x) + (-1)^{k+1} \log f(-x)\right) dx -$$

$$-\frac{1}{2\pi i k}\left(1 - \left(\frac{R}{r}\right)^k\right)(\log f(r) + (-1)^{k+1} \log f(-r)), \quad k \in \mathbb{Z}\backslash\{0\},$$

or

$$\begin{aligned} \frac{1}{2\pi} \int\limits_0^\pi \log f(Re^{i\theta}) e^{-ik\theta} d\theta = \int\limits_r^R \left(\sum_{z_j \in G_t} \left(\frac{t}{z_j}\right)^k \right) \frac{dt}{t} + \frac{1}{2\pi} \frac{R^k}{r^k} \int\limits_0^\pi \log f(re^{i\theta}) e^{-ik\theta} d\theta +& \\ +\frac{i}{2\pi} \int\limits_r^R \frac{R^k}{x^{k+1}} \left(\log f(x) + (-1)^{k+1} \log f(-x)\right) dx, \quad k \in \mathbb{Z}.& \end{aligned} \tag{16}$$

Since relation (16) takes place for each $k \in \mathbb{Z}$, we can write

$$\begin{aligned} \frac{1}{2\pi} \int\limits_0^\pi \log f(Re^{i\theta}) e^{ik\theta} d\theta = \int\limits_r^R \left(\sum_{z_j \in G_t} \frac{t^{-k}}{z_j^{-k}} \right) \frac{dt}{t} + \frac{1}{2\pi} \frac{R^{-k}}{r^{-k}} \int\limits_0^\pi \log f(re^{i\theta}) e^{ik\theta} d\theta +& \\ +\frac{i}{2\pi} \int\limits_r^R \frac{R^{-k}}{x^{-k+1}} \left(\log f(x) + (-1)^{k+1} \log f(-x)\right) dx, \quad k \in \mathbb{Z}.& \end{aligned} \tag{17}$$

We have $c_k(R,f) = \operatorname{Re} b_k(R,f)$, where

$$b_k(R,f) = \frac{2}{\pi}\int\limits_0^{\pi} \log f(Re^{i\theta}) \sin k\theta d\theta, \quad k \in \mathbb{N}.$$

To evaluate the coefficients $b_k(R,f)$ we use (16) and (17).

The coefficients $a_k(R,f) = \operatorname{Im} b_k^*(R,f)$, where

$$b_k^*(R,f) = \frac{2}{\pi}\int\limits_0^{\pi} \log f(Re^{i\theta}) \cos k\theta d\theta, \quad k \in \{0\} \cup \mathbb{N}$$

can be obtained similarly. Lemma 1 is proved. □

Remark 1. Let $n(t,f)$ be the zero counting function of the function f in G_t and $N(R,f) = \int_r^R \frac{n(t,f)}{t} dt$ then, taking the real parts of both sides of relation (13), we obtain the analogue of Jensen formula for the upper half-plane in the case when the real axis does not contain the zeroes of the function f:

$$N(R,f) = \frac{1}{2\pi}\int\limits_0^{\pi} \log|f(Re^{i\theta})| d\theta - \frac{1}{2\pi}\int\limits_0^{\pi} \log|f(re^{i\theta})| d\theta + \frac{1}{2\pi}\int\limits_r^{R} (\arg f(t) - \arg f(-t)) \frac{dt}{t}. \tag{18}$$

Remark 2. It should be noted that the Carleman formula ([7], p. 19) can be obtained from (2) by taking $k=1$:

$$\begin{gathered}\sum_{z_j \in G_R} \left(\frac{1}{|z_j|} - \frac{|z_j|}{R^2}\right) \sin\theta_j = \frac{1}{2\pi}\int\limits_r^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log|f(x)f(-x)| dx + \\ + \frac{1}{\pi R}\int\limits_0^{\pi} \log|f(Re^{i\theta})| \cdot \sin\theta d\theta + A_r(f,R)\end{gathered} \tag{19}$$

where $z_j = |z_j|e^{i\theta_j}$ are the zeroes of the function f, $A_r(f,R) = O(1)$ as $R \to \infty$.

Indeed, transform the integral

$$\int\limits_r^R \left(\sum_{z_j \in G_t} \left(\frac{|z_j|}{t} + \frac{t}{|z_j|}\right) \sin\theta_j\right) \frac{dt}{t}$$

using the integration by parts,

$$\int\limits_r^R \left(\sum_{z_j \in G_t} \left(\frac{|z_j|}{t} + \frac{t}{|z_j|}\right) \sin\theta_j\right) \frac{dt}{t} = \int\limits_r^R \left(\sum_{z_j \in G_t} |z_j| \sin\theta_j\right) \frac{dt}{t^2} + \int\limits_r^R \left(\sum_{z_j \in G_t} \frac{\sin\theta_j}{|z_j|}\right) dt =$$

$$= \int\limits_r^R \left(\sum_{z_j \in G_t} |z_j| \sin\theta_j\right) d\left(-\frac{1}{t}\right) + \int\limits_r^R \left(\sum_{z_j \in G_t} \frac{\sin\theta_j}{|z_j|}\right) dt =$$

$$= -\sum_{z_j\in G_R}\frac{|z_j|\sin\theta_j}{R}+\int_r^R\frac{1}{t}d\left(\sum_{z_j\in G_t}|z_j|\sin\theta_j\right)+\sum_{z_j\in G_R}\frac{R\sin\theta_j}{|z_j|}-\int_r^R td\left(\sum_{z_j\in G_t}\frac{\sin\theta_j}{|z_j|}\right).$$

Since the functions $\varphi(t)=\sum_{z_j\in G_t}|z_j|\sin\theta_j$ and $\psi(t)=\sum_{z_j\in G_t}\frac{\sin\theta_j}{|z_j|}$ have the jumps $|z_j|\sin\theta_j$ and $\sin\theta_j/|z_j|$ respectively at the points $t_j=|z_j|$ then

$$\int_r^R\frac{1}{t}d\left(\sum_{z_j\in G_t}|z_j|\sin\theta_j\right)-\int_r^R td\left(\sum_{z_j\in G_t}\frac{\sin\theta_j}{|z_j|}\right)=\sum_{z_j\in G_R}\frac{|z_j|}{|z_j|}\sin\theta_j-\sum_{z_j\in G_R}\frac{|z_j|\sin\theta_j}{|z_j|}=0$$

and

$$\int_r^R\left(\sum_{z_j\in G_t}\left(\frac{|z_j|}{t}+\frac{t}{|z_j|}\right)\sin\theta_j\right)\frac{dt}{t}=-\sum_{z_j\in G_R}\frac{|z_j|\sin\theta_j}{R}+\sum_{z_j\in G_R}\frac{R\sin\theta_j}{|z_j|}.$$

From (2) we have

$$c_1(R,f)=2\int_r^R\left(\sum_{z_j\in G_t}\left(\frac{|z_j|}{t}+\frac{t}{|z_j|}\right)\sin\theta_j\right)\frac{dt}{t}+$$

$$+\frac{1}{\pi}\int_r^R\left(\frac{1}{R}-\frac{R}{x^2}\right)(\log|f(x)|+\log|f(-x)|)dx+$$

$$+\frac{1}{\pi}\left(\frac{r}{R}+\frac{R}{r}\right)\int_0^\pi\log|f(re^{i\theta})|\cdot\sin\theta d\theta+\frac{1}{\pi}\left(\frac{r}{R}-\frac{R}{r}\right)\int_0^\pi\arg f(re^{i\theta})\cdot\cos\theta d\theta,\quad r<R,$$

or

$$2\sum_{z_j\in G_t}\left(\frac{R\sin\theta_j}{|z_j|}-\frac{|z_j|\sin\theta_j}{R}\right)=\frac{1}{\pi}\int_r^R\left(\frac{R}{x^2}-\frac{1}{R}\right)\log|f(x)f(-x)|dx+$$

$$+\frac{2}{\pi}\int_0^\pi\log|f(Re^{i\theta})|\cdot\sin\theta d\theta-\frac{1}{\pi}\left(\frac{r}{R}+\frac{R}{r}\right)\int_0^\pi\log|f(re^{i\theta})|\cdot\sin\theta d\theta-$$

$$-\frac{1}{\pi}\left(\frac{r}{R}-\frac{R}{r}\right)\int_0^\pi\arg f(re^{i\theta})\cdot\cos\theta d\theta.$$

Dividing the last equality by $2R$ we obtain the Carleman formula with

$$A_r(f,R)=-\frac{1}{2\pi}\left(\frac{r}{R^2}+\frac{1}{r}\right)\int_0^\pi\log|f(re^{i\theta})|\cdot\sin\theta d\theta-\frac{1}{2\pi}\left(\frac{r}{R^2}-\frac{1}{r}\right)\times$$

$$\times\int_0^\pi\arg f(re^{i\theta})\cdot\cos\theta d\theta=\frac{1}{2\pi}\int_0^\pi\mathrm{Im}\left(\log f(re^{i\theta})\left(\frac{e^{-i\theta}}{r}-\frac{re^{i\theta}}{R^2}\right)\right)d\theta$$

(cf. [8], p. 26).

The criterion of belonging of a subharmonic function v to the class $JS(\gamma)$ in terms of the Fourier coefficients $c_k(R,v)$:

$$c_k(R,v)=\frac{2}{\pi}\int\limits_0^{\pi} v(Re^{i\theta})\cdot\sin k\theta d\theta,\quad k\in\mathbb{N},$$

was proved in [4]. Here we formulate it as Theorem M.

Theorem M ([4]). *Let γ be a growth function and let $v\in JS$. Then the following properties are equivalent:*

(1) $v\in JS(\gamma)$;

(2) $|c_k(R,v)|\le A\gamma(BR)$, *for some positive A, B and for all $R>r, k\in\mathbb{N}$.*

If f is analytic in the closure of the upper half-plane and $|f(t)|\le 1$, $t\in\mathbb{R}$, then we obtain that $\log|f|\in JS$. To prove Theorem 1 we need the following lemmas.

Lemma 2. *Let λ be the complete measure corresponding to the function* $\log|f|$ *and λ has finite γ-density. Then*

$$2\pi\sum_{z_j\in G_R}|z_j|\sin\theta_j\le R\cdot A\gamma(BR),\tag{20}$$

$$\left|\int\limits_r^R(\log|f(x)|+\log|f(-x)|)dx\right|\le R\cdot A\gamma(BR),\tag{21}$$

$$2\pi\sum_{z_j\in G_R}\frac{\sin\theta_j}{|z_j|}\le\frac{3A}{R}\gamma(BR),\tag{22}$$

$$\left|\int\limits_r^R(\log|f(x)|+\log|f(-x)|)\frac{dx}{x}\right|\le 3A\gamma(BR)\tag{23}$$

for some positive A and B and for all $R>r$.

Proof. Inequalities (20) and (21) can be proved with the aid of the following equality, which can be easy obtained from the definition of the complete measure (1),

$$\iint\limits_{G_R}d\lambda(z)=2\pi\sum_{z_j\in G_R}|z_j|\sin\theta_j-\int\limits_r^R(\log|f(x)|+\log|f(-x)|)dx.\tag{24}$$

Then we note that $2\pi\sum_{z_j\in G_R}|z_j|\sin\theta_j\ge 0$ and $-\int_r^R(\log|f(x)|+\log|f(-x)|)dx\ge 0$. Thus each term of the right side of (24) is not greater than the left side of (24). Let us estimate the left side of (24),

$$\iint\limits_{G_R}d\lambda(z)=\int\limits_r^R d\lambda(t)=\lambda(R)-\lambda(r)\le\lambda(R)\le R\cdot A\gamma(BR).$$

Similarly, to prove (22) let us consider the equality

$$\iint\limits_{G_R} \frac{d\lambda(z)}{|z|^2} = 2\pi \sum_{z_j \in G_R} \frac{\sin\theta_j}{|z_j|} - \int\limits_r^R (\log|f(x)| + \log|f(-x)|)\frac{dx}{x^2} \tag{25}$$

Again, we may estimate the left side of (25),

$$\iint\limits_{G_R} \frac{d\lambda(z)}{|z|^2} = \int\limits_r^R \frac{d\lambda(t)}{t^2} = \frac{\lambda(R)}{R^2} - \frac{\lambda(r)}{r^2} + 2N(R,\lambda) \leq$$

$$\leq \frac{\lambda(R)}{R^2} + 2N(R,\lambda) \leq \frac{A}{R^2}R\cdot\gamma(BR) + 2\frac{A}{R}\gamma(BR) \leq \frac{3A}{R}\gamma(BR).$$

Let us prove (23). We have

$$\left|\int\limits_r^R (\log|f(x)| + \log|f(-x)|)\frac{dx}{x}\right| = \left|\int\limits_r^R x(\log|f(x)| + \log|f(-x)|)\frac{dx}{x^2}\right| \leq$$

$$\leq R\cdot\left|\int\limits_r^R (\log|f(x)| + \log|f(-x)|)\frac{dx}{x^2}\right| \leq R\cdot\iint\limits_{G_R}\frac{d\lambda(z)}{|z|^2} \leq R\cdot\frac{3A}{R}\gamma(BR) \leq 3A\gamma(BR).$$

□

Lemma 3. *If $\{z_j\}$, $z_j = |z_j|e^{i\theta_j}$ is a finite set from G_R, then*

$$\int\limits_r^R \left(\sum_{z_j\in G_t}\frac{|z_j|^k}{t^k}\sin k\theta_j\right)\frac{dt}{t} = \frac{1}{k}\sum_{z_j\in G_R}\sin k\theta_j - \frac{1}{k}\sum_{z_j\in G_R}\frac{|z_j|^k}{R^k}\sin k\theta_j, k\in\mathbb{N}, R>r. \tag{26}$$

Proof. Integrating by parts we obtain

$$\int\limits_r^R \left(\sum_{z_j\in G_t}\frac{|z_j|^k}{t^k}\sin k\theta_j\right)\frac{dt}{t} = -\frac{1}{k}\int\limits_r^R\left(\sum_{z_j\in G_t}|z_j|^k\sin k\theta_j\right)dt^{-k} =$$

$$= -\frac{1}{k}\sum_{z_j\in G_R}\frac{|z_j|^k}{R^k}\sin k\theta_j + \frac{1}{k}\int\limits_r^R t^{-k}d\left(\sum_{z_j\in G_t}|z_j|^k\sin k\theta_j\right).$$

The function $\psi(t) = \sum_{z_j\in G_t}|z_j|^k\sin k\theta_j$ has the jumps $|z_j|^k\sin k\theta_j$ at the points $t_j = |z_j|$. Then the last integral can be rewritten as

$$\frac{1}{k}\int\limits_r^R t^{-k}d\psi(t) = \frac{1}{k}\sum_{z_j\in G_R}|z_j|^{-k}|z_j|^k\sin k\theta_j = \frac{1}{k}\sum_{z_j\in G_R}\sin k\theta_j.$$

□

Lemma 4. *Under the assumptions of Lemma 1,*

$$a_k(R,f) = -c_k(R,f) + 4\int_r^R \left(\sum_{z_j\in G_t} \frac{|z_j|^k}{t^k}\sin k\theta_j\right)\frac{dt}{t} +$$

$$+\frac{2}{\pi}\int_r^R \frac{x^{k-1}}{R^k}(\log|f(x)| + (-1)^{k+1}\log|f(-x)|)dx+ \tag{27}$$

$$+\frac{2}{\pi}\frac{r^k}{R^k}\int_0^\pi \log|f(re^{i\theta})|\cdot\sin k\theta d\theta + \frac{2}{\pi}\frac{r^k}{R^k}\int_0^\pi \arg f(re^{i\theta})\cdot\cos k\theta d\theta, \quad k\in\mathbb{N}, \quad R>r.$$

Proof. Relation (27) follows immediately from (2) and (3). □

3. Proof of Theorem 1. Let (i) hold. According to the properties of $T(R,\log|f|)$ we have [5]

$$T(R,\log|f|) = T(R,\log\frac{1}{|f|}). \tag{28}$$

The measure λ has finite γ-density. This follows from (i) and (28). Further, according to Lemma 4

$$|a_k(R,f)| \le |c_k(R,f)| + |Q_k(R,f)|, \quad k\in\mathbb{N},\ R>r, \tag{29}$$

where

$$Q_k(R,f) = 4\int_r^R \left(\sum_{z_j\in G_t}\frac{|z_j|^k}{t^k}\sin k\theta_j\right)\frac{dt}{t} +$$
$$+\frac{2}{\pi}\int_r^R \frac{x^{k-1}}{R^k}(\log|f(x)| + (-1)^{k+1}\log|f(-x)|)dx+ \tag{30}$$
$$+\frac{2}{\pi}\frac{r^k}{R^k}\left(\int_0^\pi \left(\log|f(re^{i\theta})|\cdot\sin k\theta + \arg f(re^{i\theta})\cdot\cos k\theta\right)d\theta\right), \quad k\in\mathbb{N}, \quad R>r.$$

Denote the summands in the right side of (30) by I_1, I_2, I_3 respectively. We are going to estimate them.

According to Lemma 3,

$$|I_1| \le \left|\frac{4}{k}\sum_{z_j\in G_R}\sin k\theta_j\right| + \left|\frac{4}{k}\sum_{z_j\in G_R}\frac{|z_j|^k}{R^k}\sin k\theta_j\right|,$$

$$\left|\frac{4}{k}\sum_{z_j\in G_R}\sin k\theta_j\right| \le \frac{4}{k}\sum_{z_j\in G_R}|\sin k\theta_j| \le \frac{4}{k}\sum_{z_j\in G_R}k\sin\theta_j = 4\sum_{z_j\in G_R}\sin\theta_j \le$$
$$\le 4\sum_{z_j\in G_R}\frac{R}{|z_j|}\sin\theta_j \le 4R\cdot\frac{3A}{2\pi R}\gamma(BR) = \frac{6}{\pi}A\gamma(BR) \le 2A\gamma(BR).$$

The last inequality is obtained due to (22). Further,

$$\left|\frac{4}{k}\sum_{z_j\in G_R}\frac{|z_j|^k}{R^k}\sin k\theta_j\right| \le \frac{4}{k}\sum_{z_j\in G_R}|\sin k\theta_j| \le 4\sum_{z_j\in G_R}\sin\theta_j \le 2\cdot A\gamma(BR).$$

Thus

$$|I_1| \le 2A\gamma(BR)+2A\gamma(BR)=4A\gamma(BR).$$

Using the inequalities $\log|f(x)|\le 0$ and $\log|f(-x)|\le 0$ when $x\in\mathbb{R}$, we have

$$|I_2| = \left|\frac{2}{\pi}\int\limits_r^R \frac{x^{k-1}}{R^k}(\log|f(x)|+(-1)^{k+1}\log|f(-x)|)dx\right| \le$$

$$\le \frac{2}{\pi}\frac{R^{k-1}}{R^k}\left|\int\limits_r^R(\log|f(x)|+(-1)^{k+1}\log|f(-x)|)dx\right| \le$$

$$\le \frac{2}{\pi}\frac{1}{R}\left|\int\limits_r^R(|\log|f(x)||+|\log|f(-x)||)dx\right| = \frac{2}{\pi}\frac{1}{R}\left|\int\limits_r^R(-\log|f(x)|-\log|f(-x)|)dx\right|.$$

Then

$$|I_2| \le \frac{2}{\pi R}\left|\int\limits_r^R(\log|f(x)|+\log|f(-x)|)dx\right|.$$

Using (21) we obtain

$$|I_2| \le \frac{2}{\pi R}R\cdot A\gamma(BR) = \frac{2A}{\pi}\gamma(BR) \le A\gamma(BR).$$

Now we estimate I_3. We have

$$|I_3| \le \frac{2}{\pi}\left|\int\limits_0^\pi\left(\log|f(re^{i\theta})|\cdot\sin k\theta+\arg f(re^{i\theta})\cdot\cos k\theta\right)d\theta\right| = C,$$

where C is positive constant.

Hence,

$$|Q_k(R,f)| \le |I_1|+|I_2|+|I_3| \le 4A\gamma(BR)+A\gamma(BR)+C \le A_1\gamma(B_1R), \quad k\in\mathbb{N}, R>r.$$

According to the Theorem M $|c_k(R,f)| \le A_2\gamma(B_2R), k\in\mathbb{N}$, for some $A_2>0$, $B_2>0$ and for all $R>r$, $k\in\mathbb{N}$.

Returning to (29) we obtain

$$|a_k(R,f)| \le A_3\gamma(B_3R), \quad k\in\mathbb{N}, \quad R>r, \tag{31}$$

where $A_3>0$, $B_3>0$.

Let us consider $a_0(R,f)$. Using (3) we have

$$a_0(R,f)=\frac{2}{\pi}\int\limits_r^R(\log|f(x)|-\log|f(-x)|)\frac{dx}{x}+\frac{2}{\pi}\int\limits_0^\pi \arg f(re^{i\theta})d\theta,\quad R>r. \tag{32}$$

The first integral in (32) can be estimated similarly to I_2,

$$\left|\frac{2}{\pi}\int\limits_r^R(\log|f(x)|-\log|f(-x)|)\frac{dx}{x}\right|\le\frac{2}{\pi}\left|\int\limits_r^R(|\log|f(x)||+|\log|f(-x)||)\frac{dx}{x}\right|=$$

$$=\frac{2}{\pi}\left|\int\limits_r^R(\log|f(x)|+\log|f(-x)|)\frac{dx}{x}\right|.$$

Using (23) we have

$$\left|\frac{2}{\pi}\int\limits_r^R(\log|f(x)|-\log|f(-x)|)\frac{dx}{x}\right|\le\frac{2}{\pi}\cdot 3A\gamma(BR)\le 2A\gamma(BR).$$

The second integral in (32) does not depend on R and can be estimated similarly to I_3. Therefore

$$|a_0(R,f)|\le A_4\gamma(B_4R)$$

for some $A_4>0$, $B_4>0$ and for all $R>r$. The last inequality and (31) give (ii).

Now suppose that (ii) holds. At first we will prove that λ has finite γ-density. Consider

$$\begin{gathered}a_1(R,f)=2\int\limits_r^R\left(\sum_{z_j\in G_t}\left(\frac{|z_j|}{t}-\frac{t}{|z_j|}\right)\sin\theta_j\right)\frac{dt}{t}+\\ +\frac{1}{\pi}\int\limits_r^R\left(\frac{1}{R}+\frac{R}{x^2}\right)(\log|f(x)|+\log|f(x)|)dx+\\ +\frac{1}{\pi}\left(\frac{r}{R}-\frac{R}{r}\right)\int\limits_0^\pi\log|f(re^{i\theta})|\cdot\sin\theta d\theta+\frac{1}{\pi}\left(\frac{r}{R}+\frac{R}{r}\right)\int\limits_0^\pi\arg f(re^{i\theta})\cdot\cos\theta d\theta,\quad R>r.\end{gathered} \tag{33}$$

We denote $\psi(t)=\sum\limits_{z_j\in G_t}|z_j|\sin\theta_j$. Then we can write

$$\int\limits_r^R\left(\sum_{z_j\in G_t}\frac{|z_j|}{t^2}\sin\theta_j\right)dt=\int\limits_r^R\frac{\psi(t)}{t^2}dt,$$

$$\int\limits_r^R\left(\sum_{z_j\in G_t}\frac{\sin\theta_j}{|z_j|}\right)dt=\int\limits_r^R\left(\sum_{z_j\in G_t}\frac{|z_j|\sin\theta_j}{|z_j|^2}\right)dt=\int\limits_r^R\left(\int\limits_r^t\frac{d\psi(\tau)}{\tau^2}\right)dt=$$

$$= \int_r^R \left(\frac{\psi(t)}{t^2} + 2\int_r^t \frac{\psi(\tau)d\tau}{\tau^3} \right) dt = \int_r^R \frac{\psi(t)}{t^2} dt + 2\int_r^R \left(\int_r^t \frac{\psi(\tau)d\tau}{\tau^3} \right) dt =$$

$$= \int_r^R \frac{\psi(t)}{t^2} dt + 2R\int_r^R \frac{\psi(t)dt}{t^3} - 2\int_r^R t\frac{\psi(t)}{t^3} dt = -\int_r^R \frac{\psi(t)}{t^2} dt + 2R\int_r^R \frac{\psi(t)dt}{t^3},$$

and

$$2\int_r^R \left(\sum_{z_j \in G_t} \left(\frac{|z_j|}{t} - \frac{t}{|z_j|} \right) \sin\theta_j \right) \frac{dt}{t} = 2\left(2\int_r^R \frac{\psi(t)}{t^2} dt - 2R\int_r^R \frac{\psi(t)dt}{t^3} \right). \tag{34}$$

Similarly, denoting $\varphi(t) = \int_r^t (\log|f(x)| + \log|f(-x)|)dx$ and integrating by parts we obtain

$$\begin{aligned} \frac{1}{\pi}\int_r^R \left(\frac{1}{R} + \frac{R}{x^2} \right) (\log|f(x)| + \log|f(x)|)dx = \frac{1}{\pi}\int_r^R \left(\frac{1}{R} + \frac{R}{x^2} \right) d\varphi(t) = \\ = \frac{2\varphi(R)}{\pi R} + \frac{2R}{\pi}\int_r^R \frac{\varphi(t)dt}{t^3}. \end{aligned} \tag{35}$$

Further,

$$\frac{1}{\pi}\left(\frac{r}{R} - \frac{R}{r} \right)\int_0^\pi \log|f(re^{i\theta})| \cdot \sin\theta d\theta + \frac{1}{\pi}\left(\frac{r}{R} + \frac{R}{r} \right)\int_0^\pi \arg f(re^{i\theta}) \cdot \cos\theta d\theta =$$

$$= \frac{C_1}{R} + C_2 \cdot R \quad \text{where} \quad C_1, C_2 \quad \text{are} \quad \text{constant}. \tag{36}$$

Since $\liminf_{R\to\infty} \gamma(R)/R = \alpha > 0$, there exists some r_0 such that for all $R > r_0$ the following inequality holds:

$$\frac{\gamma(R)}{R} > \frac{\alpha}{2} \quad \text{or} \quad R < \frac{2}{\alpha}\gamma(R).$$

The function $\gamma(R)/R$ is continuous and positive on $[r; r_0]$. Then there exists $\beta > 0$ such that $\gamma(R)/R \geq \beta$ for all $R \in [r; r_0]$. Thus we have

$$R \leq \delta\gamma(R) \tag{37}$$

for all $R > r$, where $\delta = \max\{2/\alpha, 1/\beta\}$. Taking into account (33)–(36) we obtain

$$a_1(R, f) = 4\int_r^R \frac{\psi(t)}{t^2} dt - 4R\int_r^R \frac{\psi(t)dt}{t^3} + \frac{2\varphi(R)}{\pi R} + \frac{2R}{\pi}\int_r^R \frac{\varphi(t)dt}{t^3} + \frac{C_1}{R} + C_2 \cdot R.$$

Let

$$\Lambda(R, f) := 4R\int_r^R \frac{\psi(t)dt}{t^3} - 4\int_r^R \frac{\psi(t)}{t^2} dt - \frac{2\varphi(R)}{\pi R} - \frac{2R}{\pi}\int_r^R \frac{\varphi(t)dt}{t^3} =$$

$$= 4\int_r^R \left(\frac{R}{t} - 1\right)\frac{\psi(t)dt}{t^2} - \frac{2\varphi(R)}{\pi R} - \frac{2R}{\pi}\int_r^R \frac{\varphi(t)dt}{t^3}.$$

So $\Lambda(R, f)$ is nonnegative as φ is nonpositive. Since (ii) and (37) hold, we have

$$\Lambda(R, f) \le |a_1(R, f)| + \frac{|C_1|}{R} + |C_2| \cdot R \le A\gamma(BR)$$

for some $A > 0$, $B > 0$ and for all $R > r$.

$\Lambda(R, f)$ consists of three nonnegative terms, and we can write the inequalities

$$-\frac{2R}{\pi}\int_r^R \frac{\varphi(t)dt}{t^3} \le A\gamma(BR) \tag{38}$$

$$4R\int_r^R \frac{\psi(t)dt}{t^3} - 4\int_r^R \frac{\psi(t)}{t^2}dt \le A\gamma(BR) \tag{39}$$

with A, B mentioned above, and for all $R > r$.

Further,

$$\int_r^R \frac{\psi(t)}{t^2}dt = \int_r^{R/2} \frac{t\psi(t)}{t^3}dt + \int_{R/2}^R \frac{\psi(t)}{t^2}dt \le \frac{R}{2}\int_r^{R/2} \frac{\psi(t)}{t^3}dt + \int_{R/2}^R \frac{\psi(t)}{t^2}dt$$

Then

$$A\gamma(BR) \ge 4R\int_r^R \frac{\psi(t)dt}{t^3} - 4\int_r^R \frac{\psi(t)}{t^2}dt \ge$$

$$\ge 4\frac{R}{2}\int_r^{R/2} \frac{\psi(t)dt}{t^3} + 4R\int_{R/2}^R \frac{\psi(t)dt}{t^3} - 4\int_{R/2}^R \frac{\psi(t)}{t^2}dt \ge 4\frac{R}{2}\int_r^{R/2} \frac{\psi(t)dt}{t^3} \ge 0.$$

That is

$$4R\int_r^R \frac{\psi(t)dt}{t^3} \le A\gamma(2BR) \tag{40}$$

for all $R > r$ and A, B mentioned above.

From (38) and (40) we obtain

$$\frac{2R}{\pi}\int_r^R \frac{\lambda(t)dt}{t^3} = 4R\int_r^R \frac{\psi(t)dt}{t^3} - \frac{2R}{\pi}\int_r^R \frac{\varphi(t)dt}{t^3} \le A_1\gamma(B_1R),$$

or

$$\int_r^R \frac{\lambda(t)dt}{t^3} \le \frac{A_1}{R}\gamma(B_1R) \tag{41}$$

for all $R > r$ and some $A_1 > 0$, $B_1 > 0$.

Inequality (41) means that the measure λ is of finite γ-density.

Using (27) we can write

$$|c_k(R,f)| \le |a_k(R,f)| + |Q_k(R,f)|, \quad k \in \mathbb{N}, \quad R > r,$$

where $Q_k(R,f)$ is given by (30). We obtain the needed estimate for $|Q_k(R,f)|$ with the aid of reasoning given above. Then we have

$$|c_k(R,f)| \le A\gamma(BR), \quad k \in \mathbb{N},$$

for some $A > 0$, $B > 0$ and for all $R > r$. According to Theorem M this is equivalent to the relation $\log|f| \in JS(\gamma)$. Theorem 1 is proved.

4. Proof of Theorem 2. If (i) holds then according to Theorem 1

$$|a_k(R,f)| \le A_1\gamma(B_1R), \quad k \in \{0\} \cup \mathbb{N}$$

for some positive A_1 , B_1 and for all $R > r$, and according to Theorem M

$$|c_k(R,f)| \le A_2\gamma(B_2R), \quad k \in \{0\} \cup \mathbb{N}$$

for some positive A_2 , B_2 and for all $R > r$. Thus (ii) holds, because

$$l_k(R,f) = \frac{i}{2}(a_{|k|}(R,f) - \operatorname{sgn} k \cdot c_{|k|}(R,f)), \quad k \in \mathbb{Z}.$$

It also follows from the last equality that $a_k(R,f) = (l_k(R,f) + l_{-k}(R,f))/i, k \in \{0\} \cup \mathbb{N}$. So, if (ii) holds then from Theorem 1 we obtain $\log|f| \in JS(\gamma)$. Theorem 2 is proved.

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