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A GAME CHARACTERIZATION OF LIMIT-DETECTING SEQUENCES IN LOCALLY COMPACT G-SPACES

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It is proved that under some mild conditions a set $S \subset X$ is strongly limit-detecting if and only if S is ω-controlling in X if and only if S is asymptotically dense in the sense that for any neighborhood U of the unit in G the set US has bounded complement in X. On the other hand, $S \subset X$ is limit-detecting if and only if S is 1-controlling and splittable. In its turn, a set $S \subset X$ is 1-controlling if the product KS is ω-controlling for some compact countable set $K \subset G$. These results are proved with help of some infinite game resembling the Telgárski game characterizing $\mathcal{K}$-scattered properties.

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Доказано, что при некоторых (нежестких) условиях, множество $S \subset X$ является сильно определяющим тогда и только тогда, когда S является ω-контролирующим в X; и тогда и только тогда, когда S является асимптотически плотным в следующем смысле: для любой окрестности U единицы в G множество US имеет ограниченное дополнение в X. С другой стороны, $S \subset X$ является определяющим тогда и только тогда, когда S — 1-контролирующее и расщепляемое. В свою очередь, множество $S \subset X$ — 1-контролирующее, если произведение $\mathcal{K}S$ является ω-контролирующим для некоторого счетного компактного множества $\mathcal{K} \subset G$. Эти результаты доказаны при помощи некоторой бесконечной игры, аналогичной игре Telgárski, описывающей свойства $\mathcal{K}$-рассеяности.

The textbook [4] of problems in Mathematical Analysis contains the following Problem III.2.59*: *Prove that a continuous function $f \colon [0, +\infty) \to \mathbb{R}$ has limit $\lim_{x \to +\infty} f(x) = 0$ at infinity if and only if $\lim_{n \to \infty} f(an) = 0$ for each real $a > 0$.* For the solution the author of [4] refers the reader to a rather inaccessible paper [10] and another inaccessible textbook [6]. This problem is indicated as a problem of higher complexity (for students) and serves as an example of application of Baire Theorem. In fact, it describes a particular case of the following general result: *if (x_n) is an increasing unbounded sequence of positive real numbers with $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 1$, then a continuous function $f \colon [0, +\infty) \to \mathbb{R}$ has the limit $\lim_{x \to +\infty} f(x)$ if and only if for each $a > 0$ the limit $\lim_{n \to \infty} f(ax_n)$ exists,* see [1]. In this context the following problem arises naturally:

Problem 1. *Describe the geometric and arithmetic structure of increasing real sequences* $(x_n)_{n=1}^{\infty}$ *detecting limits of continuous functions at infinity in the sense that a continuous function* $f\colon [0, +\infty) \to \mathbb{R}$ *has the limit* $\lim_{x \to +\infty} f(x)$ *if and only if for each* $a > 0$ *the limit* $\lim_{n \to \infty} f(ax_n)$ *exists. In the sequel such sequences* (x_n) *will be called limit-detecting.*

In this paper we shall give an answer to this problem. In fact, we shall work with a more general version of this problem.

Let X be a locally compact topological space endowed with a continuous action $\cdot\colon G \times X \to X$ of a topological group G. In this case we shall say that X is a G-*space*. For an element g of the group G and a point x of X we shall write gx in place of $\cdot(g, x)$ and say that gx is the image of x under the action of the element g. By definition, the action satisfies two laws:

- the *associativity*: $\forall g, h \in G \ \forall x \in X \quad g(hx) = (gh)x$ and

- the *neutrality*: $\forall x \in X \ ex = x$,

where e stands for the neutral element of the group G. In the sequel by $\mathcal{N}(e)$ we shall denote the neighborhood base of the group G at e. Among basic examples of G-spaces we shall keep in mind the half-line $[0, +\infty)$ endowed with the action of the multiplicative group $\mathbb{R}_+$ of positive real numbers and the Euclidean space $\mathbb{R}^n$ endowed with the natural action of the isometry group $\mathrm{Iso}(\mathbb{R}^n)$ or some its subgroups (e.g. the subgroup of translations of $\mathbb{R}^n$).

Given a function $f\colon X \to \mathbb{R}$ and a subset $S \subset X$ we shall write $\lim_{S \ni x \to \infty} f(x) = a$ if for any neighborhood $O(a) \subset \mathbb{R}$ of a there is a compact subset $K \subset X$ such that $f(S \setminus K) \subset O(a)$. We shall write $\lim_{x \to \infty} f(x)$ in place of $\lim_{X \ni x \to \infty} f(x)$. The principal concepts of the paper are defined as follows.

Definition 1. A subset S of a locally compact G-space X is called

- *limit-detecting* if a bounded continuous function $f\colon X \to \mathbb{R}$ has limit $\lim_{x \to \infty} f(x)$ at infinity, provided for each $g \in G$ the limit $\lim_{S \ni x \to \infty} f(gx)$ exists;

- *strongly limit-detecting* if for any neighborhood U of the unit e in G a bounded continuous function $f : X \to \mathbb{R}$ has limit $\lim_{x \to \infty} f(x)$, provided for each $g \in U$ the limit $\lim_{S \ni x \to \infty} f(gx)$ exists.

According to [1] for an increasing sequence $S = \{x_n\}_{n \in \omega}$ in the half-line $[0, +\infty)$ endowed with the action of the multiplicative group $\mathbb{R}_+$ the strong limit detecting property is equivalent to $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 1$. In fact these two properties of (x_n) are equivalent to the asymptotical density as well as the ω-controlling property defined as follows.

Definition 2. A subset S of a locally compact G-space X is defined to be

- *bounded* if it has compact closure in X;

- *cobounded* if $X \setminus S$ is bounded in X;

- *asymptotically dense* if for any neighborhood U of the unit e in G the set US is cobounded in X;

- κ-*controlling* for a cardinal κ if for any collection $\mathcal{U}$ of open unbounded subsets of X with $|\mathcal{U}| \le \kappa$ there is an element $g \in S$ such that for each $U \in \mathcal{U}$ the intersection $gS \cap U$ is unbounded in X;

- *splittable* if for any disjoint open unbounded subsets $U, V \subset X$ with cobounded union $U \cup V$ in X there is $g \in G$ such that the intersections $gS \cap U$ and $gS \cap V$ are unbounded.
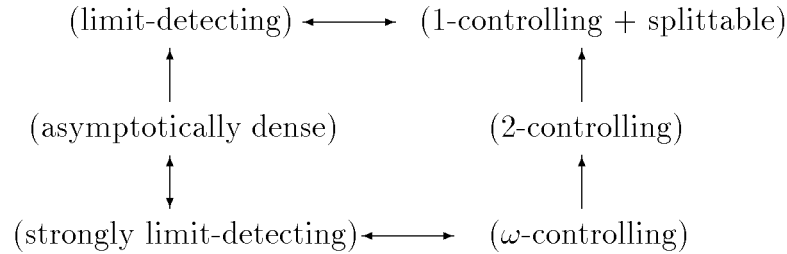
Under some natural restrictions on the action of G on X we shall prove that the strong limit-detecting property of a subset $S \subset X$ is equivalent to the ω-controlling property and also to the asymptotical density of S while the limit-detecting property of S is equivalent to the 1-controlling property combined with the splittability of S and follows from the asymptotic density of the product KS for some scattered compacts subset $K \subset X$.

To describe these restrictions let us say that an action of a topological group G on a topological space X is

- *open on a set* $L \subset X$ if for any neighborhood U of the unit e in G and any point $x \in L$ the set Ux is open in X;

- *asymptotically open* if it is open on some open cobounded subset of X;

- *almost open* if it is open on some open subset $L \subset X$ with cobounded closure $\bar{L}$ in X;

- *discrete* if there is an open subset $L \subset X$ with cobounded closure $\bar{L}$ in X such that for each $x \in L$ the stabilizer $\mathcal{S}t(x) = \{g \in G : gx = x\}$ is a discrete subgroup of G.

A topological group G is called ω-*bounded* if for each neighborhood U of the unit in G there is a countable subset $F \subset G$ with $G = FU = UF$, see [7], [14]. As expected, by a *Baire topological group* we understand a topological group whose underlying topological space is Baire, which means that the intersection of any countable family of dense open subsets of G is dense. A topological space is *Polish* if it is homeomorphic to a separable complete metric space.

For a closed discrete subset S of a locally compact σ-compact space X endowed with a discrete almost open action of a ω-bounded locally compact group G we shall prove the following implications:

$$
\begin{array}{ccc}
\text{(limit-detecting)} & \longleftrightarrow & \text{(1-controlling + splittable)} \\
\uparrow & & \uparrow \\
\text{(asymptotically dense)} & & \text{(2-controlling)} \\
\uparrow & & \uparrow \\
\text{(strongly limit-detecting)} & \longleftrightarrow & \text{(ω-controlling)}
\end{array}
$$

These implications will be proved in the subsequent two characterizing theorems. First, we recall some information concerning the small cardinal $\mathrm{cov}(\mathcal{M})$. By definition, it is equal to the smallest size of a cover of the real line by nowhere dense subsets. It is clear that $\aleph_1 \leq \mathrm{cov}(\mathcal{M}) \leq \mathfrak{c}$, where $\mathfrak{c}$ stands for the size of continuum. Martin Axiom implies that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ but there are models of ZFC in which $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$. In fact, the equality $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ is equivalent to $\mathrm{MA}_{\text{countable}}$, the Martin Axiom for countable posets, see [3], [16].

Theorem 1. *For a subset S of a locally compact space X endowed with a continuous action of a Baire topological group G the following conditions are equivalent:*

1) S is strongly limit-detecting;

2) for any neighborhood $U \in \mathcal{N}(e)$ the closure of US has bounded complement in X;

   3) for any open unbounded subset $U \subset X$ the set $\{g \in G : gS \cap U$ is unbounded in $X\}$ is dense G_δ in G.

Moreover, if the action of G on X is asymptotically open, then (1)–(1) are equivalent to

   4) the set S is asymptotically dense in X.

If the group G is ω-bounded, then conditions (1)–(1) are equivalent to

   5) S is ω-controlling;

If the group G is Polish, then conditions (1)–(1) are equivalent to

   6) S is κ-controlling for any infinite cardinal $\kappa < \mathrm{cov}(\mathcal{M})$.

The following proposition shows that strongly limit-detecting closed discrete subsets exist in all locally compact σ-compact spaces endowed with an asymptotically open action of a Polish group.

Proposition 1. *If X is a locally compact σ-compact G-space endowed with an asymptotically open action of a Baire metrizable group G, then X contains a closed discrete strongly limit-detecting subset S. If the group G is Polish, then the set S is κ-controlling for all $\kappa < \mathrm{cov}(\mathcal{M})$.*

Proof. Write X as a countable union, $X = \bigcup_{n \in \omega} X_n$, of compact subsets such that each X_n lies in the interior $\overset{\circ}{X}_{n+1}$ of X_{n+1}. Fix also a decreasing neighborhood base $(U_n)_{n \in \omega}$ at the identity of G. Find a bounded open subset $W \subset X$ such that the action of G on X is open on $X \setminus W$.

For every $n \in \omega$ consider the compact subset $K_n = X_{n+1} \setminus (W \cup \overset{\circ}{X}_n)$ and its open cover $\{U_n x : x \in K_n\}$. By the compactness of K_n, there is a finite subset $S_n \subset K_n$ such that $K_n \subset U_n S_n$. Then the set $S = \bigcup_{n \in \omega} S_n$ is closed and discrete in X. Being asymptotically dense, S is strongly limit-detecting by Theorem 1. If the group G is Polish, then S is κ-controlling for all cardinals $\kappa < \mathrm{cov}(\mathcal{M})$. $\qquad\square$

Next, we turn to characterization of limit-detecting subsets in locally compact G-spaces.

Theorem 2. *Let X be a σ-compact locally compact G-space endowed with a discrete almost open action of a σ-compact locally compact group G. A closed countable subset $S \subset X$ is limit-detecting if and only if it is 1-controlling and splittable.*

Let us remark that for an unbounded subset S of a locally compact G-space X the splittability is automatic if X contains no unbounded disjoint open subsets U, V with cobounded union $U \cup V$. This happens if each bounded subset of X lies in a bounded subset with connected complement in X (in fact, the latter property is equivalent to the connectedness of the remainder $\beta X \setminus X$ of the Stone-Čech compactification of X). Among spaces having the latter property there are the half-line $[0, +\infty)$ and the Euclidean spaces $\mathbb{R}^n$ for $n > 1$. For such spaces the limit-detecting property is equivalent to the 1-controlling property.

Another extreme happens in the case of discrete G-spaces endowed with an action of a discrete group G. The structure of *strongly* limit-detecting subsets in such spaces is not interesting: all of them are cofinite. The situation with limit-detecting property is much more interesting. Following [11] we shall say that a subset S of a G-space X is *large* if for some finite subset $F \subset G$ the set FS has finite complement in X.

Proposition 2. *Let X be a discrete space endowed with an action of a discrete countable group G. A subset $S \subset X$ is 1-controlling if and only if it is large in X.*

In spite of the fact that this proposition follows from the subsequent more general Theorem 3, we shall present a short (and direct) proof.

Proof. The "if" part is obvious. Assuming that $S \subset X$ is not large in X, write $G = \bigcup_{n \in \omega} F_n$ as a countable union of finite subsets $F_n \subset F_{n+1}$ and for every $n \in \omega$ select a point $x_n \in X \setminus F_n S$ distinct from $x_0, \ldots, x_{n-1}$. Then $U = \{x_n : n \in \omega\}$ is an open unbounded subset of X having finite intersection with each shift gS, $g \in G$, of S. This shows that S fails to be 1-controlling. $\square$

To characterize limit-detecting sets in discrete G-spaces X, to each subset $S \subset X$ assign the graph Γ_S whose vertices are elements of the group G and two vertices $g, h \in G$ are linked by an edge in Γ_S if the intersection $gS \cap hS$ is infinite. Observe that for each large (equivalently, 1-controlling) subset $S \subset X$ the graph Γ_S has only finitely many connected components.

Proposition 3. *Let X be a discrete space endowed with an action of a discrete countable group G. A subset $S \subset X$ is limit-detecting if and only if S is large in X and the graph Γ_S is connected.*

Proof. Assume that S is limit-detecting. Then S is 1-controlling and hence large according to the previous proposition. Assuming that the graph Γ_S is not connected, take any its connected component $C \subset G$. Let $\chi_C \colon G \to \{0,1\}$ be the characteristic function of the set C and $G = \{g_n : n \in \mathbb{N}\}$ be an enumeration of G. For every $n \in \mathbb{N}$ consider the subset $S_n = g_n S \setminus \bigcup_{1 \leq i < n} S_i$ of X. Define a function $f \colon X \to \{0,1\}$ letting $f|S_n \equiv \chi_C(g_n)$ for $n \in \mathbb{N}$ and $f|X \setminus GS \equiv 0$. This function f has no limit at infinity. Nonetheless, for any $g \in G$ the restriction $f|gS$ has limit at infinity (this limit equals 0 if $g \in C$ and 1 otherwise). This contradiction shows that the graph Γ_S is connected.

Now assume conversely that S is large in X and the graph Γ_S is connected. Fix any function $f \colon X \to \mathbb{R}$ such that for any $g \in S$ the restriction $f|gS$ has limit $\Psi(g)$ at infinity. The connectedness of Γ_S implies that all the limits $\Psi(g)$, $g \in G$, coincide and are equal to some number A. Now the 1-controlling property of the large set S implies that A equals to $\lim_{x \to \infty} f(x)$. $\square$

In contrast to the above simple arguments, the proof of Theorem 2 in the general (non-discrete) case is not trivial and relies on a game characterization of the 1-controlling property. This characterization will be given in terms of the existence of winning strategies in some infinite game resembling the infinite topological game $G(\mathcal{K}, X)$ introduced and studied by Telgársky [12], see also [17]. The Telgársky game $G(\mathcal{K}, X)$ is played by two persons called Player I and Player II on a topological space X endowed with some collection $\mathcal{K}$ of closed subsets. Player I starts the game selecting a subset $K_1 \in \mathcal{K}$ while Player II responds with an open neighborhood OK_1 of K_1 in X. Continuing in this fashion, at the n-th inning the Player I selects a subset $K_n \in \mathcal{K}$ and Player II responds with an open neighborhood OK_n of K_n in X. At the end of the game $G(\mathcal{K}, X)$ the players construct a sequence $(OK_n)_{m \in \mathbb{N}}$ of open sets of X. If $\bigcup_{i < n} OK_i = X$ for some finite n, then Player I is declared the winner. Otherwise, Player II wins the game.

If the class $\mathcal{K}$ consists of all one-point subsets of X, then the Telgársky game $G(\mathcal{K}, X)$ is nothing else but the classical game "Point-Open", well-known in topology, see [15], [13]. The "Point-Open" game is used to characterize scattered compacta. We recall that a topological space X is *scattered* if each non-empty subset of X contains an isolated point. According to [12] a compact Hausdorff space X is scattered if and only if Player I has a winning strategy in the "Point-Open" game $G(\mathcal{P}, X)$, where $\mathcal{P} = \{A \subset X : |A| = 1\}$. In a similar way for more general classes $\mathcal{K}$ the existence of a winning strategy of Player I in the Telgársky game $G(\mathcal{K}, X)$ characterizes the $\mathcal{K}$-scatteredness property of X, where X is defined to be $\mathcal{K}$-*scattered* if each non-empty subset $A \subset X$ contains a non-empty relatively open subset U that lies in some $K \in \mathcal{K}$. More detailed information on infinite games and strategies can be found in [9, §20], [15], [13], or [17].

In light of this information about the Telgársky game the rules of the game $\mathfrak{G}_\mathcal{K}(S)$ detecting 1-controlling subsets S in locally compact G-spaces will be natural. So, let S be a subset of a locally compact G-space X and $\mathcal{K}$ be a class of subsets of the group G (in the sequel we will be interested in the classes $\mathcal{K}$ consisting of all one-point, finite, or compact scattered subsets of the group G). Like the Telgársky game $G(\mathcal{K}, X)$, the game $\mathfrak{G}_\mathcal{K}(S)$ is played by two persons called Player I and Player II. Player I starts the game selecting a subset $K_1 \in \mathcal{K}$ while player II responds with an open neighborhood U_1 of the unit in G. Continuing in this fashion, at the n-th inning Player I selects a subset $K_n \in \mathcal{K}$ and Player II responds with an open neighborhood $U_n \in \mathcal{N}(e)$. At the end of the game $\mathfrak{G}_\mathcal{K}(S)$ the players jointly construct two sequences (K_n) and (U_n). If for some finite n the set $\bigcup_{i \leq n} U_i K_i S$ has bounded complement in X, then Player I is declared the winner. Otherwise, Player II wins the game.

By a strategy of Player I in the game $\mathfrak{G}_\mathcal{K}(S)$ we understand a function $\$_I$ assigning to a finite sequence $(U_1, \ldots, U_n)$ of neighborhoods of e in G a subset $K_{n+1} = \$_I(U_1, \ldots, U_n) \in \mathcal{K}$. Such a strategy $\$_I$ is winning in the game $\mathfrak{G}_\mathcal{K}(S)$ if for any infinite sequence $(U_n) \subset \mathcal{N}(e)$ there is $m \in \mathbb{N}$ such that the set $\bigcup_{n < m} U_n \$_I(U_1, \ldots, U_{n-1})S$ has bounded complement in X.

Dually, a strategy of Player II in $\mathfrak{G}_\mathcal{K}(S)$ is a function $\$_{II}$ assigning to each finite sequence $(K_1, \ldots, K_n) \in \mathcal{K}^n$ a neighborhood $U_n = \$_{II}(K_1, \ldots, K_n) \in \mathcal{N}(e)$. Such a strategy $\$_{II}$ is winning in the game $\mathfrak{G}_\mathcal{K}(S)$ if for any infinite sequence $(K_n) \in \mathcal{K}^\mathbb{N}$ the set $\bigcup_{n < m} U_n K_n S$ has unbounded complement in X for all m.

Note that if $\mathcal{K} \subset \mathcal{C}$, then each winning strategy of Player I in the game $\mathfrak{G}_\mathcal{K}(S)$ will be winning in $\mathfrak{G}_\mathcal{C}(S)$ while each winning strategy of Player II in the game $\mathfrak{G}_\mathcal{C}(S)$ will be winning in $\mathfrak{G}_\mathcal{K}(S)$. Thus constructing a winning strategy for Player I (resp. Player II) it is desirable to shrink (resp. enlarge) the class $\mathcal{K}$. There will be two extremal choices of the collection $\mathcal{K}$: the largest is the collection of all scattered compacta in G while the smallest is the collection of all one-point subsets of some G_δ-subset $D \subset G$ with cobounded closure $\bar{D}$ in G. Now we are able to give a game characterization of 1-controlling sets.

Theorem 3. *Let S be a closed countable subset of a σ-compact locally compact G-space X endowed with a continuous discrete almost open action of a σ-compact locally compact group G. Suppose that $\mathcal{K}$ is a collection of scattered compact subsets of G whose union $\cup \mathcal{K}$ contains a G_δ-set $D \subset G$ with cobounded closure $\bar{D}$ in G. Then the following conditions are equivalent:*

1) the set S is 1-controlling;

2) the first player has a winning strategy in the game $\mathfrak{G}_\mathcal{K}(S)$;

3) the second player has no winning strategy in the game $\mathfrak{G}_\mathcal{K}(S)$.

In fact, the implication (3)$\Rightarrow$(1) is true in a more general situation where X is a σ-compact locally compact space endowed with a continuous action of a Čech complete group G, see Lemmas 2 and 3. We recall that a Tychonov space X is *Čech-complete* if X is a G_δ-set in some/any compactification of X. This observation allows us to prove

Corollary 1. *Let X be a locally compact G-space endowed with a continuous action of a Čech-complete group G. A subset $S \subset X$ is 1-controlling if KS is asymptotically dense in X for some scattered compact subset $K \subset G$.*

Proof. It suffices to verify that Player II has no winning strategy in the game $\mathfrak{G}_\mathcal{K}(S)$ where $\mathcal{K}$ is the family of all scattered compact subsets of G. For this we describe a simple winning strategy of Player I in the game. His first move is the scattered compact subset $K_1 = K$. Since KS is asymptotically dense in X, for any response $U_1 \in \mathcal{N}(e)$ of Player II, the set $U_1 K_1 S$ is cobounded in X, which means that Player I wins the game. $\square$

In contrast to Theorem 1 characterizing ω-controlling sets in geometric terms as asymptotically dense subsets, Theorem 3 gives a game-theoretic characterization of 1-controlling sets but provides no information on the geometric structure of such sets. Now we shall try to fill this gap.

First we recall some necessary information on (ultra)filters. As expected, by a *filter* on a set X we understand a collection $\mathcal{F}$ of non-empty subsets of X, closed under finite intersections and taking supersets. A filter $\mathcal{F}$ on X is called an *ultrafilter* if any filter on X containing $\mathcal{F}$ coincides with $\mathcal{F}$. It follows from Zorn Lemma that each filter can be enlarged to an ultrafilter. It is well-known that a filter $\mathcal{F}$ on X is an ultrafilter if and only if for any subset $A \subset X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. We shall say that an (ultra)filter $\mathcal{F}$ on a locally compact space X *converges to* ∞ if the complement $X \setminus B$ of any bounded set $B \subset X$ belongs to $\mathcal{F}$. The following theorem shows that 1-controlling sets are asymptotically dense in a "locally-scattered" sense.

Theorem 4. *Suppose that S is a 1-controlling subset of a locally compact G-space X satisfying the conditions of Theorem 3. Then for any convergent to ∞ ultrafilter $\mathcal{F}$ on X there is a nonempty open set $U \subset G$ such that for each $W \in \mathcal{N}(e)$ the set $\{x \in X : Ux \subset WS\}$ belongs to $\mathcal{F}$.*

Applying Theorems 1–4 to the real line $\mathbb{R}$ endowed with the natural action of the additive group $G = \mathbb{R}$ we get

Corollary 2. *Let $S = \{x_n : n \in \omega\}$ be an increasing unbounded sequence in $[0, +\infty)$. Then*

1) *The sequence $S_\pm = \{\pm x_n : n \in \omega\}$ is ω-controlling iff it is strongly limit-detecting iff $S_\pm$ is asymptotically dense iff $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$;*

2) *The sequence $S_\pm$ is limit-detecting iff $S_\pm$ is 1-controlling if $C + S_\pm$ is asymptotically dense for some compact countable set $C \subset \mathbb{R}$;*

3) *If S is 1-controlling, then*

$$\lim_{n \in \omega} \frac{x_n}{n} = 0 = \liminf_{n\to\infty}(x_{n+1} - x_n) \leq \limsup_{n\to\infty}(x_{n+1} - x_n) < \infty.$$

Proof. The first two items easily follow from Theorems 1, 2, Proposition 1 and the fact that the set $S_\pm$, being unbounded in both directions, is splittable in $\mathbb{R}$.

To prove the last item, assume that $S_\pm$ is 1-controlling and apply Theorem 3 to conclude that Player II has no winning strategy in the game $\mathfrak{G}_{\mathcal{K}}(S_\pm)$, where $\mathcal{K}$ is the collection of all singletons of the group $G = \mathbb{R}$.

Assuming that $\limsup_{n\to\infty} \frac{x_n}{n} > C$ for some positive constant $C > 0$, find an increasing number sequence (n_k) such that $x_{n_k} > Cn_k$ for all k. To derive a contradiction it suffices to construct a winning strategy for Player II in the game $\mathfrak{G}_{\mathcal{K}}(S_\pm)$. For every $n \in \mathbb{N}$ let $\varepsilon_n = 2^{-(n+2)}C$.

Player I starts the game selecting a one-point subset $\{y_1\} \subset \mathbb{R}$ and Player II answers with the neighborhood $U_1 = (-\varepsilon_1, \varepsilon_1)$ of zero in the group $G = \mathbb{R}$. At the n-th inning Player I selects a one-point subset $\{y_n\}$ of $\mathbb{R}$ and Player II answers with the neighborhood $U_n = (-\varepsilon_n, \varepsilon_n)$ of zero (as we see the moves of Player II do not depend on the choices of Player I). Let us show that the described strategy of Player II is winning. Take any $n \in \omega$ and assume that the complement of the set $W = \bigcup_{i<n} U_i + y_i + S_\pm$ in $\mathbb{R}$ lies in some closed interval $[-a, a]$, $a > 0$. Let $b = \sum_{i<n} |y_i|$ and find k such that $a + b + \frac{C}{2} < \frac{C}{2}n_k$. To derive a contradiction it suffices to verify that the set $[a, Cn_k] \setminus W$ is not empty. This will be established as soon as we prove that the Lebesgue measure of this set is strictly positive. For this observe that $x_{n_k} > Cn_k$ implies $|S \cap [0, Cn_k]| < n_k$. Denote by λ the standard Lebesgue measure on $\mathbb{R}$ and observe that for each $i < n$ we get

$$\lambda\big([0, Cn_k] \cap (U_i + S)\big) \leq 2\varepsilon_i + \lambda\big([\varepsilon_i, Cn_k - \varepsilon_i] \cap (U_i + S)\big) \leq$$
$$\leq 2\varepsilon_i + \lambda(U_i)|[0, Cn_k] \cap S| < 2\varepsilon_i + 2\varepsilon_i n_k = 2^{-(i+1)}C(1 + n_k).$$

Next,

$$\lambda\big([0, Cn_k] \cap (U_i + y_i + S)\big) = \lambda\big([-y_i, Cn_k - y_i] \cap (U_i + S)\big) \leq$$
$$\leq |y_i| + \lambda\big([0, Cn_k] \cap (U_i + S)\big) < |y_i| + 2^{-(i+1)}C(1 + n_k).$$

Then

$$\lambda([0, Cn_k] \cap W) = \lambda\big([0, Cn_k] \cap \bigcup_{i<n} U_i + y_i + S\big) \leq \sum_{i<n} \lambda\big([0, Cn_k] \cap (U_i + y_i + S)\big) <$$
$$< \sum_{i<n} |y_i| + 2^{-(i+1)}C(1 + n_k) = b + \frac{1}{2}C(1 + n_k)$$

and hence

$$\lambda([a, Cn_k] \setminus W) = (Cn_k - a) - \lambda([a, Cn_k] \cap W) \geq$$
$$\geq Cn_k - (a + b + \frac{1}{2}C(1 + n_k)) = \frac{1}{2}Cn_k - (a + b + \frac{1}{2}C) > 0.$$

This finishes the proof of the equality $\lim_{n\to\infty} \frac{x_n}{n} = 0$. This equality trivially implies another equality $\liminf_{n\to\infty} x_{n+1} - x_n = 0$ of the last item of the corollary.

Finally, assume that $\limsup_{n\to\infty} x_{n+1} - x_n = +\infty$ and find a number sequence (n_k) such that $\lim_{k\to\infty} x_{n_k+1} - x_{n_k} = +\infty$. For each $k \in \omega$ let $a_k = \frac{1}{2}(x_{n_k+1} + x_{n_k})$ and note that for any $C > 1$ the intersection $S \cap (a_k - C, a_k + C)$ is empty for sufficiently large k. Then for the filter $\mathcal{F} = \{F \subset [0, +\infty) : F \supset \{a_k : k > n\}$ for some $n\}$ the conclusion of Theorem 3 fails, which is a contradiction. $\qquad\square$

Applying to the previous corollary the transformation $\ln\colon \mathbb{R} \to \mathbb{R}_+$ we get the following properties of (strongly) limit-detecting sequences in the half-line $X = [0, +\infty)$ endowed with the action of the multiplicative group $\mathbb{R}_+$.

Corollary 3. *Let $S = \{x_n : n \in \omega\}$ be an increasing unbounded sequence in $[0, +\infty)$. Then*

1. *The sequence S is strongly limit-detecting iff S is ω-controlling iff $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.*

2. *The sequence S is limit-detecting iff it is 1-controlling if CS is strongly limit-detecting for some countable compact set $C \subset \mathbb{R}_+$.*

3. *If S is limit-detecting, then $1 = \liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n} < \infty$ and $\lim_{n\to\infty} \frac{\ln x_n}{\ln n} = 0$.*

Question 1. *Let $S \subset [0, +\infty)$ be a limit-detecting sequence. Does there exist $N \in \mathbb{N}$ such that $\lim_{x\to+\infty} |S \cap [x, Nx]| = \infty$?*

Let us finish the introduction with some counterexamples distinguishing the limit-detecting and κ-controlling properties.

Example 1. The sequence $\{\pm \ln n : n \in \mathbb{N}\}$ is strongly limit-detecting in the real line $\mathbb{R}$ endowed with the natural action of its isometry group. On the other hand, the sequence $S = \{\ln n : n \in \mathbb{N}\}$ is 1-controlling but not limit-detecting. This set S is not splittable.

Proof. Let e be the identity transformation of $\mathbb{R}$ and $h\colon x \to -x$ be the central symmetry of $\mathbb{R}$. The set $\pm S = \{h, e\} \cdot S = \{\pm \ln n : n \in \mathbb{N}\}$ is strongly limit-detecting by Corollary 2.

Since $\{h, e\} \cdot S$ is strongly limit-detecting, the set S is 1-controlling according to Corollary 1. This set is not limit-detecting since it cannot detect the non-existence of the limit $\lim_{|x|\to\infty} \arctan x$. Also this set fails to be splittable since there is no isometry f of $\mathbb{R}$ such that $f(S)$ has unbounded intersection with the sets $(-\infty, 0)$ and $(0, +\infty)$.   □

Example 2. There is a limit-detecting closed discrete subset $S \in \mathbb{R}$ which is not 2-controlling with respect to the natural action of the additive group $\mathbb{R}$.

Proof. In the real line $\mathbb{R}$ endowed with the natural action of the additive group $\mathbb{R}$ consider two subsets $A = \{\pm \ln n : n \in \mathbb{N}\}$ and $B = \bigcup_{n\in\mathbb{Z}}(4n, 4n + 1)$. The intersection $A \cap B$ is 1-controlling since the set $\{0, 1, 2, 3\} + (A \cap B)$ is asymptotically dense in $\mathbb{R}$, see Corollary 2. Since $A \cap B$ is unbounded in both directions, it is splittable. Applying Theorem 2 we conclude that $A \cap B$ is limit-detecting. To see that the set $A \cap B$ is not 2-controlling, observe that the shifts of $A \cap B$ cannot simultaneously intersect the open sets B and $2 + B$.   □

Next, we shall construct κ-controlling sets which fail to be n-controlling for $n < \kappa$. Our construction will work also for infinite cardinals $\kappa \le \mathrm{cov}(\mathcal{M})$.

Example 3. For a non-zero cardinal $\kappa \le \mathrm{cov}(\mathcal{M})$ let S_κ be the group of all bijections of κ. Consider the product $X = \mathbb{R} \times \kappa$ as a G-space with the coordinatewise action of the group $G = \mathbb{R} \times S_\kappa$. Using the asymptotical density of the set $\{\pm \ln n : n \in \mathbb{N}\}$ in $\mathbb{R}$ and Theorem 1, it is easy to show that the set $S = \{\pm \ln n : n \in \mathbb{N}\} \times (\kappa \setminus \{0\})$ is λ-controlling for any cardinal $\lambda < \kappa$ in $\mathbb{R} \times \kappa$. On the other hand, it is obvious that S fails to be κ-controlling.

According to Proposition 1 any finite-dimensional Banach space X contains a sequence $S = \{x_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} \|x_n\| = \infty$ which is 1-controlling in the sense that for any unbounded open set $U \subset X$ the intersection $(a + S) \cap U$ is unbounded for some $a \in X$. It is interesting to note that such 1-controlling sequences do not exist in infinite-dimensional Banach spaces.

Proposition 4. *If X is an infinite-dimensional Banach space and $S = \{x_n : n \in \omega\} \subset X$ is a sequence such that $\lim_{n\to\infty} \|x_n\| = \infty$, then there is an unbounded open set $U \subset X$ such that the intersection $(a + S) \cap U$ is bounded for any $a \in X$.*

Proof. Denote by $B_n = \{x \in X : \|x\| \leq n\}$ the n-ball around centered at zero. For every $n \in \omega$ let $S_n = S \cap B_{2n+1}$.

We shall apply a classical Riesz almost orthogonality Lemma [8, p. 11] asserting that for any finite-dimensional subspace F of an infinite-dimensional normed space X there is an element $x \in X$ with $\|x\| = 3$ and $\mathrm{dist}(x, F) > 2$. Apply the Riesz Lemma to construct a sequence $(y_n)_{n \geq 3}$ such that $\|y_n\| = n$ and $\mathrm{dist}(y_n, F_n) > 2$ for the finite set

$$F_n = S_n + \bigcup_{i < n} y_i - S_i.$$

Consider the unbounded open set $U = \bigcup_{n \geq 3} y_n + B_1$. We claim that for every $g \in X$ the intersection $(g + S) \cap U$ is finite. Assuming the converse find $m > n > 3$ such that $g \in B_n$ and $(g + S) \cap (y_n + B_1) \neq \varnothing \neq (g + S) \cap (y_m + B_1)$. Consequently, we can find points $g + s_n = y_n + b_n$ and $g + s_m = y_m + b_m$, where $s_n, s_m \in S$ and $b_n, b_m \in B_1$. Subtracting the second equality from the first one, we get $y_m - (y_n + s_m - s_n) = b_n - b_m$. Note that $\|s_n\| = \|y_n + b_n - g\| \leq 2n+1$ and $\|s_m\| = \|y_m + b_m - g\| \leq m+n+1 < 2m+1$. Consequently, $\|y_m - (y_n + s_m - s_n)\| \leq 2$ and $y_n + s_m - s_n \in y_n + S_m - S_n \subset F_m$, which contradicts to the choice of the point y_m. $\square$

Finally let us formulate some open problems.

Question 2. *Let S be a limit-detecting subset S of the half-line $[0, +\infty)$ endowed with the natural action of the multiplicative group $\mathbb{R}_+$. Is KS asymptotically dense in $[0, +\infty)$ for some scattered compact set $K \subset \mathbb{R}_+$?*

In other words, does the existence of a winning strategy of Player I in the game $\mathfrak{G}_{\mathcal{K}}(S)$ (where $\mathcal{K}$ is the family of scattered compact subsets of $\mathbb{R}_+$) imply that Player I can always win at the first move?

Consider the Euclidean space $X = \mathbb{R}^n$ endowed with the natural action of the matrix group $G = GL(n)$. The action of G on $\mathbb{R}^n$ is open on the complement of zero, but fails to be discrete. Thus Theorems 2, 3 are not applicable.

Question 3. *Is any 1-controlling unbounded sequence in $X = \mathbb{R}^n$ limit-detecting?*

1. Proof of Theorem 1

Let S be a subset of a locally compact space X endowed with a continuous action of a Baire topological group G.

At first, we prove the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

$(1) \Rightarrow (2)$ If the set $S \subset X$ fails to satisfy condition (1), then for some $U \in \mathcal{N}(e)$ the closure of the product US in X has unbounded complement $W = X \setminus \overline{US}$. It is easy to construct a continuous function $f : X \to \mathbb{R}$ which has no limit $\lim_{x \to \infty} f(x)$ but vanishes on the closed set $X \setminus W = \overline{US}$, see the proof of Lemma 6. In this case $\lim_{S \ni x \to \infty} f(gx) = 0$ for all $g \in U$ and, consequently, S fails to be strongly limit-detecting.

(2) $\Rightarrow$ (3) Assuming (2) we first show that for each open unbounded subset $W \subset X$ the set $H = \{g \in G : gS \cap W \neq \varnothing\}$ is open and dense in G. The openness of H trivially follows from the continuity of the action G on X. To prove the density of H in G, fix arbitrary point $g \in G$ and an open neighborhood $U \in \mathcal{N}(e)$. It follows from (1) that the closure of the set US has bounded complement in X. The unboundedness of the open set W and its shift gW implies that the intersection $US \cap g^{-1}W$ is not empty and thus contains some point $g^{-1}w$. Then $gUS \cap W \ni w$ and hence $gU \cap H \neq \varnothing$, which means that H is dense in G. It remains to prove that the set $\{g \in G : gS \cap W$ is unbounded in $X\}$ is dense G_δ in G. For this, write the locally compact and σ-compact space X as a countable union of compact subsets, $X = \bigcup_{n \in \omega} X_n$, such that each X_n lies in the interior of X_{n+1}. Then each bounded subset of G lies in some X_n.

For every $n \in \omega$ the set $W \setminus X_n$ is unbounded and open in X. The above argument yields that the sets $H_n = \{g \in G : gS \cap (W \setminus X_n) \neq \varnothing\}$ are open and dense in G for all n. Since the group G is Baire, the countable intersection $\bigcap_{n \in \omega} H_n = \{g \in G : gS \cap W$ is unbounded in $X\}$ is dense G_δ in G.

(3)$\Rightarrow$(1) Let $f : X \to \mathbb{R}$ be a continuous function having no limit $\lim_{x \to \infty} f(x)$. Then there are two open subsets $U_1, U_2 \subset \mathbb{R}$ with disjoint closures whose preimages $f^{-1}(U_1), f^{-1}(U_2)$ are unbounded in X. Condition (1) implies that the set $A = \{g \in G : gS \cap f^{-1}(U_1)$ and $gS \cap f^{-1}(U_2)$ are unbounded in $X\}$ is dense G_δ in X and consequently, meets each open neighborhood $W \in \mathcal{N}(e)$. Then for each $g \in A \cap W$ the limit $\lim_{S \ni x \to \infty} f(gx)$ fails to exist. This means that the set S is strongly limit-detecting.

The implication (4)$\Rightarrow$(2) is trivial. To prove the inverse implication, assume that the action of the group G on X is open on the complement of some bounded set $B \subset X$. Fix any neighborhood $U \in \mathcal{N}(e)$ and find a neighborhood $W \in \mathcal{N}(e)$ such that $W^{-1}W \subset U$. Condition (1) implies that the complement $X \setminus \overline{WS}$ lies in some compact subset $K \subset X$. We claim that $X \setminus (B \cup K) \subset US$. Take any point $x \in X \setminus (B \cup K)$. Since the action of the group G on X is open on $X \setminus B$, Wx is a neighborhood of x. The density of WS in $X \setminus K$ implies that the intersection $WS \cap Wx$ is not empty. Then $x \in W^{-1}WS \subset US$ and hence the set US has bounded complement in X.

The implication (3)$\Rightarrow$(5) trivially follows from the Baireness of G.

Next, assuming that the group G is ω-bounded we shall prove the implication (5)$\Rightarrow$(1). It suffices to show that for any open unbounded subsets $U_1, U_2 \subset X$ and a neighborhood $W \in \mathcal{N}(e)$ of the unit in G there is an element $g \in W$ such that the intersections $gS \cap U_1$, $gS \cap U_2$ are unbounded. The group G, being ω-bounded, contains a countable subset C such that $CW = G$. It follows from (5) that there is an element $a \in G$ such that the intersections $aS \cap cU_1$, $aS \cap cU_2$ are unbounded for any $c \in C$. Taking into account that $a \in G = CW$, find an element $c \in C$ such that $a \in cW$. For this element c, the intersections $aS \cap cU_i$, $i = 1, 2$, are unbounded. Then the intersections $c^{-1}aS \cap U_i$, $i = 1, 2$, are unbounded as well. It remains to observe that the element $g = c^{-1}a$ belongs to the set W.

Finally, assume that the group G is Polish. The implication (6)$\Rightarrow$(5) is trivial. To finish the proof of the theorem it suffices to verify that (3)$\Rightarrow$ (6). Fix a family $\mathcal{U}$ of unbounded open subsets of X of size $|\mathcal{U}| < \mathrm{cov}(\mathcal{M})$ and for each $U \in \mathcal{U}$ consider the set $A_U = \{g \in G : gS \cap U$ is unbounded$\}$ which is dense G_δ-set in G by (3). It follows from the definition of the cardinal $\mathrm{cov}(\mathcal{M})$ that the intersection $\bigcap_{U \in \mathcal{U}} A_U$ contains some point $g \in G$. For this point, all the intersections $gS \cap U$, $U \in \mathcal{U}$, are unbounded, which finishes the proof.

2. Proof of Theorem 3

Theorem 3 follows from the cycle of implications $(2) \Rightarrow (3) \overset{L2.4}{\Longrightarrow} (1) \overset{L2.5}{\Longrightarrow} (2)$, the first of which is trivial (the last two implications will be proved in the indicated lemmas).

We shall need the game characterization of scattered compacta due to Galvin, see [12]. This characterization asserts that a compact space is scattered if and only if Player I has a winning strategy in the game "Point-Finite". For the convenience of the reader, we give a short proof of the "only if" part of this characterization (which will be used in the proof of the subsequent Lemma 2).

Lemma 1. *If K is a compact Hausdorff scattered space, then Player I has a winning strategy in the "Point-Open" game on K.*

Proof. Let $K^{(1)} = K$ be the set of all non-isolated points of K. For each ordinal α by recursion define the α-th derived set $K^{(\alpha)} = \bigcap_{\beta < \alpha} (K^{(\beta)})^{(1)}$. Since K is scattered, each non-empty $K^{(\alpha)}$ has an isolated point, which means that $K^{(\alpha)} \neq K^{(\alpha+1)}$. Consequently, $K^{(\alpha)} = \varnothing$ for some ordinal α. The smallest ordinal α for which $K^{(\alpha)}$ is finite is called *the scattered height* of K.

Now we are able to define a winning strategy of Player I in the game "Point-Open" on K. He looks at the finite set $F_1 = K^{(\alpha_0)}$, where α_0 is the scattered height of K and his first $n_1 = |F_1|$ moves in the "Point-Open" game are just the points $x_1, \ldots, x_{n_1}$ of the set F_1. During these n_1 moves the second player answers with some open neighborhoods $Ox_1, \ldots, Ox_{n_1}$ of these points. Then the first player looks at the set $W_1 = \bigcup_{i \le n_1} Ox_i$. If this set equals K, then Player I wins the game. Otherwise, he calculates the scattered height α_1 of the scattered compactum $K_1 = K \setminus W_1$ (it is important to note that $\alpha_1 < \alpha_0$). Then Player I looks at the finite set $F_2 = K_1^{(\alpha_1)}$ and his next $n_2 = |F_2|$ moves are just points $x_{n_1+1}, \ldots, x_{n_1+n_2}$ of the set F_2. Player I receives the answers Ox_i, $n_1 < i \le n_1 + n_2$, of Player II and looks at the set $W_2 = \bigcup_{1 \le i \le n_1+n_2} Ox_i$. If this set equals K, then Player I wins.

Otherwise, he calculates the scattered height $\alpha_3 < \alpha_2$ of the scattered compactum $K_3 = K \setminus W_3$ and so on. Since no strictly decreasing sequence of ordinals is infinite, the game must stop at some finite step by the victory of Player I. □

The Galvin Lemma allows us to prove the following useful reduction result.

Lemma 2. *Let S be a subset of a locally G-space X, $\mathcal{K}$ be a collection of scattered compact subsets of X, and $\mathcal{P} = \{\{x\} : x \in \cup \mathcal{K}\}$. Player I or II has a winning strategy in the game $\mathfrak{G}_{\mathcal{K}}(S)$ if and only if he has a winning strategy in the game $\mathfrak{G}_{\mathcal{P}}(S)$.*

Proof. To prove this lemma use the winning strategy of Player I in the "Point-Open" game on scattered compacta and also the well-known fact that for any open neighborhood $O(K)$ of a compact subset K in a topological group G there is an open neighborhood U of the unit in G such that $US \subset O(K)$. □

Lemma 3. *Let X be a σ-compact locally compact G-space endowed with a continuous action of a Baire group G. Let U be an open non-empty subset of G and O be an open subset of X such that $gS \cap O$ is bounded in X for each $g \in U$. Then there are a point $g \in U$ and a neighborhood $V \in \mathcal{N}(e)$ of the unit in G such that the intersection $VS \cap g^{-1}O$ is bounded in X.*

Proof. Write $X = \bigcup_{n\in\omega} X_n$ as a countable union of an increasing sequence of compact subsets X_n of X. For every $n \in \omega$ consider the closed subset $F_n = \{g \in U : gS \cap O \subset X_n\}$ of U and observe that $U = \bigcup_{n\in\omega} F_n$. Applying Baire Theorem to the locally compact space U, find $n \in \omega$ such that the set F_n has non-empty interior. This allows us to find $V \in \mathcal{N}(e)$ and $g \in F_n$ with $gV \subset F_n$. Then $gVS \cap O \subset X_n$ and hence $VS \subset g^{-1}X_n$ is bounded in X. $\qquad\square$

The following lemma in combination with Lemma 2 proves the implication $(3) \Rightarrow (1)$ of Theorem 3.

Lemma 4. *Let X be a σ-compact locally compact G-space endowed with a continuous action of a Čech-complete group G. If a set $S \subset X$ is not 1-controlling, then the player II has a winning strategy in the game $\mathfrak{G}_{\mathcal{K}}(S)$, where $\mathcal{K}$ is the family of all one-point subsets of X.*

Proof. Assuming that S fails to be 1-controlling, find an open unbounded subset O of X such that $gS \cap O$ is bounded in X for each $g \in G$.

The space X, being σ-compact and locally compact, can be written as a union $X = \bigcup_{n\in\mathbb{N}} X_n$ of an increasing sequence of compact subsets such that each X_n lies in the interior of X_{n+1}.

The Čech-complete group G, being a G_δ-set in its Stone-Čech compactification βG, is the intersection $G = \bigcap_{n=1} G_n$ of a decreasing sequence (G_n) of open subsets of βG.

Now we describe a winning strategy for Player II in the game $\mathfrak{G}_{\mathcal{K}}(S)$. Playing the game he will construct two sequences (U_n) and (V_n) of neighborhoods of the unit in G and sequences (g_n) and (x_n) of points of G and X, respectively. The set U_n will be the answers of Player II in the n-th inning while other objects play an auxiliary role in the inductive construction. Let $U_0 = G$.

Given the first move $\{t_1\} \subset G$ of Player I, Player II applies Lemma 3 to find a point $g_1 \in U_0$ and a neighborhood $V_1 \subset U_0$ of the unit in G such that $V_1(t_1 S) \cap g_1^{-1}O$ is bounded in X. This allows him to select a point $x_1 \in O \setminus (X_1 \cup g_1 V_1 t_1 S)$ and a neighborhood U_1 of the unit in G such that $U_1 = U_1^{-1}$, $U_1^4 \subset V_1$, $U_1 x_1 \subset O$, and closure of $g_1 U_1^2$ in βX lies in G_1. The neighborhood U_1 is the answer of II in the 1-st inning.

Continuing in this fashion, at the n-th inning the second player receives the n-th move $\{t_n\} \subset G$ of Player I and applies Lemma 3 to find a point $g_n \in U_{n-1}$ and a neighborhood $V_n \subset U_{n-1}$ of the unit in G such that the intersection $V_n t_n S \cap (g_1 \cdots g_n)^{-1}O$ is bounded in X. Then he selects a point $x_n \in O \setminus (X_n \cup \bigcup_{k\le n}(g_1 \cdots g_k)V_k t_k S)$ and a neighborhood U_n of the unit in G such that $U_n = U_n^{-1}$, $U_n^4 \subset V_n$, $U_n x_n \subset O$, and the closure of $g_1 \cdots g_n U_n^2$ in βG lies in G_n. The neighborhood U_n is the answer of II in his n-th move.

We claim that the described strategy of Player II is winning in the game $\mathfrak{G}_S$. We have to verify that for each n the set $\bigcup_{i<n} U_i t_i S$ has unbounded complement in X.

Let g_∞ be a cluster point of the sequence $(g_1 \cdots g_n)_{n\in\mathbb{N}}$ in βG. First we show that $g_\infty \in G$. Indeed, for any $n < m$ we get

$$g_1 \cdots g_m \in g_1 \cdots g_{m-2}U_{m-2}U_{m-1} \subset g_1 \cdots g_{m-2}U_{m-2}^2 \subset g_1 \cdots g_{m-3}U_{m-3}U_{m-2}^2 \subset$$
$$\subset g_1 \cdots g_{m-3}U_{m-3}^2 \subset \cdots \subset g_1 \cdots g_n U_n^2 \subset \mathrm{cl}_{\beta G}(g_1 \cdots g_n U_n^2) \subset G_n$$

and thus $g_\infty \in \bigcap_{n\ge1} \mathrm{cl}_{\beta G}(g_1 \cdots g_n U_n^2) \subset \bigcap_{n\ge1} G_n = G$.

Assuming that for some n the set $\bigcup_{i<n} U_i t_i S$ is cobounded in X, we would find $m > n$ such that $g_\infty^{-1}x_m \in \bigcup_{i<n} U_i t_i S$. Then

$$x_m \in \bigcup_{i<n} g_\infty U_i t_i S \subset \bigcup_{i<n}(g_0 \cdots g_i)\overline{U_i^2}U_i t_i S_i \subset \bigcup_{i<n}(g_0 \cdots g_i)U_i^3 U_i t_i S_i \subset \bigcup_{i<n}(g_0 \cdots g_i)V_i t_i S,$$

which contradicts to the choice of the point x_m.   □

Thus the implication (3)⇒(1) of Theorem 3 has been proved. The following lemma in combination with Lemma 2 proves the implications (1) ⇒ (2) of Theorem 3.

Lemma 5. *Let X be a σ-compact locally compact space endowed with a continuous discrete almost open action of a σ-compact locally compact group G. Let $D \subset G$ be a G_δ-set with cobounded closure in G and let $\mathcal{K}$ be the collection of all compact scattered subsets of D. A countable closed subset $S \subset X$ fails to be 1-controlling if Player I has no winning strategy in the game $\mathfrak{G}_\mathcal{K}(S)$.*

Proof. Assume that Player I has no winning strategy in the game $\mathfrak{G}_\mathcal{K}(S)$.

The group G, being locally and σ-compact, can be written as a union, $G = \bigcup_{n \in \omega} G_n$, of an increasing sequence of compact neighborhoods of the unit such that $G_n = G_n^{-1}$ and $G_n G_n \subset G_{n+1}$ for all n. Similarly, the space X can be written as a union, $X = \bigcup_{n \in \omega} X_n$, of an increasing sequence of compact subsets such that $G_n X_n$ lies in the interior of X_{n+1} for all n.

Since the action of G on X is discrete and almost open, there is a G-invariant open set $L \subset X$ with cobounded closure $\bar{L}$ in X such that for each $x \in L$ the shift $g \mapsto gx$ is a local homeomorphism of G onto an open subset of X.

Consider the set $E = L \setminus ((G \setminus D) \cdot S)$, which is a dense G_δ in L. To see that it is indeed so, write $G \setminus D$ as a countable union $G \setminus D = \bigcup_{n \in \omega} K_n$ of compact subsets $K_n \subset G$ and observe that

$$E = \bigcap_{s \in S, n \in \omega} L \setminus (K_n s).$$

Each set $L \setminus K_n s$ is open, being the complement of the compact set $K_n s$. To see that $L \setminus K_n s$ is dense in L, use the fact that K_n is nowhere dense in G and for each $x \in L$ the map $g \mapsto gx$ is a local homeomorphism. Now Baire theorem implies that the set E, being a countable intersection of open dense subsets in the locally compact space L, is dense G_δ in L.

Now let us describe a strategy of Player I in the game $\mathfrak{G}_\mathcal{K}(S)$. He starts the game picking any element $x_1 \in E$ and considers the set $F_1 = \{g \in G_1 : x_1 \in gS\}$. This set belongs to D by the choice of $E \ni x_1$. Since both S and the stabilizer $\mathcal{S}t(x_1) = \{g \in G : gx_1 = x_1\}$ of x_1 are countable, the set F_1 is a closed countable subset of the compactum G_1. Being countable and compact, the set F_1 is scattered. The second player answers with a neighborhood U_1 of the unit in G.

At the n-th move, Player I receives the $(n-1)$-th move U_{n-1} of Player II and chooses an open neighborhood $V_{n-1} \in \mathcal{N}(e)$ with $V_{n-1}^2 \subset U_{n-1}$. Then he looks at the set $R_n = E \setminus (X_n \cup \bigcup_{i<n} F_i V_i S)$. If this set is empty, then the n-th move of the first player is the empty set $F_n = \varnothing$. In the opposite case, he picks any element x_n of the set R_n and his n-th move is the scattered compact subset $F_n = \{g \in G_n : x_n \in gS\}$ of D.

By our hypothesis, the described strategy of the first player cannot be winning, which means that for some sequence $\{U_n\}_{n \in \omega} \subset \mathcal{N}(e)$ the sets R_n constructed in the process of the game are not empty for all n. As a result, following the described strategy, the first player constructs the unbounded sequence of points $x_n \in R_n$ and neighborhoods V_n with $V_n^2 \subset U_n$, $n \in \omega$.

Take any decreasing sequence (W_n) of neighborhoods of the unit in G such that $W_0 \subset G_0$, $W_n^{-1} W_n \subset V_n \cap W_{n-1}$ and $W_n x_n \in X \setminus X_n$ for all n. Finally, consider the unbounded open subset $O = \bigcup_{n \in \omega} W_n x_n$ of X.

We claim that for each $g \in G$ the intersection $gS \cap O$ is bounded. Assuming that it is not so, we would find numbers $n < m$ such that $g \in G_{n-1}$ and gS meets both $W_n x_n$ and $W_m x_m$. Since $gS \cap W_n x_n \neq \varnothing$ and hence $W_n^{-1} gS \ni x_n$, there is an element $g_n \in W_n^{-1} g$ such that $g_n S \ni x_n$. Observe that $g_n \in W_n^{-1} g \in G_0 G_{n-1} \subset G_n$ and thus $g_n \in F_n$. Then $x_m \in W_m^{-1} gS \subset W_m^{-1} W_n g_n S \subset V_n F_n S$, which contradicts to the choice of x_m. Therefore for each $g \in G$ the intersection $gS \cap O$ is bounded, which means that the set S fails to be 1-controlling. $\qquad\square$

3. Proof of Theorem 2

The "only if" part of Theorem 2 is almost trivial and follows from

Lemma 6. *Suppose S is a limit-detecting subset of a σ-compact locally compact G-space X. Then S is 1-controlling and splittable.*

Proof. The space X, being σ-compact and locally compact, can be written as the union, $X = \bigcup_{n \in \mathbb{N}} X_n$, of an increasing sequence of compact subsets such that each X_n lies in the interior of X_{n+1}. Assuming that S fails to be 1-controlling, find an open unbounded subset $U \subset X$ such that for each $g \in G$ the intersection $gS \cap U$ is bounded. For every $n \in \omega$ select a point $x_n \in U \setminus X_n$ and find a number $m(n) \in \omega$ such that $x_n \in X_{m(n)}$. Then take any neighborhood $O x_n \in X_{m(n)+1} \setminus X_n$ and using the Tychonov property of locally compact spaces, find a continuous function $f_n : X \to [0,1]$ such that $f_n(x_n) = 1$ and $f_n^{-1}(0,1] \subset O x_n$.

Replacing (x_n) by a suitable subsequence we can assume that the neighborhoods $O x_n$, $n \in \omega$, are pairwise disjoint. Then $f = \sum_{n \in \omega} f_{2n} : X \to [0,1]$ is a continuous function having no limit $\lim_{x \to \infty} f(x)$ at ∞. On the other hand, for each $g \in S$ the intersection $gS \cap f^{-1}(0,1] \subset gS \cap U$ is bounded and thus $\lim_{S \ni x \to \infty} f(gx) = 0$. This shows that the set S fails to be limit-detecting.

To show that S is splittable, fix two disjoint unbounded open subsets $U, V \subset X$ with cobounded union $U \cup V$ in X. Find a compact subset $K \subset X$ such that $K \cup U \cup V = X$. Then the set $K \cup U = X \setminus (V \setminus K)$ is closed in X. Using the compactness of K and the Tychonov property of the locally compact space $K \cup U$, construct a continuous function $f : K \cup U \to [0,1]$ such that $f(K) = \{0\}$ and $f^{-1}[0,1)$ is bounded in X. Define f on V letting $f|V \equiv 0$. Then $f : X \to [0,1]$ is a continuous function which has no limit $\lim_{x \to \infty} f(x)$. Since S is limit-detecting, there is $g \in G$ such that the restriction $f|gS$ has no limit at infinity. It follows that both the intersections $gS \cap U$ and $gS \cap V$ are unbounded and hence S is splittable. $\qquad\square$

The proof of the "if" part of Theorem 2 is less trivial and uses the characterizing Theorem 3 as well as the Baire Theorem on continuity points of F_σ-measurable functions.

We recall that a function $f : T \to \mathbb{R}$ on a topological space T is called F_σ-*measurable* if the preimage $f^{-1}(U)$ of each open set $U \subset \mathbb{R}$ is of type F_σ in X, which means that $f^{-1}(U)$ is a countable union of closed subsets of T. By $C(f)$ we shall denote the set of continuity points of f. The following classical result belongs to Baire, see [9, 24.14].

Lemma 7. *If $f : T \to \mathbb{R}$ is an F_σ-measurable function on a Baire topological space T, then the set $C(f)$ of continuity points of f is dense G_δ-set in T.*

The following Lemma proves the "if" part of Theorem 2.

Lemma 8. *Let X be a σ-compact locally compact G-space endowed with a continuous discrete almost open action of a σ-compact locally compact group G. A subset $S \subset X$ is limit-detecting provided it is 1-controlling and splittable.*

Proof. Assume that S is 1-controlling and splittable. By Lemmas 2 and 5 for any dense G_δ-set $D \subset G$ Player I has a winning strategy in the game $\mathfrak{G}_{\mathcal{K}}(S)$ where $\mathcal{K}$ is the family of all one-point subsets of D.

To show that S is limit-detecting, fix any bounded continuous function $f : X \to \mathbb{R}$ having no limit $\lim_{x \to \infty} f(x)$. Let $\bar{f} : \beta X \to \mathbb{R}$ be the continuous extension of f onto the Stone-Čech compactification of X. Then $\bar{f}(\beta X \setminus X)$ is compact in $\mathbb{R}$ and we can put $a = \min \bar{f}(\beta X \setminus X)$, $A = \max \bar{f}(\beta X \setminus X)$. Since f has no limit at infinity, $a < A$.

If $\bar{f}(\beta X \setminus X) \neq [1, A]$, then we can find two points $c < C$ such that $[c, C] \subset [a, A] \setminus \bar{f}(\beta X \setminus X)$. Now consider two disjoint open unbounded subsets $U = f^{-1}(-\infty, c)$ and $V = f^{-1}(C, +\infty)$ of X. It is easy to see that $U \cup V$ is cobounded in X. Since S is splittable, there is $g \in G$ such that the intersections $gS \cap U$ and $gS \cap V$ are unbounded. Then

$$\liminf_{S \ni x \to \infty} f(gx) \leq c < C \leq \limsup_{S \ni x \to \infty} f(gx),$$

which means that the restriction $f|gS$ has no limit at infinity.

Next, we consider the case $\bar{f}(\beta X \setminus X) = [a, A]$. Multiplying f by a suitable constant, we can assume that $A - a > 1$. To derive a contradiction, assume that for each $g \in G$ the limit $\Phi(g) = \lim_{S \ni x \to \infty} f(gx)$ exists. Thus we obtain a bounded function $\Phi : G \to \mathbb{R}$. We claim that this function is F_σ-measurable. Indeed, given an open subset $U \subset \mathbb{R}$ write $U = \bigcup_{n=1}^\infty U_n$ as a union of open subsets of U such that $\bar{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$. Let also $S = \{s_k : k \in \mathbb{N}\}$ be any enumeration of the sequence S. Then

$$\Phi^{-1}(U) = \{g \in G : \lim_{m \to \infty} f(gs_m) \in U\} = \bigcup_{n,k=1}^\infty \bigcap_{m \geq n} \{g \in G : f(gs_m) \in \bar{U}_k\},$$

being a countable union of the closed sets $\bigcap_{m \geq n}\{g \in G : f(gs_m) \in \bar{U}_k\}$, $n, k \in \mathbb{N}$, is an F_σ-set in G.

Thus the function $\Phi \colon G \to \mathbb{R}$ is F_σ-measurable and the set $D = C(\Phi)$ of continuity points of Φ is dense G_δ in G according to the Baire Lemma 7. According to Lemmas 2 and 5, Player I has a winning strategy in the game $\mathfrak{G}_{\mathcal{K}}(S)$, where $\mathcal{K}$ is the family of all one-point subsets of D.

To derive a final contradiction it suffices to describe a winning strategy of Player II in the game $\mathfrak{G}_{\mathcal{K}}(S)$. After the first move $\{g_1\} \subset D$ of the first player, the second player chooses a neighborhood U_1 of the unit in G such that $\operatorname{diam}\Phi(U_1 g_1) < \frac{1}{2}$. Continuing in this way, in response to the n-th move $\{g_n\} \subset D$ of the first player, the second player chooses a neighborhood U_n of the unit in G such that $\operatorname{diam}\Phi(U_n g_n) < 2^{-n}$.

We claim that the described strategy of the second player is winning, which means that $\bigcup_{i < n} U_i g_i S$ has unbounded complement for each n. Assuming the converse we would find $n \in \mathbb{N}$ such that the closure of $\bigcup_{i < n} W_i g_i S$ in βX contains the remainder $\beta X \setminus X$. Observe that $\overline{\bigcup_{i < n} W_i g_i S} = \bigcup_{i < n} \overline{W_i g_i S}$ and $\operatorname{diam}\bar{f}(\overline{W_i g_i S} \setminus X) \leq \operatorname{diam}\Phi(W_i g_i) < 2^{-i}$ for all $i < n$. Then

$$1 < A - a = \operatorname{diam}\bar{f}(\beta X \setminus X) \leq \sum_{i < n} \operatorname{diam}\bar{f}(\overline{W_i g_i S} \setminus X) \leq \sum_{i < n} \frac{1}{2^i} < 1,$$

which is a contradiction. $\square$

4. PROOF OF THEOREM 4

Let S be a 1-controlling subset of a locally compact G-space satisfying the conditions of Theorem 3. This theorem implies that Player II has no winning strategy in the game $\mathfrak{G}_{\mathcal{K}}(S)$, where $\mathcal{K}$ is the collection of all one-point subsets of G. We shall show that for each convergent to ∞ ultrafilter $\mathcal{F}$ on X there is a nonempty open subset $U \subset G$ such that for any $W \in \mathcal{N}(e)$ the set $\{x \in X : Ux \subset WS\} \in \mathcal{F}$.

Assuming the converse we would find an ultrafilter $\mathcal{F}$ such that

$$(\star) \quad \forall U \underset{op}{\subset} G \; \forall y \in X \; \exists W \in \mathcal{N}(e) \quad \{x \in X : Ux \not\subset W^{-2}WyS\} \in \mathcal{F}.$$

Here we used a well-known property of ultrafilters asserting that $A \notin \mathcal{F}$ is equivalent to $X \setminus A \in \mathcal{F}$.

To get a contradiction, it suffices to describe a winning strategy of Player II in the game $\mathfrak{G}_{\mathcal{K}}(S)$, where $\mathcal{K}$ is the collection of one-point subsets of G. Fix any bounded neighborhood W_0 of the unit in G.

The game $\mathfrak{G}_{\mathcal{K}}(S)$ starts with the 1-st move $\{y_1\} \subset G$ of Player I. Using the property $(\star)$ for the open set $U = W_0$ and the point y_1 Player II find a neighborhood $W_1 \in \mathcal{N}(e)$ such that $W_1^2 \subset W_0$ and $F_1 = \{x \in X : W_0x \not\subset W_1^{-2}W_1y_1S\} \in \mathcal{F}$. The neighborhood W_1 is the 1-st move of Player II in the game $\mathfrak{G}_{\mathcal{K}}(S)$. Continuing in this way, after the n-th move $\{y_n\} \subset G$ of Player I, Player II applies $(\star)$ to find a neighborhood $W_n \in \mathcal{N}(e)$ with $W_n^2 \subset W_{n-1}$ and $F_n = \{x \in X : W_{n-1}x \not\subset W_n^{-2}W_ny_nS\} \in \mathcal{F}$. The set W_n is the answer of Player II in the n-th inning.

To show that the described strategy is winning, assume that for some $n \in \omega$ the set $\bigcup_{i<n} W_iy_iS$ has bounded complement B in X. Since $\mathcal{F}$ converges to ∞, the set $\bigcap_{i\leq n} F_n \setminus W_0^{-2}B$ contains some point x_0. For this point we get $W_0^2x_0 \cap B = \varnothing$.

It follows from the choice of W_1 that there is a point $x_1 \in W_0x_0 \setminus W_1^{-2}W_1y_1S$. For this point we get $W_1^2x_1 \cap W_1y_1S = \varnothing$. Continuing by induction we shall construct a finite sequence of points $x_0, x_1, \ldots, x_n$ such that $x_i \in W_{i-1}x_{i-1} \setminus W_i^{-2}W_iy_iS$ and thus $W_i^2x_i \cap W_iy_iS = \varnothing$.

Observe that

$$x_n \in W_{n-1}x_{n-1} \subset W_{n-1}W_{n-2}x_{n-2} \subset W_{n-2}^2x_{n-2} \subset W_{n-2}^2W_{n-3}x_{n-3} \subset W_{n-3}^2x_{n-3} \subset \cdots \subset W_i^2x_i$$

for all $i < n$. Since $W_i^2x_i \cap W_iy_iS = \varnothing$, we conclude that $x_n \notin \bigcup_{i<n} W_iy_iS$ and, consequently, $x_n \in B$, which contradicts to $x_n \in W_0^2x_0 \subset X \setminus B$.

5. ACKNOWLEDGEMENTS

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