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EXACT REDUCTION OF LIOUVILLE INTEGRABLE HAMILTONIAN SYSTEMS WITH POLYNOMIAL ADDITIONAL INTEGRALS

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The paper is devoted to the problem of exact reduction of some special Hamiltonian systems. The existence of the full collection of first integrals in involution for the reduced systems is proved for the case of first integrals which are polynomial in the momenta.

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Данная работа посвящена проблеме точной редукции некоторых конечномерных гамильтоновых систем. Для случая систем с полиномиальными по импульсам первыми интегралами доказано существование полного инволютивного набора первых интегралов для приведенной системы.

Consider the natural Hamiltonian system with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + W(x_1, x_2, \ldots, x_n) \tag{1}$$

which is integrable by Liouville and

$$\{F_1, \ldots, F_n\} \tag{2}$$

is the correspondent full involutory and functionally independent collection of first integrals. Suppose that this system also admits the first integral (the full momentum)

$$\mathcal{P} = \sum_{i=1}^{n} p_i$$

which does not belong to collection (2). The following problem arises: *does this system also admit a full involutory and functionally independent collection of integrals containing the full momentum $\mathcal{P}$?*

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In this paper we consider the class of three-dimensional Hamiltonian systems with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + W(x_1, x_2, x_3), \tag{3}$$

which admits two additional functionally independent first integrals that are polynomials in p_1, p_2, p_3.

Using the linear canonical transformation with the matrix

$$\mathbf{A} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix},$$

the Hamiltonian $\mathcal{H}$ can be modified into

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{3} p_i^2 + V(x_1, x_2), \tag{4}$$

where the new canonical coordinates and momenta are again denoted by (x, p). The integral of full momentum for system (4) is now equal to p_3. The problem of investigation of the Liouville integrability of the Hamiltonian system (4) is now reduced to the problem of discovering an additional polynomial first integral $R = R(x_1, x_2, p_1, p_2)$ for the system with 2 degrees of freedom and the Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) + V(x_1, x_2). \tag{5}$$

Lemma 1. *Let $F(x,p)$ be the first integral of the system with Hamiltonian (4), polynomial in the momenta p_1, p_2, p_3. Then*

1. *$\frac{\partial F}{\partial p_3}$ is also the first integral of this system.*
2. *F is a polynomial in x_3.*

The first statement of the lemma follows from the equality $\{\frac{\partial F}{\partial x_3}, \mathcal{H}\} = \frac{\partial}{\partial x_3}\{F, \mathcal{H}\} = 0$.

As $F(x,p)$ is a polynomial in the momenta, it can be represented in the form

$$F = F_n + F_{n-1} + \cdots + F_0,$$

where the component F_k is a homogeneous of degree k polynomial in the momenta p_1, p_2, p_3. Then for the upper part F_n we obtain, that $\{F_n, \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)\} = 0$, i.e.

$$\sum_{i=1}^{3} \frac{\partial F_n}{\partial x_i} p_i = 0.$$

Therefore F_n is a polynomial of degree n in the variables $(p_1, p_2, p_3, x_1p_2 - x_2p_1, x_2p_3 - x_3p_2, x_3p_1 - x_1p_3)$ (see [1]). Obviously, the degree in the momenta of the first integral $(\frac{\partial}{\partial x_3})^{n+1}F$ is less than n. This remark proves the second statement of the lemma.

Lemma 2. *Let* $\operatorname{rank}(\mathcal{H}, F, p_3) = 2$ *in a neighborhood of some point in the phase space (x,p). Then $F = Q(\mathcal{H}, p_3)$, where Q is the some polynomial in two variables.*

Note that under the conditions of Lemma 2 the functions $\mathcal{H}, F, p_3$ are polynomials in the momenta. In accordance with the theorem about rank in a neighborhood of some point in the phase space (x,p) there exists an $\mathbb{R}$-analytic function φ of two variables such that $F = \varphi(\mathcal{H}, p_3)$. On the other hand, it follows from Lemma 1 that the first integral F can be represented in the form

$$F = Q_n(\mathcal{H}_0)p_3^n + \cdots + Q_0(\mathcal{H}_0).$$

As all terms $Q_k(\mathcal{H}_0)$ are polynomials in the momenta, they are also polynomials in $\mathcal{H}_0$.

Theorem. *Let the Hamiltonian system with Hamiltonian (4) admit a full collection of the first integrals* $\{\mathcal{H}, F, G\}$ *where the integrals* F, G *are polynomials in the momenta* p_1, p_2, p_3. *Then this system also admits a full collection of integrals* $\{\mathcal{H}, R, p_3\}$, *where the integral* R *is a polynomial in the momenta* p_1, p_2, p_3.

Under the conditions of the theorem, $\operatorname{rank}(\mathcal{H}, F, p_3) = 3$ in all points of the phase space. Consider the following 2 cases:

Case 1. $\operatorname{rank}(\mathcal{H}, F, G, p_3) = 3$. Then, obviously, for any point $(x_1, x_2, x_3, p_1, p_2, p_3)$ from the phase space the vectors $\operatorname{grad}\mathcal{H}, \operatorname{grad}F, \operatorname{grad}G, \operatorname{grad}p_3$ are functionally dependent. Therefore the Poisson brackets $\{F, p_3\}$ and $\{F, p_3\}$ are equal to zero on the phase space. Let N be a natural number such that $N > \max(\deg_p F, \deg_p G)$. Then the expression $R = p_3^N + G$ satisfies all conditions of the theorem, i.e. the Hamiltonian system with Hamiltonian (4) admits a full collection of the first integrals in the form $(\mathcal{H},\ R,\ p_3)$.

Indeed, assume that in some neighborhood U in the phase space $\operatorname{rank}(\mathcal{H}, R, p_3) = 2$. Then in this neighborhood $\operatorname{rank}(\mathcal{H}, F, p_3) = \operatorname{rank}(\mathcal{H}, G, p_3) = 2$. Consequently, there exist two nontrivial linear combinations,

$$L_1 = a_1 \operatorname{grad}\mathcal{H} + a_2 \operatorname{grad}F + a_3 \operatorname{grad}p_3,$$
$$L_2 = b_1 \operatorname{grad}\mathcal{H} + b_2 \operatorname{grad}G + b_3 \operatorname{grad}p_3,$$

such that $L_1 = L_2 = 0$ identically in U and at least one of the coefficients a_3, b_3 is not equal to zero. Therefore there exists a nontrivial linear combination of $\operatorname{grad}\mathcal{H}, \operatorname{grad}F, \operatorname{grad}G$ which is identically zero in U. This contradiction proves that $(\mathcal{H},\ R,\ p_3)$ is a full collection of the first integrals for Hamiltonian (4).

Case 2. $\operatorname{rank}(\mathcal{H}, F, G, p_3) = 4$. The first integrals F and G can be represented in the following form:

$$F = F_m + \cdots + F_0 x_3^m, \qquad G = G_n + \cdots + G_0 x_3^n,$$

where $F_m, \ldots, G_0$ are independent in x_3. Without loss of generality we can assume that $\operatorname{rank}(\mathcal{H}, F_0, p_3) = \operatorname{rank}(\mathcal{H}, G_0, p_3) = 2$.

Definition. The collection $(\mathcal{H}, F, G, p_3)$ is said to be a *minimal collection of the first integrals*, if $\operatorname{rank}(\mathcal{H}, F, G, p_3) = 4$, and for any other collection of the first integrals $(\mathcal{H}, \tilde{F}, \tilde{G}, p_3)$ such that $\operatorname{rank}(\mathcal{H}, \tilde{F}, \tilde{G}, p_3) = 4$ we have $\deg_{x_3} F + \deg_{x_3} G \le \deg_{x_3} \tilde{F} + \deg_{x_3} \tilde{G}$.

We will also say that the collection $(\mathcal{H}, \tilde{F}, \tilde{G}, p_3)$ is "less" than $(\mathcal{H}, \tilde{F}, \tilde{G}, p_3)$, if $\deg_{x_3} F + \deg_{x_3} G < \deg_{x_3} \tilde{F} + \deg_{x_3} \tilde{G}$.

Without loss of generality we can assume, that the collection $(\mathcal{H}, F, G, p_3)$ under consideration is minimal, $\deg_{x_3} F = m, \deg_{x_3} G = n$ and $m \le n$. If $m = 0$, then the full collection $(\mathcal{H}, F, p_3)$ satisfies the statement of the theorem.

Naturally, if $\operatorname{rank}(\mathcal{H}, F_0, p_3) = 3$ ($\operatorname{rank}(\mathcal{H}, F_0, p_3) = 3$), then the corresponding collection of the first integrals satisfies the statement of the theorem. Let $\operatorname{rank}(\mathcal{H}, F_0, p_3) = \operatorname{rank}(\mathcal{H}, G_0, p_3) = 2$. From Lemma 2 it follows that F_0 and G_0 are polynomials in $\mathcal{H}, p_3$. It is easy to see that for the minimal collection the equality $m = n > 0$ is impossible, otherwise the collection $(\mathcal{H}, F_0G - G_0F, F, p_3)$ is "less" than $(\mathcal{H}, F, G, p_3)$. Consider the remaining two cases:

Case 1. $1 = m < n$. This case is impossible, because the collection $(\mathcal{H}, F, F_0^nG - G_0F^n, p_3)$ is "less" than $(\mathcal{H}, F, G, p_3)$ and $\operatorname{rank}(\mathcal{H}, F, F_0^nG - G_0F^n, p_3) = 4$.

Case 2. $1 < m < n$. According to the definition,

$$\operatorname{rank}\Big(\mathcal{H}, \frac{\partial F}{\partial x_3}, G, p_3\Big) < 4, \quad \operatorname{rank}\Big(\mathcal{H}, \frac{\partial F}{\partial x_3}, F, p_3\Big) < 4.$$

Taking into account the equalities $\operatorname{rank}(\mathcal{H}, G, p_3) = \operatorname{rank}(\mathcal{H}, F, p_3) = 3$, we obtain that there exist two nontrivial linear combinations,

$$L_1 = a_1 \operatorname{grad}\mathcal{H} + a_2 \operatorname{grad}\tfrac{\partial F}{\partial x_3} + a_3 \operatorname{grad} G + a_4 \operatorname{grad} p_3,$$
$$L_2 = b_1 \operatorname{grad}\mathcal{H} + b_2 \operatorname{grad}\tfrac{\partial F}{\partial x_3} + b_3 \operatorname{grad} F + b_4 \operatorname{grad} p_3,$$

such that $L_1 = L_2 = 0$ in U and both of the coefficients a_2, b_2 are not equal to zero. Besides, the coefficients a_3, b_3 are also nonzero, because $m > 1$ and, therefore, $\operatorname{grad}(\partial F/\partial x_3)$ essentially depends on x_3. The expression $b_2L_1 - a_2L_2$ is a nontrivial linear combination of $\operatorname{grad}\mathcal{H}, \operatorname{grad} F, \operatorname{grad} G, \operatorname{grad} p_3$, which is equal to zero. This contradiction with the condition $\operatorname{rank}(\mathcal{H}, F, G, p_3) = 4$ proves the theorem.

The similar statement for Hamiltonian (3) with a full collection of the first integrals being analytical in the momenta (at least one of them) is still open.

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