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AN EXAMPLE OF IRREVERSIBLE WILLIAMS' SOLENOID

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An example of a generalized Williams' solenoid is given in the case of irreversible dynamical systems with expanding attractors.

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Приведен пример обобщенного соленоида Уильямса для случая необратимых динамических систем с растягивающими аттракторами.

Introduction. Let M be a compact manifold, $f\colon M \longrightarrow M$ be a continuous map.

Definition 1. A set Λ is called *attractor*, if there exists a neighbourhood $U(\Lambda)$ of Λ such that $\bigcap_{n\geq 0} f^n(U(\Lambda)) = \Lambda$.

R. Williams [3, 4] defined and studied expanding attractors for hyperbolic diffeomorphisms. The main goal of this paper is to expand Williams' theory onto the case of irreversible maps. We provide an example of an irreversible map and an analog of Williams' solenoids.

Hyperbolic maps and their hyperbolic sets mentioned there are irrelevant for understanding, so they are not defined there. See, e.g. [1] for their definition and properties.

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Manifold of the dynamical system. Let $\mathbb{T}$ be the complex torus $\mathbb{C}/\mathbb{Z}\times i\mathbb{Z}$, where $\mathbb{Z}\times i\mathbb{Z}$ is the lattice $\{a+bi|a,b\in\mathbb{Z}\}$, and $\mathbb{T}_1$ be the complex torus $\mathbb{C}/\mathbb{Z}\times 2i\mathbb{Z}$, where $\mathbb{Z}\times 2i\mathbb{Z}$ is the lattice $\{a+2bi|a,b\in\mathbb{Z}\}$. Let $\mathbb{D}=\{z\in\mathbb{C} : |z|<1\}$ be the complex unit disc. Let $g(w)$ be a homeomorphism of $\mathbb{D}$ swapping two subdiscs $\mathbb{D}_1=\{z\in\mathbb{C} : |z-\frac{i}{2}|<\frac{1}{4}\}$ and $\mathbb{D}_2=\{z\in\mathbb{C} : |z+\frac{i}{2}|<\frac{1}{4}\}$. For example, let $g(w)=-w$.

Introduce an action of the group $G_1=\mathbb{Z}\times\mathbb{Z}$ on $\mathbb{C}\times\mathbb{D}$. Let $(n,m)\in\mathbb{Z}\times\mathbb{Z}$, $(z,w)\in\mathbb{C}\times\mathbb{D}$. Then $(n,m)\circ(z,w)=(z+ln+im, g^n(w))$. The group acts freely and discrete. Hence, the quotient set $N=\mathbb{C}\times\mathbb{D}/\mathbb{Z}\times i\mathbb{Z}$ is a manifold. N is a fiber bundle over the torus $\mathbb{T}$ with the fiber $\mathbb{D}$, because the group acts on $\mathbb{C}\subset\mathbb{C}\times\mathbb{D}$ exactly as the lattice $\mathbb{Z}\times i\mathbb{Z}$. N is diffeomorphic (but not holomorphic) to $\mathbb{T}\times\mathbb{D}$ as a fiber bundle. Let h_N be corresponding diffeomorphism preserving fiber bundle structure.

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Let N_1 be a submanifold of N such that in each fiber $\mathbb{D}$ of N it consists of two mentioned above discs $\mathbb{D}_1 \cup \mathbb{D}_2 \subset \mathbb{D}$. Obviously, it is a fiber bundle over the torus $\mathbb{T}$ with the fiber $\mathbb{D}_1 \cup \mathbb{D}_2$. On the other hand, since $g(w)$ glues $\mathbb{D}_1$ and $\mathbb{D}_2$, N_1 is a connected submanifold. It is diffeomorphic (even holomorphic) to $\mathbb{T}_1 \times \mathbb{D}_1$ as a fiber bundle. Let h_{N_1} be a corresponding diffeomorphism.

Mapping. Let $f(z) = 2z$ be a map $f : \mathbb{C} \to \mathbb{C}$. Note that $f(z)$ preserves the lattice of $\mathbb{T}$ and hence it can be considered as a map of $\mathbb{T}$. Consider a map $\grave{l} : \mathbb{C} \times \mathbb{D} \to \mathbb{C} \times \mathbb{D}_1$, defined by $\grave{l}(z, w) = \left(f(z), \frac{i}{2} + \frac{w}{4}\right)$. Show that $\grave{l}$ commutes with both lattices.

Let $g_1 = (n, m) \in \mathbb{Z} \times i\mathbb{Z}$, $(z, w) \in \mathbb{C} \times \mathbb{D}$.

$$\grave{l} \circ g_1 \circ (z, w) = \left(f(z + 2n + mi), \frac{w}{4}\right) = \left(2z + 4n + 2mi), \frac{w}{4}\right) = g_2 \circ \grave{l} \circ (z, w)$$

where $g_2 = (n, 2m) \in \mathbb{Z} \times 2i\mathbb{Z}$. Hence, $\grave{l}$ induces a map $\mathbb{T} \times \mathbb{D} \to \mathbb{T}_1 \times \mathbb{D}_1$.

Since h_N and h_{N_1} coordinate with the fiber bundle structure, l induces a map $l : N \to N_1$ as shown on the following diagram

$$\begin{array}{ccc}
\mathbb{C} \times \mathbb{D} & \xrightarrow{\grave{l}} & \mathbb{C} \times \mathbb{D}_1 \\
\downarrow & & \downarrow \\
\mathbb{T} \times \mathbb{D} & \xrightarrow{l_{\mathbb{T}\times\mathbb{D}}} & \mathbb{T}_1 \times \mathbb{D}_1 \\
\downarrow{\scriptstyle h_N} & & \downarrow{\scriptstyle h_{N_1}} \\
N & \xrightarrow{l} & N_1 \subset N.
\end{array}$$

Let ρ and ψ be expanding maps on $\mathbb{T}$: $\rho(z) = \mathrm{Re} z + 2i\mathrm{Im} z$, $\psi(z) = 2\mathrm{Re} z + i\mathrm{Im} z$. Note that $(\rho \circ \psi)(z) = 2z$.

Remark 1. $l_{\mathbb{T}\times\mathbb{D}}$ can be written as the composition of the maps $\bar{\rho}$ and $\bar{\psi}$, where $\bar{\rho}(z, w) = \left(\rho(z), \frac{w}{4}\right)$ is a homeomorphism, and $\bar{\psi}(z, w) = (\psi(z), w)$ is a "pure irreversible" part of $l_{\mathbb{T}\times\mathbb{D}}$. $(z, w) \in \mathbb{C} \times \mathbb{D}$. They induce "pure reversible" map $\rho_N : N \to N_1$ and "pure irreversible" map $\psi_N : N \to N_1$.

Attractor of l.

Remark 2. N is a hyperbolic set of the map $l : N \to N$ because N is a fiber bundle and l is a contraction on $\mathbb{D}$ and expansion on $\mathbb{T}$.

Lemma 1. *The dynamic system (N, l) has an attractor with N as the basin of attraction.*

Proof. Let ξ be a point of $\mathbb{T}$ and let $\mathbb{D}_\xi$ be a fiber disc over ξ. The image $l(\mathbb{D}_\xi)$ of $\mathbb{D}_\xi$ is one of discs D_i contained in the fiber $\mathbb{D}_{l_{\mathbb{T}}(\xi)}$. The second image $l^2(\mathbb{D}_\xi)$ is a disc contained in one of the discs D_i and so on with diameters of images monotonely decreasing to 0.

The limit set of the images of N under the map l is a local direct product of an imbedded into $\mathbb{D}$ Cantor set and $\mathbb{T}$. $\square$

Solenoidal representation. Let $\mathbb{T}$ be as defined above and ρ and ψ be expanding maps on it: $\rho(z) = \mathrm{Re} z + 2i\mathrm{Im} z$, $\psi(z) = 2\mathrm{Re} z + i\mathrm{Im} z$. Note that $\rho = \pi\rho_N$ and $\psi = \pi\psi_N$, where $\pi : N \to \mathbb{T}$ is a projection.

Consider the sequence

$$\mathbb{T} \xleftarrow{\rho} \mathbb{T} \xleftarrow{\rho} \dots \xleftarrow{\rho} \mathbb{T} \xleftarrow{\rho} \dots .$$

The inverse limit of the sequence can be written as the set

$$\Sigma = \{(\xi_1, \xi_2, \dots, \xi_n, \dots) | \, \xi_k \in \mathbb{T}, \xi_k = \rho(\xi_{k+1})\}$$

There exists a shift map $h : \Sigma \to \Sigma$,

$$h(\xi_1, \xi_2, \dots, \xi_n, \dots) = (\psi(\rho(\xi_1)), \psi(\xi_1), \psi(\xi_2), \dots, \psi(\xi_{n-1}), \dots). \tag{1}$$

It is natural to call the obtained dynamical system (Σ, h) the irreversible Williams' solenoid. If ψ is Id then the so introduced shift map h will coincide with the shift map introduced by Williams in [3].

Theorem 1. *The obtained dynamical system* (Λ, l) *is topologically conjugate to the dynamical system* (Σ, h).

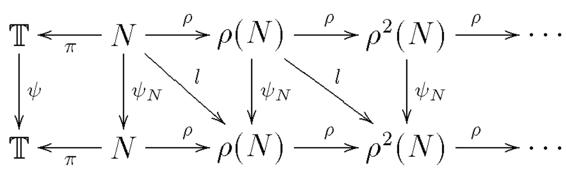
Proof. First consider the dynamical system (Λ, ρ_N). It is a "pure reversible" Williams' solenoid. So, according to Williams' theorem A in [3], there is a homeomorphism R_ρ to the solenoid $\Sigma = \mathbb{T} \xleftarrow{\rho} \mathbb{T} \xleftarrow{\rho} \dots \xleftarrow{\rho} \mathbb{T} \xleftarrow{\rho} \dots$ with the shift h_ρ such that (Λ, ρ_N) is topologically conjugate to (Σ, h_ρ). Then

$$F_\rho(x) = \left(\pi(x), \pi(\rho^{-1}(x), \dots\right), \tag{2}$$

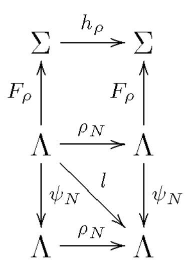
$$h_\rho(\xi_1, \xi_2, \dots, \xi_n, \dots) = (\rho(\xi_1), \xi_1, \xi_2, \dots, \xi_{n-1}, \dots). \tag{3}$$

Note that Λ is the common space for l and ρ_N.

From the commutate diagram



we obtain the diagram



which shows that (Λ, l) is topologically conjugate to $(\Sigma, F_\rho \psi_N F_\rho^{-1} h_\rho)$.

Using (1),(2) and (3) we get $h = F_\rho \psi_N F_\rho^{-1} h_\rho$. □

REFERENCES

1. J. Palis, W. de Melo, Geometric Theory of Dynamical Systems. Springler-Verlag. 1982.
2. R. F. Williams, *One-dimentional non-wandering sets*, Topology. **6** (1967), 473–487.
3. R. F. Williams, *Expanding attractors*, Inst. Hautes Etudes Sci. Publ. Math. **43** (1973), 169–203.
4. C. Robinson, R. Williams, *Classification of expanding attractors: an example*, Topology **15** (1976), 321–323.

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