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A. L. Golberg

QUASIISOMETRIC HOMEOMORPHISMS AND P-MODULI OF SEPARATING SETS

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The quasi-invariance of p-module is a characteristic property for quasiconformal mappings for $p = n$ and for quasiisometric mappings for $p \neq n$. The theorem provide a condition which is more general than the quasi-invariance. This condition completely characterizes quasiisometric homeomorphisms and can be considered as a new definition.

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Квазиинвариантность p-модуля является характеристическим свойством квазиконформных отображений для $p = n$ и квазиизометрических отображений при $p \neq n$. Теорема обеспечивает условие более общее, чем квазиинвариантность. Это условие полностью характеризует квазиизометрические и гомеоморфизмы и может рассматриваться как новое определение.

1. Let G and G^* be two bounded domains in $\mathbb{R}^n$, $n \geq 2$.

A homeomorphism $f\colon G \to G^*$ is called *quasi-isometric* if for any $x, z \in G$ and $y, t \in G^*$ the inequalities

$$\limsup_{z\to x} \frac{|f(x) - f(z)|}{|x - z|} \leq K, \quad \limsup_{t\to y} \frac{|f^{-1}(y) - f^{-1}(t)|}{|y - t|} \leq K, \tag{1}$$

hold, with a constant K, $0 < K < \infty$, depending only on G and G^*. (See [6], [2], [3].)

We now define a quasi-isometry of a homeomorphism in other terms (geometric or modular). Let $\mathcal{S}^k$ be a family of k-dimensional surfaces $\mathcal{S}$ in $\mathbb{R}^n$, $1 \leq k \leq n-1$ (curves for $k = 1$). $\mathcal{S}$ is a k-dimensional surface if $\mathcal{S}\colon D_s \to \mathbb{R}^k$ is a continuous image of the closed domain $D_s \subset \mathbb{R}^k$.

The p-module of $\mathcal{S}^k$ is defined as

$$M_p(\mathcal{S}^k) = \inf \int\limits_{\mathbb{R}^n} \rho^p \, dx, \quad p \geq 1,$$

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where the infinum is taken over all Borel measurable functions $\rho \geq 0$ and such that

$$\int\limits_{\mathcal{S}} \rho^k \, d\sigma_k \geq 1$$

for every $\mathcal{S} \in \mathcal{S}^k$. We call each such ρ an *admissible function* for $\mathcal{S}^k$.

A ring domain $D \subset \mathbb{R}^n$ is defined as a finite domain whose complement consists of two components C_0 and C_1. We set $F_0 = \partial C_0$ and $F_1 = \partial C_1$. Then F_0 and F_1 are simply the components of ∂D. For convenience of notations, we always assume that $\infty \in C_1$.

We say that a curve γ *joins the boundary components in* D if γ lies in D, except for its endpoints, and if one of these endpoints lies in F_0 and the other one in F_1. A compact set Σ is said to *separate the boundary components of* D if $\Sigma \subset D$ and if C_0 and C_1 lie in different components of $C\Sigma$. Denote by Γ_D the family of all locally rectifiable curves γ that join the boundary components of D and by Σ_D the family of all compact piecewise smooth $(n-1)$-dimensional surfaces Σ that separate the boundary components of D.

The following proposition was given in [4] (see, also, [6]) in the terms of p-capacity. On the hand, the p-capacity and the p-modulus $M_p(\Gamma_D)$ as is well-known (see, e. g., [7]) are equivalent.

Proposition. *Let $1 \leq p < \infty$, $p \neq n$ and let a homeomorphism $f\colon G \to G^*$ satisfy:*

$$Q_p^{-1} M_p(\Gamma_D) \leq M_p(f(\Gamma_D)) \leq Q_p M_p(\Gamma_D) \tag{2}$$

for any ring domain $D \subset G$ with Q_p not depending on D.

Then f is quasi-isometric.

The relations between the p-capacities and the p-moduli of families of separating sets were obtained by W. P. Ziemer [8] and by P. Caraman [1]. W. P. Ziemer has considered the condition

$$\int\limits_{s} \rho \, d\sigma_{n-1} \geq 1$$

and established that

$$M_p(\Gamma_D) = M^{1-p}_{\frac{p}{p-1}}(\Sigma_D).$$

It follows from Caraman's paper that

$$M_p(\Gamma_D) = M^{1-p}_p(\Sigma_D),$$

assuming a metric ρ to be admissible if

$$\int\limits_{s} \rho^{p-1} \, d\sigma_{n-1} \geq 1.$$

In the case $\int\limits_{s} \rho^{p-1} \, d\sigma_{n-1} \geq 1$ we have

$$M_p(\Gamma_D) = M^{1-p}_{\frac{p(n-1)}{p-1}}(\Sigma_D). \tag{3}$$

In addition, the following relations hold:

$$1 < p < n \iff n < p(n-1)/(p-1) < \infty,$$

$$p = n \iff p(n-1)/(p-1) = n,$$

$$n < p < \infty \iff 1 < p(n-1)/(p-1) < n.$$

Denote by $m(A) = m_n(A)$ the n-dimensinal Lebesgue measure of a set A. Our main result is the following

Theorem. *Suppose that $f\colon G \to G^*$ is a homeomorphism. Then the following conditions are equivalent:*

1^0. *f is quasiisometric;*

2^0. *For fixed real number $\alpha, \beta, \gamma, \delta$ such that*

$$n-1 < \alpha < \beta < n \quad \text{and} \quad n-1 < \gamma < \delta < n$$

or

$$n < \alpha < \beta < (n-1)^2/(n-2) \quad \text{and} \quad n < \gamma < \delta < (n-1)^2/(n-2),$$

there exists a constant K such that for any ring domain $D \subset G$ the inequalities

$$M_\alpha^\beta(f(\Sigma_D)) \le K^\alpha \big[m(D^*)\big]^{\beta-\alpha} M_\beta^\alpha(\Sigma_D), \tag{4}$$

$$M_\gamma^\delta(\Sigma_D) \le K^\gamma \big[m(D)\big]^{\delta-\gamma} M_\delta^\gamma(f(\Sigma_D)), \tag{5}$$

hold, where $D^* = f(D)$.

Proof. The implication $1^0 \Rightarrow 2^0$ follows from Proposition and Hölder's inequality. Indeed, for p, q, s, t such that $p < q$ and $s < t$ we have from (2) the inequalities

$$M_p^{\frac{q}{p}}(f(\Gamma_D)) \le [m(D^*)]^{\frac{q-p}{p}} M_q(f(\Gamma_D)) \le Q_q\, [m(D^*)]^{\frac{q-p}{p}} M_q(\Gamma_D), \tag{6}$$

and

$$M_s^{\frac{t}{s}}(\Gamma_D) \le [m(D)]^{\frac{t-s}{s}} M_t(f(\Gamma_D)) \le Q_t\, [m(D)]^{\frac{t-s}{t}} M_t(f(\Gamma_D)). \tag{7}$$

Suppose that

$$\alpha = \frac{q(n-1)}{q-1}, \quad \beta = \frac{p(n-1)}{p-1}, \quad \gamma = \frac{t(n-1)}{t-1}, \quad \delta = \frac{s(n-1)}{s-1}.$$

Substituting these values into (6)–(7) and applying (3) we obtain inequalities (4)–(5).

The inverse implication $2^0 \Rightarrow 1^0$ will be proved only for inequality (5). The second inequality in (1) follows in the same way if one applied f^{-1} insteed of f.

Fix a point $x \in D$ and a ball $B^n(x,r)$ of the radius r so that $0 < r < \operatorname{dist}(x, \partial D)$. Let x_1 be a point of the $(n-1)$-dimensional sphere $S^{n-1}(x,r)$. For $p > n-1$, $p \ne n$ we have

$$M_p(\Sigma_D) = C_1 |x_1 - x|^{n-p}. \tag{8}$$

Here C_1 depends only on p and n. According to Lemma 3 ([4]) we obtain the estimate

$$M_p(f(\Sigma_D)) \le C_2|f(x_1) - f(x)|^{n-p}, \tag{9}$$

where C_2 is a positive constant which depends only on n and p.

Substituting (8) and (9) into (5) yields

$$|x_1 - x|^{(n-\gamma)\delta} \le C_3 r^{n(\delta-\gamma)} |f(x_1) - f(x)|^{(n-\delta)\gamma}.$$

Thus for $n - 1 < \gamma < \delta < n$

$$\frac{|x_1 - x|}{|f(x_1) - f(x)|} \le M,$$

and for $n < \gamma < \delta < (n-1)^2/(n-2)$

$$\frac{|f(x_1) - f(x)|}{|x_1 - x|} \le M,$$

where $M = C_3^{\frac{1}{|n-\delta|\gamma}}$ depends only on γ, δ and n. This completes the proof of Theorem. □

Remark. A similar result for the planar case with $\alpha = \gamma = 1$ was given in [5].

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Department of Mathematics and Statistics
Bar-Ilan University, 52900 Ramat-Gan, Israel
golbera@macs.biu.ac.il

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