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A. A. Kondratyuk, I. P. Kshanovskyy

# ON THE LOGARITHMIC DERIVATIVE OF A MEROMORPHIC FUNCTION

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We will prove that the inequality

$$m\left(r, \frac{f'}{f}\right) \le \log^+\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right) + 4.8517,$$

where $\rho > r$, holds for all meromorphic functions such that $f(0) = 1$. This is an improvement of the earlier results by Gol'dberg and Grinshtein, Benbourenane and Korhonen.

А. А. Кондратюк, И. П. Кшановский. *О логарифмической производной мероморфной функции* // Математичні Студії. – 2004. – Т.21, №1. – С.98–100. Доказывается, что неравенство

$$m\left(r, \frac{f'}{f}\right) \le \log^+\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right) + 4.8517,$$

где $\rho > r$, имеет место для всех мероморфных функций таких ,что $f(0) = 1$. Эта оценка является более точной, чем предыдущие результаты Гольдберга и Гринштейна, Бенбуренана и Корхонена.

Gol'dberg and Grinshtein obtained an upper bound for the proximity function of a meromorphic function, $f$

$$m(r, f) \stackrel{def}{=} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi})| d\varphi,$$

in [1]. Their result is stated as follows:

$$m\left(r, \frac{f'}{f}\right) \le \log^+\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right) + 5.8501, \tag{1}$$

where $\rho > r$ and $f$ is a meromorphic function such that $f(0) = 1$. Then, Benbourenane and Korhonen [2] improved the result (1). They obtained the constant 5.3078 instead of 5.8501. In this paper we will deal with some corollary of the Jensen inequality, in order to find a better constant than 5.3078.

Our result is the following one.

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**Theorem.** *Let $f(z)$ be a meromorphic function in $\{z : |z| < R\}$, where $0 < R \le \infty$, such that $f(0) = 1$. Then for any $r$ and $\rho$ with $0 < r < \rho < R$,*

$$m\left(r, \frac{f'}{f}\right) \le \log^+\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right) + 4.8517. \tag{2}$$

The proof of this Theorem is based on the following inequality from [2] and one lemma. Under the conditions of the Theorem for all $\alpha, \beta,\ 0 < \alpha < 1,\ 0 < \beta < 1$,

$$\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi} \le C(\alpha, \beta)\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right)^\alpha, \tag{3}$$

where

$$C(\alpha, \beta) = \left(\frac{2}{1-\beta}\right)^\alpha + \sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{4 + (2^{(1+\alpha)/(1-\alpha)} + 2^{(2+\alpha)/(1-\alpha)})^{1-\alpha}}{\beta^\alpha}\right).$$

**Lemma.** *If $u(t) \ge 0$ on $[0, T]$ and $I = \frac{1}{T}\int\limits_0^T u(t)dt$ exists, then*

$$\frac{1}{T}\int\limits_0^T \log^+ u(t)dt \le \max(1, \log I).$$

*Proof.* Denote $E = \{t : u(t) \ge 1\} \subset [0, T]$.

If meas $E = |E| = 0$ then $\log^+ u(t) = 0$ and Lemma is proved. Let $|E| > 0$. By the definition of $E$ we have $u(E) \subset [1, +\infty)$. The function $\log u$ is concave on $[1, +\infty)$. Applying the Jensen inequality (see, for example, [3, p. 58]) $\frac{1}{|E|}\int_E \log u\, dt \le \log\left(\frac{1}{|E|}\int_E u\, dt\right)$ we have

$$\frac{1}{T}\int\limits_0^T \log^+ u\, dt = \frac{1}{T}\int\limits_E \log u\, dt \le \frac{|E|}{T}\log\frac{TI}{|E|}. \tag{4}$$

Put $x = |E|/T$ and consider the function $x\log\frac{I}{x}$. Its unique maximum point on $(0, +\infty)$ is $I/e$. We consider $0 < x \le 1$. Then

$$x\log\frac{I}{x} \le \begin{cases} I/e & \text{if} \quad I/e \le 1, \\ \log I & \text{if} \quad I/e > 1. \end{cases}$$

So, $x\log\frac{I}{x} \le \max(1, \log I)$, $0 < x \le 1$. From this inequality and relation (4) we obtain the statement of Lemma. □

*Proof of Theorem.* Denote

$$I = \int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi}.$$

Then using (3) we have

$$\log I \le \log C(\alpha, \beta) + \alpha\log\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right).$$

Next by means of Lemma and the inequality $\log t \le \log^+ t$

$$m\left(r, \frac{f'}{f}\right) \le \frac{1}{\alpha} \max\left(1, \log^+ C(\alpha,\beta) + \alpha \log^+ \left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho-r}\right)\right).$$

Since

$$\log^+ C(\alpha,\beta) + \alpha \log^+ \left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho-r}\right) > 1 \qquad \left(C(\alpha,\beta) > \sec\left(\frac{\alpha\pi}{2}\right)\frac{4}{\beta^\alpha} > 4\right),$$

we see that

$$m\left(r, \frac{f'}{f}\right) \le \frac{1}{\alpha} \log C(\alpha,\beta) + \log^+ \left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho-r}\right).$$

This inequality holds for all $\alpha$, $\beta$, $0 < \alpha < 1$ and $0 < \beta < 1$. Applying some mathematical software Mathcad we see that the minimum of $(\log^+ C(\alpha,\beta))/\alpha$ is 4.8517 attained at $\alpha \approx 0.797184$, $\beta \approx 0.841914$. □

**Remark.** Our Theorem allows us to sharpen the constants in the error terms of the second main theorem in [4], and in the lemma on the logarithmic derivative in [5]. By using our result instead of inequality (1), we may replace the constant 6.7 in the second main theorem by 5.71, and the constant 7.55 in the lemma on the logarithmic derivative by 6.55.

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Faculty of Mechanics and Mathematics
Lviv Ivan Franko National University
kshanovskyy@ukr.net

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