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UNIVERSAL MAPS OF K_ω-SPACES

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We introduce counterparts of the spaces $\mathbb{R}^\infty$ and Q^∞ in the classes of spaces that are countable direct limits of compacta of prescribed weights. The counterpart of the space Q^∞ is a countable direct limit of Tychonov cubes, and that of $\mathbb{R}^\infty$ is a countable direct limit of Dranishnikov universal spaces. A universal map, which is a generalization of the universal map between $\mathbb{R}^\infty$ and Q^∞ defined by the second-named author, between these spaces is constructed.

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Вводятся аналоги пространств $\mathbb{R}^\infty$ и Q^∞ в классах пространств, являющихся прямыми пределами компактов предписанного веса. Аналогом пространства Q^∞ является счетный прямой предел тихоновских кубов, аналогом $\mathbb{R}^\infty$ — счетный прямой предел универсальных пространств Дранишникова. Построено универсальное отображение между этими пространствами, являющееся аналогом универсального отображения между $\mathbb{R}^\infty$ и Q^∞, определенного вторым автором.

1. Introduction. Recall that a topological space X is called a *k_ω-space* if X is the direct limit of a sequence (X_i) of compact Hausdorff spaces and embeddings.

The space $\mathbb{R}^\infty$ is the direct limit of the sequence

$$\mathbb{R} \to \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \dots.$$

The space Q^∞ is the direct limit of the sequence

$$Q \to Q \times \{0\} \hookrightarrow Q \times Q \to Q \times Q \times \{0\} \hookrightarrow Q \times Q \times Q \to \dots,$$

where Q denotes the Hilbert cube. It is known that Q^∞ is homeomorphic to the space (ℓ^2, bw), where bw stands for the bounded weak topology (see [1]).

The theory of $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds has been developed by different authors (see, e.g. [3]–[13]). The following characterization theorem for the spaces $\mathbb{R}^\infty$ and Q^∞ is proved by K. Sakai [3].

Theorem 1.1. (Sakai [3]) *A countable direct limit X of a sequence of metrizable (respectively metrizable finite-dimensional) compact spaces is homeomorphic to Q^∞ (respectively $\mathbb{R}^\infty$) if and only if the following holds: for every compact metrizable (respectively finite-dimensional compact metrizable) pair (A, B) and every embedding $f\colon B \to X$ there exists an embedding $\bar{f}\colon A \to X$ that extends f.*

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In [13] a special map $\varphi\colon \mathbb{R}^\infty \to Q^\infty$ is defined and it is proved that this map can be uniquely characterized by its universality properties.

It is easy to find a counterpart of the space Q^∞ that plays the same role for the class of compact Hausdorff spaces of higher weight as Q^∞ does for the class of compact metrizable spaces. Namely, we replace the Hilbert cubes by the Tychonov cubes in the definition of Q^∞ (note that perhaps the first example of such a space is considered in [2]). However, a construction of a counterpart of the space $\mathbb{R}^\infty$ is considerably less straightforward. It is based on universal maps of Dranishnikov universal spaces.

In this note we introduce generalizations of the spaces Q^∞ and $\mathbb{R}^\infty$ and prove characterization theorems for these spaces. We also define and characterize universal maps of these spaces.

The manifolds modeled on universal spaces introduced in this note will be considered in another publication.

The authors are indebted to the referee for useful remarks and suggestions.

2. Preliminaries. All spaces are assumed to be Tychonov, all maps continuous.

By $w(X)$ we denote the weight of a topological space X.

By I we denote the unit segment $[0,1]$. For $\tau > \omega$, the space I^τ is a Tychonov cube of weight τ. Below, $0 \in I^\tau$ also denotes the point with all coordinates equal to zero.

Lemma 2.1. *Let (X,Y) be a compact Hausdorff pair and $w(X) = \tau$. For every embedding $f\colon Y \to I^\alpha$, for some α, there exists an embedding $\bar f\colon Y \to I^\alpha \times I^\tau$ such that $\bar f(x) = (f(x), 0)$ for every $x \in Y$.*

Proof. Since I^α is an absolute extensor, there exists a continuous extension $f'\colon X \to I^\alpha$ of the map f. Denote by X/Y the quotient space of X obtained by identification of all the points of Y and let $q\colon X \to X/Y$ be the quotient map. There exists an embedding $i\colon X/Y \to I^\tau$ such that $i(Y) = 0$. Put $\bar f(x) = (f'(x), i \circ q(x))$. □

By *dimension* we mean the covering dimension dim. Given $n \in \omega$, we denote by AE(n) the class of *absolute extensors in dimension n*, i.e. spaces X with the following property: for any map $f\colon A \to X$ defined on a closed subset A of a compact Hausdorff space Y with $\dim Y \le n$ there is an extension $\bar f\colon Y \to X$ of f.

A space X is a *k_ω-space* if $X = \varinjlim X_i$, for an increasing sequence $X_1 \subset X_2 \subset \dots$ of its compact Hausdorff subspaces.

A map $f\colon X \to Y$ is said to be *(m,n)-soft* if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{\psi} & Y, \end{array}$$

where Z is a paracompact space of dimension $\le n$, A is a closed subset of Z of dimension $\le m$, there exists a map $h\colon Z \to X$ such that $fh = \psi$ and $h|A = \varphi$ (see [17]). The (n,n)-soft maps are called *n-soft.* A map is *soft* if we drop the dimension restrictions in the above definition.

Finally, $\cong$ means ‘homeomorphic to’.

3. Characterization theorems. In this section we introduce and characterize spaces $I^{(\alpha)}$ and $I_\omega^{(\alpha)}$.

3.1 Spaces $I^{(\alpha)}$. Denote by $\mathbf{A}$ the set of all nondecreasing sequences of uncountable cardinal numbers. Given $\alpha = (\alpha_i) \in \mathbf{A}$, define the space $I^{(\alpha)}$ as the direct limit of the sequence

$$I^{\alpha_0} \to I^{\alpha_0} \times \{0\} \hookrightarrow I^{\alpha_0} \times I^{\alpha_1} \to I^{\alpha_0} \times I^{\alpha_1} \times \{0\} \hookrightarrow I^{\alpha_0} \times I^{\alpha_1} \times I^{\alpha_2} \to \dots .$$

Recall that, for any cardinal number τ, by τ^+ we denote the *successor* of τ. Given $\alpha = (\alpha_i) \in \mathbf{A}$, let $\tilde{\alpha} = \sup\{\alpha_i^+ \mid \alpha \in \omega\}$.

Theorem 3.1. *A k_ω-space X is homeomorphic to $I^{(\alpha)}$ if and only if the following two conditions hold:*

1. *$X = \varinjlim X_i$, where $w(X_i) < \tilde{\alpha}$, for every i;*
2. *for every compact Hausdorff pair (A, B) with $w(A) < \tilde{\alpha}$ and every embedding $f\colon B \to X$ there exists an embedding $\bar{f}\colon A \to X$ such that $\bar{f}|B = f$.*

Proof. By the definition, the space $I^{(\alpha)}$ satisfies condition 1). That $I^{(\alpha)}$ also satisfies condition 2) is a consequence of Lemma 2.1.

Suppose now that a k_ω-space X satisfies conditions 1) and 2). Let (X_i) be the direct system from condition 1). We naturally identify every $I^{\alpha_0} \times I^{\tau_0} \times \dots \times I^{\tau_i}$ with the subspace $I^{\alpha_0} \times I^{\tau_0} \times \dots \times I^{\tau_i} \times \{0\}$ of the space $I^{\alpha_0} \times I^{\tau_0} \times \dots \times I^{\tau_i} \times I^{\tau_{i+1}}$. Also, every space $I^{\alpha_0} \times I^{\alpha_0} \times \dots \times I^{\alpha_i}$ is naturally identified with a subspace of the space $I^{(\alpha)}$. For the sake of brevity, we put $Y_i = I^{\alpha_0} \times I^{\alpha_0} \times \dots \times I^{\alpha_i}$.

We follow the proof of Sakai's characterization theorem.

Let $n_1 = 1$. There exists an embedding $f_1\colon X_{n_1} \to Y_{m_1}$, for some m_1. Note that $w(Y_{m_1}) = \alpha_{m_1}$ and there exists an embedding $g_1\colon Y_{m_1} \to X$ that extends the embedding $f_1^{-1}\colon f_1(X_{n_1}) \to X_{n_1} \subset X$. Since X is a k_ω-space, there exists $n_2 > n_1$ such that $g_1(Y_{m_1}) \subset X_{n_2}$. By Lemma 2.1, there exists an embedding $f_2\colon X_{n_2} \to Y_{m_2}$, for some $m_2 > m_1$, such that $f_2 g_1 | Y_{m_1} = 1_{Y_{m_1}}$. Proceeding similarly, we obtain a commutative diagram of spaces and embeddings,

$$\begin{array}{ccccccc}
X_{n_1} & \hookrightarrow & X_{n_2} & \hookrightarrow & X_{n_2} & \hookrightarrow & \dots, \\
{\scriptstyle f_1}\downarrow & \nearrow{\scriptstyle g_1} & {\scriptstyle f_2}\downarrow & \nearrow{\scriptstyle g_2} & {\scriptstyle f_3}\downarrow & \nearrow{\scriptstyle g_3} & \\
Y_{m_1} & \hookrightarrow & Y_{m_2} & \hookrightarrow & Y_{m_3} & \hookrightarrow & \dots
\end{array} \tag{3.1}$$

from which we conclude that

$$X = \varinjlim X_i \cong \varinjlim\{X_{n_1} \xrightarrow{f_1} Y_{m_1} \xrightarrow{g_1} X_{n_2} \xrightarrow{f_2} \dots\} \cong \varinjlim Y_j = I^{(\alpha)}$$

□

Corollary 3.2. *If $\alpha_1, \alpha_2 \in \mathbf{A}$ are such that $\tilde{\alpha}_1 = \tilde{\alpha}_2$, then the spaces $I^{(\alpha_1)}$ and $I^{(\alpha_2)}$ are homeomorphic.*

A space X is said to be an *absolute extensor* for (finite-dimensional) compact Hausdorff spaces, written $X \in \mathrm{AE}$ ($X \in \mathrm{AE(fd)}$) if for any (finite-dimensional) compact Hausdorff pair (A, B) and any map $f\colon B \to X$ there is an extension of f over A. It easily follows from Characterization Theorem 3.1 that $I^{(\alpha)} \in \mathrm{AE}$.

Proposition 3.3. *Let $X \in \mathrm{AE}$ and $X = \varinjlim X_i$, where $X_1 \subset X_2 \subset \dots$ be a sequence of compact Hausdorff spaces embeddable in $I^{(\alpha)}$. Then $X \times I^{(\alpha)} \cong I^{(\alpha)}$.*

Proof. We have

$$X \times I^{(\alpha)} = (\varinjlim X_i) \times (\varinjlim I^{\alpha_0} \times \dots \times I^{\alpha_i}) = \varinjlim X_i \times I^{\alpha_0} \times \dots \times I^{\alpha_i}.$$

Let (A, B) be a compact Hausdorff pair and $w(A) \le \alpha_k$, for some k. Given an embedding $f\colon B \to X \times I^{(\alpha)}$, one can find i such that $f(B) \subset X_i \times I^{\alpha_0} \times \dots \times I^{\alpha_i}$. Since $X \times I^{(\alpha)} \in \mathrm{AE}$, there is an extension $f'\colon A \to X \times I^{(\alpha)}$ of f. There exists $j \ge i$ such that $f(A) \subset X_j \times I^{\alpha_0} \times \dots \times I^{\alpha_j}$. Let $l = \max\{j+1, k\}$. Denote by $q\colon A \to A/B$ the quotient map and let $g\colon A/B \to I^{\alpha_{j+1}} \times \dots \times I^{\alpha_m}$ be an embedding such that $g(B) = 0$. Then the map $\bar{f}\colon X_m \times I^{\alpha_0} \times \dots \times I^{\alpha_m}$ defined by the formula $\bar{f}(a) = (f'(a), gq(a))$, $a \in A$, is an embedding that extends f. □

Recall that a space is said to be *locally self-similar* if there exists a base of its topology consisting of sets homeomorphic to the whole space.

Proposition 3.4. *The space $I^{(\alpha)}$ is a locally self-similar, topologically homogeneous space.*

Proof. First show that $I^{(\alpha)}$ is topologically homogeneous. Let $x, y \in I^{(\alpha)}$. Proceeding as in the proof of the characterization theorem for $I^{(\alpha)}$, we represent $I^{(\alpha)}$ as the direct limit of a sequence (X_i) of compact Hausdorff spaces, where $X_1 = \{x\}$. In diagram (3.1) one can assume that $f_1(x) = y$. Then $f = \varinjlim f_i$ is a homeomorphism with the property that $f(x) = y$.

We are going to show that the space $I^{(\alpha)}$ is locally self-similar. A subset $Y \subset I^\tau = \prod_{i<\tau}[0,1]_i$ is said to be *cubic* if Y is of the form $\prod_{i<\tau}[a_i, b_i]$, for some segments $[a_i, b_i] \subset [0,1]_i$, $i < \tau$. In the sequel, we will use the following simple observation. For any closed cubic subset Y of I^τ and any neighborhood W of Y, there exists a closed cubic neighborhood Y_1 of Y in I^τ such that $Y_1 \subset W$.

Now, let $x \in I^{(\alpha)}$ and let U be a neighborhood of x. Then $x \in I^{\alpha_{i_1}}$, for some $i_1 \in \mathbb{N}$. There exists a closed cubic neighborhood V_1 of x in $I^{\alpha_{i_0}}$ such that $V_1 \subset U$. Using the above remark, find a sequence (V_j) of subsets with the following properties:

1. V_j is a cubic subset of $I^{\alpha_{i_0+j}}$;
2. V_{j+1} is a closed neighborhood of V_j in $I^{\alpha_{i_0+j}}$;
3. $V_{j+1} \subset U$.

Let $V = \varinjlim V_j$. It follows from conditions 2) and 3) that V is a neighborhood of x and $V \subset U$. We leave to the reader an easy verification that the space V satisfies the conditions of the characterization theorem for $I^{(\alpha)}$.

□

Proposition 3.5. *The spaces $I^{(\alpha)}$ and $I^{(\alpha)} \setminus \{*\}$ are not homeomorphic.*

Proof. Suppose the contrary and let $h\colon I^{(\alpha)} \to I^{(\alpha)} \setminus \{*\}$ be a homeomorphism. There exists $i \in \mathbb{N}$ such that $* \in Y_i = I^{\alpha_0} \times I^{\alpha_0} \times \cdots \times I^{\alpha_i}$ (see the proof of Theorem 3.1). Then $Y_i \setminus \{*\} = \cup_{j=1}^{\infty}(Y_i \cap h^{-1}(Y_j))$, i.e. $Y_i \setminus \{*\}$ is a σ-compact space and $*$ has a countable character in Y_i, a contradiction. □

One can similarly prove that, for any nonempty compact subset $K \subset I^{(\alpha)}$ the spaces $I^{(\alpha)}$ and $I^{(\alpha)} \setminus K$ are not homeomorphic.

For $n \in \mathbb{N}$, by $\exp_n X$ we denote the *n-th hypersymmetric power* of X (according to the terminology of [14]). The elements of $\exp_n X$ are the nonempty subsets of X of cardinality $\leq n$. The topology of $\exp_n X$ is the quotient topology of X^n under the natural map $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$. Given a k_ω-space $X = \varinjlim X_i$, for an increasing sequence $X_1 \subset X_2 \subset \dots$ of its compact subspaces, one can easily verify that $\exp_n X = \varinjlim \exp_n X_i$ (see, e.g., [16]).

It is proved in [15] that $\exp_n Q^\infty \cong Q^\infty$. The following proposition demonstrates that this result does not have a counterpart for the spaces $I^{(\alpha)}$.

The proof of the following proposition is suggested by the referee.

Proposition 3.6. *Let $\alpha \in \mathbf{A}$ and $n \geq 2$.*

The spaces $I^{(\alpha)}$ and $\exp_n I^{(\alpha)}$ are not homeomorphic.

Proof. Note that, for some uncountable τ, the Tychonov cube I^τ is a retract of the space $I^{(\alpha)}$. Then also the space $\exp_n I^\tau$ is a retract of the space $\exp_n I^{(\alpha)}$. Assuming that $I^{(\alpha)}$ and $\exp_n I^{(\alpha)}$ are homeomorphic, we see that then $\exp_n I^{(\alpha)} \in \mathrm{AE}$. Therefore, $\exp_n I^\tau \in \mathrm{AE}$, which contradicts to the results of [14]. □

Note that results similar to Proposition 3.6 can be also proved for some other functors of finite degree in the category of Tychonov spaces (see [16]).

3.2. Space $I_\omega^{(\alpha)}$. A. Dranishnikov [17] proved that for every $n = 0, 1, 2, \dots$ and every cardinal τ there exists a map $f_n^\tau\colon D_n^\tau \to I^\tau$ with the following properties:

(i) D_n^τ is a compact Hausdorff space of weight τ and dimension n;

(ii) $D_n^\tau \in \mathrm{AE}(n-1)$;

(iii) f_n^τ is $(n-1)$-soft and $(n-2, n)$-soft.

Given $\alpha = (\alpha_i) \in \mathbf{A}$, construct a space $I_\omega^{(\alpha)}$ as follows. We put $Y_i = D_{2i+1}^{\alpha_i}$ and define embeddings $s_i\colon Y_i \to Y_{i+1}$ as follows. We regard Y_i as a subset of I^{α_i}, then the graph of the map $f_{2i+1}^{\alpha_i}\colon Y_i = D_{2i+1}^{\alpha_i} \to I^{\alpha_i}$ is a subset of $I^{\alpha_i} \times I^{\alpha_i}$. In its turn, the set $I^{\alpha_i} \times I^{\alpha_i}$ is identified with the subset $I^{\alpha_i} \times I^{\alpha_i} \times \{0\}$ of $I^{\alpha_i} \times I^{\alpha_i} \times I^{\alpha_{i+1}}$. Let

$$f_{2i+3}^{\alpha_{i+1}}\colon Y_{i+1} = D_{2i+3}^{\alpha_{i+1}} \to I^{\alpha_i} \times I^{\alpha_i} \times I^{\alpha_{i+1}} \cong I^{\alpha_{i+1}}$$

be a Dranishnikov map. Since $f_{2i+3}^{\alpha_{i+1}}$ is $(2i+31, 2i+1)$-soft, there exists a map $s_i\colon Y_i \to Y_{i+1}$ such that $f_{2i+3}^{\alpha_{i+1}} s_i(y) = ((y, f_{2i+1}^{\alpha_i}(y)), 0)$, $y \in Y_i$.

Let $I_\omega^{(\alpha)} = \varinjlim(Y_i, s_i)$. In the sequel, we identify every Y_i with the corresponding subspace in $I_\omega^{(\alpha)}$.

Remark 3.7. Since the diagram

$$\begin{array}{ccccc} D_1^{\alpha_1} & \xrightarrow{s_1} & D_2^{\alpha_2} & \xrightarrow{s_2} & \cdots \\ \downarrow{\scriptstyle f_1^{\alpha_1}} & & \downarrow{\scriptstyle f_2^{\alpha_2}} & & \\ I^{\alpha_1} & \hookrightarrow & I^{\alpha_2} & \hookrightarrow & \cdots \end{array}$$

is commutative, it determines the map of the direct limits, $f^{(\alpha)}\colon I_\omega^{(\alpha)} \to I^{(\alpha)}$.

Theorem 3.8. *A k_ω-space X is homeomorphic to $I_\omega^{(\alpha)}$ if and only if the following two conditions hold:*

1. *$X = \varinjlim X_i$, where X_i is a finite-dimensional compact Hausdorff space with $w(X_i) < \tilde{\alpha}$, for every i;*
2. *for every finite-dimensional compact Hausdorff pair (A, B) with $w(A) < \tilde{\alpha}$ and every embedding $f\colon B \to X$ there exists an embedding $\bar{f}\colon A \to X$ such that $\bar{f}|B = f$.*

Proof. First, show that the space $I_\omega^{(\alpha)}$ satisfies properties 1) and 2) (with X replaced by $I_\omega^{(\alpha)}$). Suppose that (A, B) is a finite-dimensional compact Hausdorff pair with $w(A) < \tilde{\alpha}$ and $f\colon B \to I_\omega^{(\alpha)}$ is an embedding. Since $I_\omega^{(\alpha)}$ is a k_ω-space, there exists i such that $f(B) \subset Y_i \subset I_\omega^{(\alpha)}$. There exists $j > i$ such that $w(A) \le \alpha_{j+1}$ and $j > \dim A + 1$. Then there exists a map $f'\colon A \to I^{\alpha_j} \times I^{\alpha_j}$ extending $f\colon B \to Y_i \subset Y_j \subset I^{\alpha_j} \times I^{\alpha_j}$ and, by Lemma 2.1, there exists an embedding $f''\colon A \to I^{\alpha_j} \times I^{\alpha_j} \times I^{\alpha_{j+1}}$ that extends

$$f\colon B \to Y_i \subset Y_j \subset I^{\alpha_j} \times I^{\alpha_j} = I^{\alpha_j} \times I^{\alpha_j} \times \{0\} \subset I^{\alpha_j} \times I^{\alpha_j} \times I^{\alpha_{j+1}}.$$

Since the map $f_{j+1}^{\alpha_{j+1}}$ is $(j-1, j+1)$-soft and $\dim A \le j-1$, there exists a map $\bar{f}\colon A \to D_{j+1}^{\alpha_{j+1}}$ making the diagram

$$\begin{array}{ccccc} B & \xrightarrow{f} & Y_i & \hookrightarrow & D_{j+1}^{\alpha_{j+1}} \\ \downarrow & & & \nearrow{\scriptstyle \bar{f}} & \downarrow{\scriptstyle f_{j+1}^{\alpha_{j+1}}} \\ A & & \xrightarrow{f''} & & I^{\alpha_j} \times I^{\alpha_j} \times I^{\alpha_{j+1}} \end{array}$$

commutative. Obviously, $\bar{f}$ is an embedding.

If X satisfies properties 1) and 2), then, applying back and forth arguments similar as in the proof of Theorem 3.1 one can show that $X \cong I_\omega^{(\alpha)}$. We leave the details to the reader. □

Corollary 3.9. *If $\alpha_1, \alpha_2 \in \mathbf{A}$ are such that $\tilde{\alpha}_1 = \tilde{\alpha}_2$, then the spaces $I_\omega^{(\alpha_1)}$ and $I_\omega^{(\alpha_2)}$ are homeomorphic.*

Proposition 3.10. *Let $X \in \mathrm{AE(fd)}$ and $X = \varinjlim X_i$, where $X_1 \subset X_2 \subset \dots$ be a sequence of compact Hausdorff spaces embeddable in $I_\omega^{(\alpha)}$. Then $X \times I_\omega^{(\alpha)} \cong I_\omega^{(\alpha)}$.*

Proof. An easy modification of the proof of Proposition 3.3. □

Note that the space $\mathbb{R}^\infty$ is an AE. The following proposition demonstrates that this is no longer true for its counterparts $I_\omega^{(\alpha)}$. Recall that a space is said to be *strongly countable-dimensional* if it can be represented as a countable union of its closed finite-dimensional subspaces.

Proposition 3.11. *The space $I_\omega^{(\alpha)}$ is not an AE.*

Proof. Let $f\colon D^{\omega_1} \to I_\omega^{(\alpha)}$ be an embedding (here $D = \{0,1\} \subset [0,1]$). Assuming, on the contrary, that $I_\omega^{(\alpha)} \in \mathrm{AE}$ we conclude that there exists an extension $\bar f\colon I^{\omega_1} \to I_\omega^{(\alpha)}$ of f. By a result of Shchepin [18], there exists an embedding $g\colon I^{\omega_1} \to I^{\omega_1}$ such that $\bar f g$ is also an embedding. Then $\bar f g(I^{\omega_1})$ turns out to be a countable union of its closed finite-dimensional subspaces, $\bar f g(I^{\omega_1}) = \cup_{i=1}^\infty \bar f g(I^{\omega_1}) \cap D_{\alpha_i}^{n_i}$. This contradicts to the fact that the space I^{ω_1} is not strongly countable-dimensional. □

A counterpart of Proposition 3.5 holds for the space $I_\omega^{(\alpha)}$: for any nonempty compact subset $K \subset I_\omega^{(\alpha)}$, the spaces $I_\omega^{(\alpha)}$ and $I_\omega^{(\alpha)} \setminus K$ are not homeomorphic.

4. Universal maps. Given a sequence $\alpha = (\alpha_i) \in \mathbf{A}$, denote by $\mathcal{K}(\alpha)$ (respectively $\mathcal{K}_\omega(\alpha)$) the class of all (respectively finite-dimensional) compact Hausdorff spaces X with $w(X) < \tilde\alpha$. Given classes $\mathcal{C}$, $\mathcal{D}$ of compact Hausdorff spaces, denote by $(\mathcal{C}, \mathcal{D})$ the class of maps $f\colon X \to Y$, where $X \in \mathcal{C}$, $Y \in \mathcal{D}$.

The category of maps is defined as follows. Its objects are the maps of compact Hausdorff spaces. Given two maps, $f\colon X \to Y$ and $f'\colon X' \to Y'$, a morphism $\bar\imath\colon f \to g$ consists of pair of morphisms $i_1\colon X \to X'$ and $i_2\colon Y \to Y'$ such that $f' i_1 = i_2 f$. If i_1, i_2 are inclusion maps, we write $f \subset g$. If i_1, i_2 are homeomorphisms, we say that f and g are *homeomorphic* and write $f \cong g$.

Given classes $\mathcal{C}$, $\mathcal{D}$ of compact Hausdorff spaces, we say that a map $f\colon X \to Y$ is *strongly* $(\mathcal{C}, \mathcal{D})$*-universal* if for every pair of maps (g, h) with $g, h \in (\mathcal{C}, \mathcal{D})$, $h \subset g$, and every embedding $\bar\imath\colon h \to f$ there exists an embedding $\bar\jmath\colon g \to f$ that extends $\bar\imath$.

Theorem 4.1. *There exists a unique, up to homeomorphism, strongly $(\mathcal{K}_\omega^{(\alpha)}, \mathcal{K}^{(\alpha)})$-universal map $\varphi\colon I_\omega^{(\alpha)} \to I^{(\alpha)}$.*

Proof. Let $\mathrm{pr}_i\colon I^{(\alpha)} \times I^{(\alpha)} \to I^{(\alpha)}$ denote the projection onto the i-th factor. Denote by φ the composition

$$I_\omega^{(\alpha)} \xrightarrow{f^{(\alpha)}} I^{(\alpha)} \xrightarrow{h} I^{(\alpha)} \times I^{(\alpha)} \xrightarrow{\mathrm{pr}_1} I^{(\alpha)},$$

where h is a homeomorphism.

We are going to show that the map φ is strongly $(\mathcal{K}_\omega^{(\alpha)}, \mathcal{K}^{(\alpha)})$-universal. Consider a commutative diagram

$$\begin{array}{ccccc} X & \hookleftarrow & A & \xrightarrow{i_1} & I_\omega^{(\alpha)} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \varphi} \\ Y & \hookleftarrow & B & \xrightarrow{i_2} & I^{(\alpha)}, \end{array}$$

where $f, g \in (\mathcal{K}_\omega^{(\alpha)}, \mathcal{K}^{(\alpha)})$. The map $\mathrm{pr}_2 h f^{(\alpha)} i_1\colon A \to I^{(\alpha)}$ can be extended to a map $r'\colon X \to I^{(\alpha)}$ such that $r'(X \setminus A) \cap r'(A) = \varnothing$ and $r'|(X \setminus A)$ is an embedding. This can be easily deduced from Theorem 3.1. Also there is an embedding $j_2\colon Y \to I^{(\alpha)}$ such that $j_2|B = i_2$. The map $r = (j_2, r')\colon X \to I^{(\alpha)} \times I^{(\alpha)}$ is an embedding that extends i_2.

Since X is compact, there exists $l \in \mathbb{N}$ such that

1. $\varphi(D_l^{\alpha_l}) \supset h^{-1}(r(X))$;
2. $i_1(A) \subset D_l^{\alpha_l}$;
3. the map $\varphi|D_l^{\alpha_l}\colon D_l^{\alpha_l} \to \varphi(D_l^{\alpha_l})$ is $\dim X$-soft.

By $\dim X$-softness of $f^{(\alpha)}$, there exists a map $j_1\colon X \to I_\omega^{(\alpha)}$ such that $f^{(\alpha)} j_1 = h^{-1} r$ and $j_1|A = i_1$. Obviously j_1 is an embedding for wthich $\varphi j_1 = j_2 g$.

The uniqueness up to homeomorphism of the strongly $(\mathcal{K}_\omega^{(\alpha)}, \mathcal{K}^{(\alpha)})$-universal map can be verified by the back and forth argument (see the proof of Theorem 3.1).

□

We say that an open surjective map $f\colon X \to Y$ is *locally self-similar* if for every $x \in X$ and every neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and $f|V\colon V \to f(V)$ is homeomorphic to f (see [8]).

Recently, in [19] it was proved that the universal map $\varphi_\omega\colon \mathbb{R}^\infty \to Q^\infty$ is not locally self-similar. This proof works also in our case. Therefore, we obtain the following

Theorem 4.2. *The map $\varphi\colon I_\omega^{(\alpha)} \to I^{(\alpha)}$ is not locally self-similar.*

5. Open problems.

Problem 5.1. Find a topological characterization of the space $I^\tau \times Q^\infty$ for $\tau > \omega$.

Note that the space $I^\tau \times Q^\infty$ is homeomorphic to the direct limit of the sequence

$$I^\tau \to I^\tau \times \{0\} \hookrightarrow I^\tau \times I^\omega \to I^\tau \times I^\omega \times \{0\} \hookrightarrow I^\tau \times I^\omega \times I^\omega \to \dots.$$

More generally, one can also consider the problem of topological characterization of the direct limits of the sequences

$$I^{\alpha_0} \to I^{\alpha_0} \times \{0\} \hookrightarrow I^{\alpha_0} \times I^{\tau_0} \to I^{\alpha_0} \times I^{\tau_0} \times \{0\} \hookrightarrow I^{\alpha_0} \times I^{\tau_0} \times I^{\tau_1} \to \dots,$$

where $\alpha_0 > \omega$ and $\tau_0, \tau_1, \dots$ are arbitrary cardinals.

In [9], it is proved, in particular, that the free topological group in the sense of Graev of the Hilbert cube is homeomotphic to Q^∞. The following question was formulated by the second author during Tiraspol Topological symposium in 1987.

Question 5.2. Is the free topological group in the sense of Graev of the Tychonov cube I^τ homeomorphic to the space $I^{(\alpha)}$, where $\alpha = (\tau, \tau, \dots)$?

In [20], the universal map $\varphi\colon \mathbb{R}^\infty \to Q^\infty$ is realized as a homomorphism of topological group.

Question 5.3. Is it possible to find a counterpart of this result for the universal map of $I_\omega^{(\alpha)}$ to $I^{(\alpha)}$?

Question 5.4. Is the space $I_\omega^{(\alpha)}$ locally self-similar?

In [11] the authors characterized topologically the bitopological space (Q^∞, Σ), where Σ stands for the radial pseudointerior of the Hilbert cube $Q = \prod_{i=1}^\infty [0,1]_i$,

$$\Sigma = \bigcup_{j=3}^\infty \prod_{i=1}^\infty \left[\frac{1}{j}, 1 - \frac{1}{j}\right]_i.$$

Let $\alpha = (\tau, \tau, \dots)$, where $\tau > \omega$, and

$$\Sigma^{(\alpha)} = \bigcup_{j=3}^\infty \prod_{i \in \tau} \left[\frac{1}{j}, 1 - \frac{1}{j}\right]_i.$$

Problem 5.5. Find a topological characterization of the bitopological space

$$\left(\varinjlim{}_j \prod_{i \in \tau} \left[\frac{1}{j}, 1 - \frac{1}{j}\right]_i, \Sigma^{(\alpha)}\right).$$

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