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APPLICATION OF GENERALIZED RADEMACHER FUNCTIONS TO INVESTIGATION OF ALGEBRAS OF SYMMETRIC ANALYTIC FUNCTIONS ON $L_P[0,1]$

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Actions of symmetric polynomials on generalized Rademacher functions are investigated. Some applications to algebras of symmetric analytic functions on the unit ball of $L_p[0,1]$, $1 \le p < \infty$, are described.

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Изучается действие симметрических полиномов на обобщенные функции Радемахера. Описаны некоторые применения к алгебрам симметрических аналитических функций на единичном шаре в $L_p[0,1]$, $1 \le p < \infty$.

1. Introduction and preliminaries. The concept of symmetric polynomials on the Hilbert space and, more general, on l_p and $L_p[0,1]$, $1 < p < \infty$, was introduced by Nemirovski and Semenov [10]. A polynomial P on l_p is said to be symmetric (with respect to the group of permutations on the symmetric basis $\{e_n\}$) if

$$P\left(\sum_{i=1}^{\infty} a_i e_i\right) = P\left(\sum_{i=1}^{\infty} a_i e_{\sigma(i)}\right)$$

for every permutation σ on the set of natural numbers $\mathbb{N}$.

The polynomial P on the $L_p[0,1]$ is called symmetric, if $P(\sigma x) = P(x)$ for any $\sigma \in \Sigma$, where Σ is the group of measurable automorphisms of $[0,1]$ interval.

Properties of symmetric polynomials and analytic functions were investigated in [8], [1]. In particular, in [8] it is given the precise representation of symmetric polynomials on Banach spaces with symmetric bases and on so-called separable rearrangement-invariant function spaces on $[0,1]$ and $[0,\infty]$ by elementary symmetric polynomials. In [1] the spectra of algebras of symmetric holomorphic functions on l_p are investigated. The sets of maximal ideals of algebras of symmetric analytic functions on the unit ball of $L_1[0,1]$ and $L_2[0,1]$ are

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described in [11]. Maximal ideals of algebras of analytic functions were studied in [3], [4], [5], [12].

Let us denote by $A_s(B_{L_p})$ the algebra of symmetric analytic functions (under measurable automorphisms of $[0,1]$) on the unit ball B_{L_p} of $L_p[0,1]$ which are uniformly continuous on $\overline{B_{L_p}}$. We shall denote by $H_s(L_p)$ the space of entire symmetric functions on L_p. Notice that in the case of $p<\infty$, functions from $H_s(L_p)$ are bounded on bounded sets.

Let us denote by $F_k(x)=\int_0^1 x^k(t)dt$ the elementary symmetric polynomials on $L_p[0,1]$, $k=1,\dots,p$. According to [8] every symmetric polynomial on L_p belongs to the algebraic span of polynomials F_k, $k\le p$.

We shall use the notion of generalized Rademacher functions which was introduced in [6]. There the authors used them to prove that every continuous multilinear form $A: c_0\times\ldots\times c_0\to\mathbb{C}$ has a trace. In [2]it was showed that these functions are quite useful in obtaining simple proofs of various estimates in several different areas of analysis. For example, a short proof of polarization formula and its generalization was obtained. In this paper we shall use the generalized Rademacher functions to investigate the set of maximal ideals of the algebra $A_s(B_{L_n})$.

For every natural number $n\ge 2$ the generalized Rademacher functions (S_j^n) are defined inductively as follows. Let $\alpha_1=1,\alpha_2,\dots,\alpha_n$ be the complex n-th roots of unity. For $j=1,\dots,n$ let $I_j=(\frac{j-1}{n},\frac{j}{n})$ and let $I_{j_1j_2}$ denote the j_2-th open subinterval of length $\frac{1}{n^2}$ of I_{j_1} $j_1,j_2=1,\dots,n$. Proceeding like this, it is clear how to define the interval $I_{j_1j_2\ldots j_k}$ for any k. Now $S_1^n:[0,1]\to\mathbb{C}$ is defined by setting $S_1^n(t)=\alpha_j$ for $t\in I_j$, where $1\le j\le n$. In general, $S_k^n(t)$ is defined to be α_j if t belongs to the subinterval $I_{j_1j_2\ldots j_k}$, where $j_k=j$. There is no harm in setting $S_k^n(t)=1$ for all endpoints t.

For example, for $n=2$ we have $j=1,2$ and the corresponding complex roots of unity $\alpha_1=1$, $\alpha_2=-1$. The generalized Rademacher function S_1^2 has a representation:

$$S_1^2(t)=\begin{cases} 1, & t\in I_1=(0,1/2);\\ -1, & t\in I_2=(1/2,1).\end{cases}$$

The generalized Rademacher function S_2^2 is determined by:

$$S_2^2(t)=\begin{cases} 1, & t\in I_{11}=(0,1/4);\\ -1, & t\in I_{12}=(1/4,1/2);\\ 1, & t\in I_{21}=(1/2,3/4);\\ -1, & t\in I_{22}=(3/4,1).\end{cases}$$

Proceeding similarly, we have:

$$S_p^2(t)=\begin{cases} 1, & t\in I_{j_1j_2\ldots j_{p-1}1};\\ -1, & t\in I_{j_1j_2\ldots j_{p-1}2},\end{cases}$$

where $j_1,\dots,j_{p-1}=1,2$.

Let us consider the case $n=k$. Set $j=1,\dots,k$ and $\alpha_1,\dots,\alpha_n$ are the complex roots of unity. The generalized Rademacher functions have a representation:

$$S_1^k = \begin{cases} \alpha_1, & t \in I_1; \\ \alpha_2, & t \in I_2; \\ \vdots & \\ \alpha_k, & t \in I_k, \end{cases} \qquad \dots \qquad S_p^k = \begin{cases} \alpha_1, & t \in I_{j_1 j_2 \dots j_{p-1} 1}; \\ \alpha_2, & t \in I_{j_1 j_2 \dots j_{p-1} 2}; \\ \vdots & \\ \alpha_k, & t \in I_{j_1 j_2 \dots j_{p-1} k}, \end{cases}$$

where $j_1, j_2, \dots, j_p = 1, \dots, k$.

Notice that in the case $n = 2$ the functions S_k^n coincide with the classical Rademacher functions (see e.g. [7, p. 10]).

2. The action of symmetric analytic functions on the generalized Rademacher functions.

Theorem 1. *Let n, j, k be defined as above and S_k^n be the generalized Rademacher functions. Then for each fixed n and arbitrary l, m, such that $l \le m$ there exists an automorphism $\sigma_{lm}^n \colon [0,1] \to [0,1]$ such that $S_l^n(t) = S_m^n(\sigma_{lm}^n(t))$.*

Proof. It is easily seen that there exists an automorphism $\sigma_{lm}^n : S_l^n \to S_m^n$ such that σ_{lm}^n maps the intervals $I_{j_1 j_2 \dots j_{l-1} j_l j_{l+1} \dots j_{m-1} j_m}$ onto the intervals $I_{j_1 j_2 \dots j_{l-1} j_m j_{l+1} \dots j_{m-1} j_l}$ and is identical on the intervals

$$I_{j_1 j_2 \dots j_{l-1} 1 j_{l+1} \dots j_{m-1} 1}, \dots, I_{j_1 j_2 \dots j_{l-1} n j_{l+1} \dots j_{m-1} n}.$$

□

Let us denote by h_n the space $\overline{\operatorname{span}}_{k \in \mathbb{N}}(S_k^n)$.

Corollary 1. *The restriction of arbitrary $f \in A_s (f \in H_s)$ onto h_n for fixed n is a symmetric analytic function with respect to the group of automorphisms on the set of lower indices.*

Lemma 1. *The sequence of functions $\{S_k^n\}$ is weakly convergent to 0 as $k \to \infty$ for every fixed n.*

Proof. The weak convergence $S_k^n \xrightarrow{w} 0$ as $k \to \infty$ means that for every $\phi \in L_n'$, $\phi(S_k^n) \to 0$ as $k \to \infty$, where each functional $\phi \in L_n'$ is determined by a function $g(t) \in L_{n^*}$, such that $1/n^* + 1/n = 1$ and

$$\phi(S_k^n) = \int_0^1 g(t) S_k^n(t) dt.$$

If $g(t) = 1$, then $\int_0^1 S_k^n(t)dt = 0$ by the property of Rademacher functions and the n-th roots of unity.

Let $[a, b]$ be an arbitrary subinterval of $[0, 1]$ and $g(t)$ be the characteristic functions of the given subinterval. Let J be a set of subintervals $I_{j_1 j_2 \dots j_k} \subset [0, 1]$ that are defined below. Let us denote by I_{ab} the closure of the set of subintervals from J which intersect I_{ab}. Then it is evident that $[a, b] \subset I_{ab}$ and $\mu(I_{ab} \setminus [a, b]) \le 2/n^k$. Therefore,

$$\left| \int_a^b g(t) S_{k+1}^n(t) dt \right| \le \left| \int_{I_{ab}} S_{k+1}^n(t) dt \right| + \frac{2}{n^k} = \frac{2}{n^k}.$$

Since this is true for all characteristic functions of intervals, the lemma is true for step functions. From the density of the step functions in L_n' it follows that $\phi(S_k^n) \to 0$ $\forall \phi \in L_n' \Longrightarrow S_k^n \xrightarrow{w} 0$ as $k \to \infty$. □

Let X be a Banach space and $1 \leq q < \infty$. We say that a sequence $\{x_n\}$ in X admits a lower q-estimate if there is a constant $c > 0$ that for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$)

$$c\left(\sum_{i=1}^{n} |a_i|^q\right)^{1/q} \leq \left\|\sum_{i=1}^{n} a_i x_i\right\|.$$

A Banach space X is said to have property T_q (for some $1 \leq q < \infty$) if every weakly null seminormalised basic sequence in X has a subsequence with a lower q-estimate.

Notice that a Banach space X has a lower q-estimate if every null seminormalised basic sequence in X has a lower q-estimate.

Proposition 1. *The space h_n has a lower n-estimate and $h_n \subset l_n$.*

Proof. According to [9], $L_n[0,1]$ has property T_q for some q and the upper index of $L_n[0,1]$

$$u(L_n[0,1]) := \inf\{q \geq 1 : L_n[0,1] \text{ has } T_q\text{-property}\} = \max\{2, n\} = n.$$

Then applying Lemma 1 we see that the sequence $\{S_k^n\}$ has a lower n-estimate, that is there exists a constant $c > 0$ such that for every $N \in \mathbb{N}$ and $a_1, \ldots, a_N \in \mathbb{K}$,

$$c\left(\sum_{k=1}^{N} |a_k|^n\right)^{1/n} \leq \left\|\sum_{k=1}^{N} a_k S_k^n\right\|.$$

According to [8], a Banach space $X \subset l_n$ if and only if the basis has a lower n-estimate. □

Corollary 2. *Let P be a symmetric polynomial on h_n, $\deg(P) = r$.*

1. *If $r < n$ then $P \equiv 0$.*
2. *If $r \geq n$ then there exists a polynomial q of $r - n$ variables such that*

$$P\left(\sum_{k=1}^{\infty} a_k S_k^n\right) = q\left(\sum_{k=1}^{\infty} a_k^n, \ldots, \sum_{k=1}^{\infty} a_k^r\right).$$

Proof. This immediately follows from Proposition 1 and [8, Theorem 1.1]. □

Let $\Phi_n(f)$ be the restriction of f on h_n.

Corollary 3. *For every $k < n$, $\Phi_n(F_k) \equiv 0$.*

Proof. According to Corollary 2, $\Phi_n(F_k)$ is a symmetric polynomial of degree n on h_n. But there is no nonzero symmetric polynomial of degree $k < n$ in a space with a lower n-estimate [8], thus $\Phi_n(F_k) = 0$. □

Proposition 2. *Φ_n is not a surjection.*

Proof. If Φ_n is a surjection, then the preimage of the polynomial $P_{n+1}(\sum a_i e_i) = \sum a_i^{n+1}$ is nontrivial. Since Φ_n is a linear mapping, the preimage must be a symmetric homogeneous polynomial of degree $n+1$ on $L_p[0,1]$. Then according to [8], every homogeneous symmetric polynomial of degree $n+1$ on $L_p[0,1]$ has a representation:

$$Q = \sum_{k_1+\cdots+k_n=n+1} a_{k_1\ldots k_n} F_1^{k_1} \ldots F_n^{k_n}.$$

Since Φ_n is a homomorphism, $\Phi_n(Q) = \sum a_{k_1\ldots k_n} \Phi_n(F_1^{k_1}) \ldots \Phi_n(F_n^{k_n}) = 0$ by Corollary 3. □

Corollary 4. *There exists a symmetric polynomial of degree* $n+1$ *on* $h_n \subset l_n \subset L_n[0,1]$ *which cannot be extended to a symmetric polynomial on* $L_n[0,1]$*. In particular, the symmetric polynomials* $P_m(\sum a_i e_i) = \sum a_i^m$, $m > n$, *cannot be extended to symmetric polynomials on* $L_n[0,1]$.

Lemma 2. *For all* n, r, m *there exist* $l \in \mathbb{N}$ *and an automorphisms* σ *such that* $(S_r^{nm}(t))^m = S_l^n(\sigma(t))$.

Proof. Let us take an arbitrary α_p, $1 \le p \le nm$. Since this is the complex p-th root of unity, it can be written:

$$\alpha_p = e^{2p\pi i/nm}.$$

Evidently, $(\alpha_p)^m = \left(e^{2p\pi i/nm}\right)^m = e^{2p\pi i/n}$ and therefore,

$$(S_r^{nm}(t))^m = S_l^n(\sigma(t)).$$

□

We shall use the notation $S_n = S_1^n$.

Lemma 3. *For every* $g(t) \in \operatorname{span}(S_1, \ldots, S_n)$, $n < k$, *and* $\lambda \in \mathbb{C}$ *holds* $F_n(g(t) + \lambda S_k(t)) = F_n(g(t))$.

Proof. By routine calculations we have

$$F_n(g(t) + \lambda S_k(t)) = F_n\left(\sum_{m=1}^{n} a_m S_m(t) + \lambda S_k(t)\right) = \int_0^1 \left(\sum_{m=1}^{n} a_m S_m(t) + \lambda S_k(t)\right)^n dt =$$

$$= \int_0^1 \left(\sum_{i=0}^{n} C_n^i \left(\sum_{m=1}^{n} a_m S_m(t)\right)^{n-i} (\lambda S_k(t))^i\right) dt =$$

$$= \int_0^1 \left(\sum_{i=0}^{n} \lambda^i (S_k(t))^i C_n^i \sum_{m_1,\ldots,m_{n-i}=1}^{n} a_{m_1} \ldots a_{m_{n-i}} S_{m_1}(t) \ldots S_{m_{n-i}}(t)\right) dt,$$

where C_n^i are the binomial coefficients. Let us consider the term

$$\sum_{m_1,\ldots,m_{n-1}=1}^{n} a_{m_1} \ldots a_{m_{n-1}} \int_0^1 S_{m_1}(t) \ldots S_{m_{n-1}}(t) S_k(t) dt.$$

Denote $m_1 m_2 \ldots m_{n-1} k = l$ and, using Lemma 1, rewrite the given integral by

$$\int_0^1 S_{m_1}(t) \ldots S_{m_{n-1}}(t) S_k(t) dt = \int_0^1 S_1^{m_1}(t) \ldots S_1^{m_{n-1}}(t) S_1^k(t) dt =$$

$$= \int_0^1 \left(S_{r_1}^l(t)\right)^{m_2 \ldots m_{n-1} k} \left(S_{r_2}^l(t)\right)^{m_1 m_3 \ldots m_{n-1} k} \ldots \left(S_{r_n}^l(t)\right)^{m_1 \ldots m_{n-1}} dt,$$

$r_1, \ldots, r_n \in \mathbb{N}$.

According to [2] the integral is equal to unit in the case when

$$m_2 \ldots m_{n-1} k \equiv m_1 m_3 \ldots m_{n-1} k \equiv \cdots \equiv m_1 \ldots m_{n-1} \equiv 0 (\operatorname{mod} l)$$

, that it is impossible (in accordance with the definition of l). Thus, the integral is equal to zero. Evidently, the same holds for the next terms. The last term:

$$\lambda^n C_n^n \int_0^1 (S_k)^n dt = \lambda^n \left(\int_0^{\frac{1}{k}} \alpha_1^n dt + \int_{\frac{1}{k}}^{\frac{2}{k}} \alpha_2^n dt + \dots + \int_{\frac{k-1}{k}}^1 \alpha_k^n dt \right) = \frac{\lambda^n}{k} (\alpha_1^n + \dots + \alpha_k^n) = 0.$$

Thus, $F_n(g(t) + \lambda S_k(t)) = \int_0^1 \left(\sum_{m=1}^n a_m S_m(t)\right)^n dt = F_n(g(t))$. □

3. Maximal ideals of the algebra $A_s(B_{L_n})$**.** Let us consider the restriction of f onto span $(S_1, \dots, S_n)$. Let $\psi\colon f \to \widetilde{f} = f|_{\text{span } (S_1,\dots,S_n)}$.

Theorem 2. *The map ψ is an injective homomorphism from $A_s(B_{L_n})$ onto $A_s(B_{\text{span } (S_1,\dots,S_n)})$.*

Proof. Let P be an arbitrary polynomial from $A_s(B_{L_n})$. As it was observed, P is an algebraic span of $F_1, \dots, F_n$, that is there exists a polynomial q of n variables such that

$$P(x) = q\left(F_1(x), \dots, F_n(x)\right).$$

Let $\widetilde{P}\,(x) = \psi\left(P(x)\right) = 0$, then $q(\widetilde{F}_1\,(x), \dots, \widetilde{F}_n\,(x)) \equiv 0$. If $q \not\equiv 0$, then there are $z_1, \dots, z_n \in \mathbb{C}$ such that $q(z_1, \dots, z_n) \neq 0$. Choose $x \in$ span $(S_1, \dots, S_n)$ such that $\widetilde{F}_1\,(x) = z_1, \dots, \widetilde{F}_n\,(x) = z_n$. Suppose that there exists $x_0 \in \mathbb{C}^n$ such that

$$F_1(x_0) = z_1, \dots, F_{n-1}(x_0) = z_{n-1}.$$

Set $x(\lambda) = \lambda S_n(t) + x_0$. According to Lemma 2 $F_k(x(\lambda)) = z_k\ \forall k < n$, for every $\lambda \in \mathbb{N}$, and $F_n(x(\lambda))$ is a polynomial of degree n of λ and $F_n(x(\lambda)) = z_n$. Let λ_0 be a solution of the equation. Then we can write: $x = \lambda_0 S_n(t) + x_0$. Thus we have proved that from $\widetilde{P}\,(x) = 0$ it follows that $P(x) = 0$ and it means that $\ker \psi = 0$. Hence ψ is an injective map. It is easily seen that ψ is a homomorphism. □

Define the set M in the following way:

$$M = \Bigg\{ x = \sum_{k=1}^n a_k S_k : \sum_{k=1}^n |a_k|^n \leq 1,\ |a_1| < 1, \dots,\ |a_n| < 1,$$

$$0 \leq \arg a_2 < \pi,\ 0 \leq \arg a_3 < \frac{2\pi}{3}, \dots,\ 0 \leq \arg a_n < \frac{2\pi}{n} \Bigg\}.$$

Proposition 3. *The functions $f \in A_s(B_{L_n})$ separate points of the set M.*

Proof. Take vectors $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$, whose coordinates satisfy the conditions from the description of the set M. Assume that $(a_1, \dots, a_n) \neq (b_1, \dots, b_n)$ and for every $f \in A_s(B_{L_n})$ $f\left(\sum a_k S_k\right) = f\left(\sum b_k S_k\right)$.

Since the vectors are not equal, there exists at least one coordinate m, $1 \leq m \leq n$ such that $a_m \neq b_m$. Take a vector $(t_1, \dots, t_n) \in \mathbb{C}^n$, $t_m \neq 0$. It is evident that $(t_1 a_1, \dots, t_n a_n) \neq (t_1 b_1, \dots, t_n b_n)$. Acting on the both parts of the inequality by the function F_m we obtain

$$F_m(t_1 a_1, \dots, t_n a_n) = F_m(t_1 b_1, \dots, t_n b_n),$$

that is

$$F_m\left(\sum_{k=1}^{n} t_k a_k S_k\right) = F_m\left(\sum_{k=1}^{n} t_k b_k S_k\right).$$

Setting here $t_m = 1$ and $t_k = 0$ for $k \neq m$, we have that $a_m^m = b_m^m$. It follows that $a_m = b_m$, a contradiction.

□

Notice that from Theorem 2 it follows that $M(A_s(B_{L_n})) \subset B_{\mathrm{span}(S_1,\dots,S_n)}$.

Theorem 3. $M(A_s(B_{L_n})) \subset M \subset B_{\mathrm{span}\ (S_1,\dots,S_n)}$

Proof. It is clear that $M \subset B_{\mathrm{span}\ (S_1,\dots,S_n)}$, because the set M was constructed by setting of special conditions on the points from $B_{\mathrm{span}\ (S_1,\dots,S_n)}$. In other words, from every set of points of $B_{\mathrm{span}\ (S_1,\dots,S_n)}$ which generate the same complex homomorphism from $M(A_s(B_{L_n}))$, we take the only point. Thus, we have the required embedding $M(A_s(B_{L_n})) \subset M$. □

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