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ASYMPTOTIC BEHAVIOR OF THE DIFFERENCE OF SUBHARMONIC FUNCTIONS

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Let μ and ν be Borel positive measures of finite upper density for order ρ in the complex plane $\mathbb{C}$ and let u_μ be a subharmonic function in $\mathbb{C}$ with the Riesz measure μ. The main result of this paper shows that a some closeness of the measures μ and ν (of order $O(r^\alpha)$, $r \to +\infty$, in every sector $\{z : |z| < r,\ 0 \le \arg z \le \psi\}$ uniformly with respect to $\psi \in [0, 2\pi]$) implies the existence of subharmonic function u_ν with the Riesz measure ν such that $|u_\mu(z) - u_\nu(z)| = O(|z|^{\rho-\alpha} \log |z|)$ as $z \to \infty$, $z \notin E$, where E is the union of a some exceptional set of disks with finite sum of radii.

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Пусть μ и ν — положительные борелевские меры конечной верхней плотности порядка ρ в комплексной плоскости $\mathbb{C}$, и пусть u_μ — субгармоническая функция в $\mathbb{C}$ с мерой Рисса μ. Основной результат этой статьи показывает, что некоторая близкость мер μ и ν (порядка $O(r^\alpha)$, $r \to +\infty$, в секторе $\{z : |z| < r,\ 0 \le \arg z \le \psi\}$ равномерно относительно $\psi \in [0, 2\pi]$) влечет существование субгармонической функции u_ν с мерой Рисса ν такой, что $|u_\mu(z) - u_\nu(z)| = O(|z|^{\rho-\alpha} \log |z|)$ при $z \to \infty$, $z \notin E$, где E — объединение некоторого исключительного множества кругов с конечной суммой радиусов.

§ 1. Introduction

A part of results of this article was announced in [1] as far back as 1986.

Let μ be a measure (resp. a charge) on the complex plane $\mathbb{C}$. All measures (resp. charges) in this article are assumed to be Borel and positive (resp. real-valued), $\operatorname{supp} \mu$ is the support of μ.

We denote by $D(z,t)$ the open disk of radius t centered at $z \in \mathbb{C}$, $D(t) \overset{\text{def}}{=} D(0,t)$, and we set $\mu(z,t) \overset{\text{def}}{=} \mu(D(z,t))$, $\mu(t) \overset{\text{def}}{=} \mu(0,t)$. For simplicity it is assumed everywhere that any measure (resp. charge) in this article vanishes in the unit disk $D(1)$, i. e., its support lies in $\mathbb{C} \setminus D(1)$.

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If $\rho \geq 0$ and $\mu(t) = O(t^\rho)$, $t \to +\infty$, then μ is the measure of finite upper density for order ρ. We denote by $\mathcal{M}_\rho$ the class of all measures of finite upper density for order ρ. We denote also by SH_ρ the class of all subharmonic functions u of finite type for order ρ in $\mathbb{C}$, i. e., $u(z) \leq C_u|z|^\rho$ for $|z| \geq 1$ where C_u is a constant. The Riesz measure of $u \in SH_\rho$ is the measure $\mu_u = \frac{1}{2\pi}\Delta u$ where Δ is the Laplacian and the equality is to be interpreted in the sense of the distribution theory. If $u \in SH_\rho$ then $\mu_u \in \mathcal{M}_\rho$, and, for noninteger ρ, vice-versa: if $\mu \in \mathcal{M}_\rho$ then there is a subharmonic function $u_\mu \in SH_\rho$ with Riesz measure μ. But the last implication is not true for each integer $\rho \geq 0$.

The main source of our investigation is the Levin-Pfluger Theorem [2, 3] on entire functions of completely regular growth. We formulate a subharmonic variant of this theorem in a form suitable for our research.

Levin-Pfluger Theorem ([2]–[5]). *Let $\rho > 0$ and let $u_\mu(z) \not\equiv -\infty$ be a subharmonic ρ-homogeneous function with Riesz measure μ, i. e. $u_\mu(tz) = t^\rho u_\mu(z)$ for every $t \geq 0$ and $z \in \mathbb{C}$. Let $\nu \in \mathcal{M}_\rho$. Then the following assertions are equivalent:*

(m) *there exists φ such that the relation*

$$\bigl|\mu(r;\varphi,\psi) - \nu(r;\varphi,\psi)\bigr| = o(r^\rho), \quad r \to +\infty, \tag{1.1}$$

holds for every $\psi \in (0, 2\pi]$ with the exception of at most countable set of values ψ, and, in addition, for integer ρ, there is the limit $\lim\limits_{r\to+\infty} \int_{|\zeta|<r} \zeta^{-\rho}\, d\nu(\zeta)$;

(u) *there are a subharmonic function u_ν and an exceptional set $E \subset \mathbb{C}$ such that*

$$\bigl|u_\mu(z) - u_\nu(z)\bigr| = o\bigl(|z|^\rho\bigr) \tag{1.2}$$

as $z \to \infty$ outside the set E, and it is possible to cover the set E by a system of disks $D(z_k, t_k)$, $k \in \{1, 2, \dots\}$, satisfying the condition

$$\sum_{|z_k|<R} t_k = o(R), \quad R \to +\infty. \tag{1.3}$$

Our results evolve the implication (m)$\Rightarrow$(u) in the direction of possible decrease of asymptotic (1.1) which have as a consequence a decrease of asymptotics (1.2) and (1.3). Our research is also connected with results of P. Z. Agranovich and V. N. Logvinenko on polynomial asymptotic representations of subharmonic functions [6]–[9].

Below we formulate the main theorem from a paper of R. S. Yulmukhametov [10, Theorem 1]. At first sight, this theorem overlaps our main results.

Let $u_1, u_2 \in SH_\rho$ with Riesz measures μ_1, μ_2, respectively. In [10, Theorem 1] the author claims that the following assertions are equivalent:

($\hat{\text{m}}$) *for every real γ there are a set $E_\gamma \subset \mathbb{C}$ and a constant C_γ such that*

$$N(z,R;\mu_1,\mu_2) \overset{\text{def}}{=} \left|\int_0^R \frac{\mu_1(z,\tau) - \mu_2(z,\tau)}{\tau}\, d\tau\right| \leq C_\gamma |z|^\sigma,\ z \notin E_\gamma,\ R \in \bigl(0, |z|\bigr), \tag{1.4}$$

and it is possible to cover the exceptional set E_γ by a system of disks $D(z_k, t_k)$, $k \in \{1, 2, \dots\}$, satisfying the condition

$$\sum_{R/2<|z_k|<2R} t_k = o(R^\gamma), \quad R \to +\infty; \tag{1.5}$$

($\hat{\mathrm{u}}$) *there is a harmonic function H of finite order such that, for every γ, there are a constant C'_γ and an exceptional set $E'_\gamma \subset \mathbb{C}$ for which*

$$\left|u_1(z) - u_2(z) + H(z)\right| \le C'_\gamma |z|^\sigma, \quad z \notin E'_\gamma,$$

and it is possible to cover the set E'_γ by a system of disks $D(z_k, t_k)$ satisfying condition (1.5).

But the implication ($\hat{\mathrm{m}}$)⇒($\hat{\mathrm{u}}$) is false. The author uses a lemma [10, Lemma 8] without proof referring to [11, § 4.1]. But we did not discover in [11, § 4.1] an assertion similar to [10, Lemma 8]. Moreover, Lemma 8 from [10] is also not true. We construct two counterexamples for the implication ($\hat{\mathrm{m}}$)⇒($\hat{\mathrm{u}}$) in § 6.

We formulate now consequences of Main Theorem of this article. Denote by $S(r, R; \varphi, \psi)$ the set $\{te^{i\theta} \in \mathbb{C} : r \le t < R, \varphi \le \theta < \psi\}$, i.e., "a polar (spherical) rectangle". In particular, $S(r, R) \overset{\text{def}}{=} S(r, R; 0, 2\pi)$ is the annulus, and $S(r; \varphi, \psi) \overset{\text{def}}{=} S(0, r; \varphi, \psi)$ is the sector, and $S(\infty; \varphi, \psi) \overset{\text{def}}{=} S(0, \infty; \varphi, \psi)$ is the angle.

For a measure or a charge μ on $\mathbb{C}$ we set $\mu(r; \varphi, \psi) \overset{\text{def}}{=} \mu\big(S(r; \varphi, \psi)\big)$.

Corollary 1. *Let μ and ν be two measures of finite upper density for an order $\rho \ge 0$. Let u_μ be a subharmonic function with the Riesz measure μ. If*

($\acute{\mathrm{m}}$) *for a some $\alpha \ge 0$ there exists φ such that the relation*

$$\big|\mu(r; \varphi, \psi) - \nu(r; \varphi, \psi)\big| = O(r^\alpha), \quad r \to +\infty, \tag{1.6}$$

holds uniformly with respect to an everywhere dense set of values ψ in $[0, 2\pi]$

then

($\acute{\mathrm{u}}$) *there exists a subharmonic function u_ν with the Riesz measure ν such that for every $\gamma \ge 0$*

$$\big|u_\mu(z) - u_\nu(z)\big| = O\big(|z|^\alpha \log|z|\big) \tag{1.7}$$

as $z \to \infty$ outside a system of disks $D(z_k, t_k) \subset \mathbb{C}$, $t_k \le |z_k|/2$, $k \in \{1, 2, \dots\}$, satisfying the condition

$$\sum_{|z_k|>R} t_k = o(R^{-\gamma}), \quad R \to +\infty. \tag{1.8}$$

If we put $\alpha = \rho$ in Corollary 1 then we obtain

Corollary 2. *If the total variation of a charge λ on $\mathbb{C}$ belongs to $\mathcal{M}_\rho$ then there exists a δ-subharmonic function v_λ, i.e. the difference of two subharmonic functions, with the Riesz charge λ such that for every $\gamma \ge 0$ the relation*

$$|v_\lambda(z)| = O\big(|z|^\rho \log|z|\big) \tag{1.9}$$

holds as $z \to \infty$ outside a system of disks $D(z_k, t_k)$, $t_k \le |z_k|/2$, $k \in \{1, 2, \dots\}$, satisfying condition (1.8).

Corollary 2 give lower bounds for subharmonic functions of finite order (see in § 5 proofs of Corollaries 1 and 3):

Corollary 3. *Let u be a subharmonic function of finite order $\rho \ge 0$ with the Riesz measure $\mu_u \in \mathcal{M}_\rho$. Then for every $\gamma \ge 0$ there exists a constant $C_\gamma \ge 0$ such that the relation*

$$u(z) \ge -C_\gamma |z|^\rho \log |z| \tag{1.10}$$

holds for all z which lie outside a system of disks $D(z_k, t_k)$, $t_k \le |z_k|/2$, $k \in \{1, 2, \dots\}$, satisfying the condition (1.8).

Remark 1. Essentially more general results are known for arbitrary δ-subharmonic functions in $\mathbb{R}^m$, $m \ge 2$, than Corollaries 2 and 3 (see [12], [13, Theorems 1 and 2]). For example, *there is a constant $C \ge 0$ such that for every subharmonic function u, $u(0) = 0$, the inequality $u(z) \ge -C \max\{u(\zeta) : |\zeta| = 2|z|\} \log(C + C|z|)$ holds outside some exceptional set of disks with finite sum of radii* [13, Corollary].

Remark 2. The example of the subharmonic function $u(z) = \log|1/\Gamma(z)|$ shows that estimate (1.10) of Corollary 3 is the best possible [2, Ch. I, § 11]. Consequently, all Corollaries 1–3 (and Main Theorem below) are sharp.

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§ 2. The Main Theorem

Let $s \ge 0$. We say that a system of polar rectangles

$$S_{n,m} = S\big(r_n, r_{n+1}; \psi_n^{(m)}, \psi_n^{(m+1)}\big), \quad n \in \mathbb{N}, \quad m \in \{1, 2, \dots, q_n\}, \tag{2.1}$$

is *s-narrow*, if this system satisfies the following two conditions:

(a) there exists a constant $a > 0$ such that

$$1 + a \le \frac{r_{n+1}}{r_n} \le 1/a, \quad n \in \mathbb{N}, \quad r_1 = 1\,; \tag{2.2}$$

(b) there exists a constant $b > 0$ such that for every n

$$br_n^{-s} \le \psi_n^{(m+1)} - \psi_n^{(m)} \le r_n^{-s}/b, \quad m \in \{1, 2, \dots, q_n\}, \tag{2.3}$$

where $\psi_n^{(q_n+1)} = \psi_n^{(1)} + 2\pi$ for all n.

According to (2.2) the condition $r \ge r_n$ implies

$$r \ge (1+a) r_{n-1} \ge (1+a)^2 r_{n-2} \dots \ge (1+a)^{n-1} r_1 = (1+a)^{n-1},$$

and so

$$n - 1 \le \frac{1}{\log(1+a)} \log r \ \text{ if } \ r_n \le r. \tag{2.4}$$

We set $S_{n,m}(t) \overset{\text{def}}{=} S_{n,m} \cap D(t)$.

Main Theorem. *Let μ_1 and μ_2 be measures of finite upper density for an order $\rho \ge 0$ and $0 \le \alpha \le \rho$. Suppose that there exist a s-narrow system* (2.1), *$s \ge \rho - \alpha$, and a constant A such that*

$$\Big|(\mu_1 - \mu_2)\Big(\bigcup_{m=1}^{q} S_{n,m}(t)\Big)\Big| \le A r_{n+1}^\alpha \ \textit{for all}\ 1 \le r_n \le t < r_{n+1}, \quad 1 \le q \le q_n. \tag{2.5}$$

Then there exist subharmonic functions u_1 and u_2 with Riesz measures μ_1 and μ_2, respectively, and constants $B, C \ge 0$ such that for every $N > 1$ the inequality

$$\big|u_1(z) - u_2(z)\big| \le BN|z|^\alpha \log|z| \tag{2.6}$$

holds outside an exceptional set

$$\bigcup_{k=1}^{\infty} D(z_k, t_k) \ \ \text{such that } t_k \le |z_k|/2, \quad \sum_{R/2\le|z_k|<2R} t_k \le \frac{C}{N} R^{1+\rho-\alpha-s} \tag{2.7}$$

for all $R \ge R_0$.

If $t \in [r_n, r_{n+1})$ then under conditions (2.2) and (2.5) we obtain

$$\big|\mu_1(t) - \mu_2(t)\big| \le \sum_{k=1}^{n-1}\Big|(\mu_1-\mu_2)\Big(\bigcup_{m=1}^{q_k} S_{k,m}(r_{k+1})\Big)\Big| + \Big|(\mu_1-\mu_2)\Big(\bigcup_{m=1}^{q_n} S_{n,m}(t)\Big)\Big| \le$$
$$\le A\sum_{k=1}^{n} r_{k+1}^\alpha \le A r_{n+1}^\alpha \sum_{m=0}^{\infty}\Big(\frac{1}{(1+a)^\alpha}\Big)^m \le Aa^{-\alpha}\frac{1}{1-1/(1+a)^\alpha}t^\alpha.$$

Thus, the union of conditions (2.2) and (2.5) implies the condition

$$\big|\mu_1(t) - \mu_2(t)\big| \le A' t^\alpha, \qquad \text{where a constant } A' \text{ is independent of } t > 0. \tag{2.8}$$

Without loss of generality, we can assume that the measures μ_1 and μ_2 are concentrated on the boundary of polar rectangles $S_{n,m}$. For that we use a particular case of the result [14, § 3, Theorem 2] which is formulated below.

Let T be a Borel-measurable mapping of $\mathbb{C}$ into itself. Then T generates a transformation in the space of Borel measures according to the rule: with each measure μ there is associated the measure μ_T such that $\mu_T(G) = \mu(T^{-1}G)$ for any Borel set G.

Let $|T(\zeta) - \zeta| = w(\zeta)$ and $\hat{w}(\zeta) = \sup\{|\zeta - z| : T(z) = \zeta\}$. Then $\hat{w}$ is a function on $T(\mathbb{C})$. Extend $\hat{w}$ to $\mathbb{C} \setminus T(\mathbb{C})$ by zero.

Let u be a subharmonic function with Riesz measure μ. We introduce the measures $\hat{\mu} = \mu + \mu_T$ and $dW = w\,d\mu + \hat{w}\,d\mu_T$ and the function

$$K(z, dW) = |z|^{q_W}\int_{|\zeta|<|z|}\frac{dW(\zeta)}{|\zeta|^{q_W}} + |z|^{q_W+1}\int_{|\zeta|\ge|z|}\frac{dW(\zeta)}{|\zeta|^{q_W+1}},$$

where $q_W \ge 0$ is the genus of dW, i. e., the smallest integer such that the last integral converges. For a function $\Phi\colon \mathbb{C} \to \mathbb{R}$ we set $\Phi_{1/2}(z) = \sup\{\Phi(\zeta) : |\zeta - z| \le |z|/2\}$, $z \in \mathbb{C}$, and $D_{1/2} = \bigcup\{D(z, |z|/2) : z \in D\}$ for $D \subset \mathbb{C}$.

Comparison Theorem ([14, Theorem 2]) *There exists a subharmonic function u_T with Riesz measure μ such that for any function $\Phi(z) > 1$, $z \in \mathbb{C}$, there is a countable set $\{D(z_k, t_k)\}_1^\infty$ of disks such that $t_k < |z_k|/2$ for all[1] $k \ge k_0$, and*

$$\sum_{z_k\in D} t_k \le \int_{D_{1/2}}\frac{dm(z)}{\Phi(z)|z|} \quad \text{for every Borel set } D, \tag{2.9}$$

[1]The inequality $t_k < |z_k|/2$ was not mentioned in the text of Theorem 2 from [14] but it can be found in first paragraph of the proof of [14, Theorem 2] (see also [14, Normal Points Lemma]).

where m is the Lebesgue measure, and for $z \notin \bigcup_k D(z_k, t_k)$

$$\bigl|u(z) - u_T(z)\bigr| \le \frac{C(q_W)}{|z|} K(z, dW)\Phi_{1/2}(z) \log\Bigl(2 + \frac{|z|\hat{\mu}\bigl(D(z, |z|/2)\bigr)}{2K(z, dW)}\Bigr), \tag{2.10}$$

where the constant $C(q_W)$ depends only on the genus q_W.

For our case consider a mapping T of $\mathbb{C} \setminus D(1)$ into itself such that $T(te^{i\psi}) = te^{i\psi_n^{(m)}}$ for each $te^{i\psi} \in S_{n,m}$, $T(z) \equiv z$ for $|z| < 1$. Then in view of (2.2) and (2.3) the mapping T satisfies the condition

$$\bigl|T(z) - z\bigr| \le C_1 |z|^{1-s}, \ z \in \mathbb{C}, \qquad \text{where } s \ge \rho - \alpha \text{ and } C_1 \text{ is a constant.} \tag{2.11}$$

The mapping T transforms the measures μ_1, μ_2 to new measures $(\mu_1)_T, (\mu_2)_T$ of finite upper density for the order ρ. Evidently, the measures μ_1, μ_2 are concentrated on the intervals

$$\bigl\{te^{i\psi_n^{(m)}} : r_n \le t < r_{n+1}\bigr\}. \tag{2.12}$$

and condition (2.5) is fulfilled for the charge $(\mu_1)_T - (\mu_2)_T$ as before.

Under the notations stated above it follows by (2.11) that $w(z) + \hat{w}(z) \le C_2 |z|^{1-s}$, $z \in \mathbb{C}$, where C_2 is a constant. Hence, for $\hat{\mu}_j = \mu_j + (\mu_j)_T$ and $dW_j = w\, d\mu_j + \hat{w}\, d(\mu_j)_T$, $j \in \{1, 2\}$, we obtain the estimates

$$\hat{\mu}_j(t) \le C_3 t^{\rho}, \quad W_j(t) \le C_3 t^{1+\rho-s}, \quad t > 0, \quad j \in \{1, 2\}, \tag{2.13}$$

where C_3 is a constant. Therefore

$$q_{W_j} \le [\rho - s] + 1, \quad K(z, dW_j) \le C_4 |z|^{1+\rho-s}, \quad z \in \mathbb{C}, \quad j \in \{1, 2\}, \tag{2.14}$$

where $[a]$ is the integral part of a, and C_4 is a constant. By Comparison Theorem, for any function $\Phi(z) > 1$ there are subharmonic functions $(u_j)_T$, $j \in \{1, 2\}$ such that for a set $\bigl\{D(z_k, t_k)\bigr\}_1^\infty$ condition (2.9) holds, and, according to (2.10), (2.13), (2.14),

$$\begin{aligned} \bigl|u_j(z) - (u_j)_T(z)\bigr| &\le \Phi_{1/2}(z) \frac{C(q_W)}{|z|} K(z, dW) \log\Bigl(2 + \frac{C_3 (3/2)^{\rho} |z|^{\rho+1}}{2K(z, dW)}\Bigr) \le \\ \le C_5 \Phi_{1/2}(z) C_4 |z|^{\rho-s} &\log\Bigl(2 + \frac{C_5 |z|^{\rho+1}}{2C_4 |z|^{1+\rho-s}}\Bigr) \le C_6 \Phi_{1/2}(z) |z|^{\rho-s} \log(3 + |z|), \end{aligned} \tag{2.15}$$

for all $z \in \mathbb{C}$, $j \in \{1, 2\}$, where we use the increase of the function $t \log(2 + a/t)$ of $t \ge 0$, $a \ge 0$, and the constants C_5, C_6 are independent of Φ.

For an arbitrary constant $N > 1$ we choose the radial function $\Phi(z) = \Phi(|z|) = \max\{N, N|z|^{\alpha-(\rho-s)}\} > 1$. Then for $D = S(R/2, 2R)$ it follows from (2.9) that

$$\sum_{z_k \in S(R/2, 2R)} t_k \le \int\limits_{z_k \in S(R/4, 3R)} \frac{dm(z)}{N|z|^{1+\alpha-(\rho-s)}} = \frac{C}{N} R^{1+\rho-\alpha-s}, \tag{2.16}$$

where a constant C depends only on α, ρ, s. Simultaneously, in view of (2.15), we get

$$\bigl|u_j(z) - (u_j)_T(z)\bigr| \le C_6 N (3/2)^{\alpha-(\rho-s)} |z|^{\alpha} \log(3 + |z|) \le BN|z|^{\alpha} \log|z| \tag{2.17}$$

for $|z| \ge 2$ with a constant B when z lies outside the exceptional set of disks from (2.16). The right-hand sides in (2.16) and (2.17) coincide with the right-hand sides in (2.7) and (2.6), respectively.

So, throughout what follows, we will be supposed that the charge $\nu = \mu_1 - \mu_2$ are concentrated on the set of intervals (2.12).

Consider an integral

$$\mathcal{I}_\nu(z) = \int_{\mathbb{C}} G\bigl(z/\zeta, [\alpha]\bigr)\, d\nu(\zeta), \quad \nu = \mu_1 - \mu_2, \tag{2.18}$$

where $G(\xi, p) = \log|1-\xi| + \operatorname{Re}\sum_{k=1}^{p} \xi^k/k$ is the logarithm of modulus of the Weierstrass primary factor of genus p.

In order to prove the Main Theorem, it is sufficient to ensure for integral (2.18) an estimate of form (2.6) outside an exceptional set from (2.7). Indeed, in this case, let u_1 and $\tilde{u}_2$ be two subharmonic functions with Riesz measures μ_1 and μ_2, respectively. Then $\Delta\bigl(u_1 - \tilde{u}_2 - \mathcal{I}_\nu\bigr) = \mu_1 - \mu_2 - \nu = 0$ in the sense of the distribution theory. Therefore, by Weil's lemma the function $u_1 - \tilde{u}_2 - \mathcal{I}_\nu = H$ is harmonic, and the subharmonic functions u_1 and $u_2 = \tilde{u}_2 + H$ are as required.

A final estimate of the integral $\mathcal{I}_\nu(z)$ of form (2.6) will be obtained in § 4.

§ 3. Two auxiliary lemmas

Lemma 1. *Let $f(t, \varphi)$ be a real-valued function which is twice continuously differentiable on the closure of a polar rectangle*

$$S = S(r_1, r_2; \psi_1, \psi_2) \ni te^{i\varphi}, \quad \psi_1 \le \psi_2 \le \psi_1 + 2\pi.$$

Let ν be a charge which is concentrated on the union of finite number of intervals

$$\{\xi \in S : \arg\xi = \varphi_l\}, \quad l \in \{1, 2, \dots, q\},$$

and the charge ν satisfies the condition

$$\bigl|\nu(r_1, t; \psi_1, \varphi)\bigr| \le M, \quad te^{i\varphi} \in S. \tag{3.1}$$

Then

$$\Bigl|\int_S f\, d\nu\Bigr| \le 2\pi M\Bigl(\bigl|f(r_2, \psi_2)\bigr| + \bigl|f(r_1, \psi_2)\bigr| + \max_\varphi\Bigl|\frac{\partial}{\partial\varphi} f(r_1, \varphi)\Bigr| + \\ + \max_\varphi\Bigl|\frac{\partial}{\partial\varphi} f(r_2, \varphi)\Bigr| + \int_{r_1}^{r_2}\Bigl(\max_\varphi\Bigl|\frac{\partial^2}{\partial\varphi\partial t} f(t, \varphi)\Bigr| + \Bigl|\frac{\partial}{\partial t} f(t, \psi_2)\Bigr|\Bigr)\, dt\Bigr)$$

Proof. Without loss of generality we can always assume that the sequence $\{\varphi_l\}$ is increasing. Set

$$\nu_l(t) = \nu\bigl(\{\tau e^{i\varphi_l} : r_1 \le \tau < t\}\bigr), \quad l \in \{1, 2, \dots, q\}, \quad \varphi_{q+1} = \psi_2, \quad \nu_{q+1}(t) \equiv 0.$$

Then it follows from (3.1) that

$$\Bigl|\sum_{m=1}^{l} \nu_m(t)\Bigr| \le M \tag{3.2}$$

for every $l \le q+1$ and $t \in [r_1, r_2)$.

Given a function $g(\varphi) \in C^1[\psi_1, \psi_2]$, we estimate the sum

$$\sum_{l=1}^{q+1} g(\varphi_l)\nu_l(t) = \sum_{l=1}^{q} \big(g(\varphi_{l+1}) - g(\varphi_l)\big)\Big(\sum_{m=1}^{l} \nu_m(t)\Big) + g(\varphi_{q+1}) \sum_{m=1}^{q+1} \nu_m(t) \tag{3.3}$$

where the right-hand side is obtained by the classical Abel transform of sum. By the mean-value theorem we get

$$\sum_{l=1}^{q} \big|g(\varphi_{l+1}) - g(\varphi_l)\big| \le \sum_{l=1}^{q} \max_{\varphi_l \le \varphi \le \varphi_{l+1}} \big|g'(\varphi)\big| \cdot |\varphi_{l+1} - \varphi_l| \le 2\pi \max_{\psi_1 \le \varphi \le \psi_2} \big|g'(\varphi)\big|. \tag{3.4}$$

In view of (3.3), (3.4) and (3.2) we obtain

$$\Big|\sum_{l=1}^{q+1} g(\varphi_l)\nu_l(t)\Big| \le 2\pi M\big(\max_{\varphi} |g'(\varphi)| + |g(\psi_2)|\big). \tag{3.5}$$

Using the integration by parts, we obtain

$$\int_S f\,d\nu = \sum_{l=1}^{q+1} \int_{r_1}^{r_2} f(t, \varphi_l) d\nu_l(t) = \sum_{l=1}^{q+1} f(r_2, \varphi_l)\nu_l(r_2) -$$
$$- \sum_{l=1}^{q+1} f(r_1, \varphi_l)\nu_l(r_1) - \int_{r_1}^{r_2} \sum_{l=1}^{q+1} \Big(\frac{\partial}{\partial t} f(t, \varphi_l)\Big)\nu_l(t)\,dt.$$

Further we use inequality (3.5) in order to estimate two first sums in the right-hand side with $g(\varphi)$ at significance level $f(r_2, \varphi)$ and $f(r_1, \varphi)$, respectively. In order to estimate the integrand of the last integral we use also inequality (3.5) with $g(\varphi) = \frac{\partial}{\partial t} f(t, \varphi)$ for each fixed t. Thus,

$$\Big|\int_S f\,d\nu\Big| \le 2\pi M\Big(\max_{\varphi}\Big|\frac{\partial}{\partial \varphi} f(r_2, \varphi)\Big|\Big) + \big|f(r_2, \psi_2)\big|\Big) +$$
$$+ 2\pi M\Big(\max_{\varphi}\Big|\frac{\partial}{\partial \varphi} f(r_1, \varphi)\Big| + \big|f(r_1, \psi_2)\big|\Big) +$$
$$+ 2\pi M \int_{r_1}^{r_2} \Big(\max_{\varphi}\Big|\frac{\partial^2}{\partial \varphi \partial t} f(t, \varphi)\Big| + \Big|\frac{\partial}{\partial t} f(t, \psi_2)\Big|\Big)\,dt,$$

and Lemma 1 is proved. □

Lemma 2. *Assume that the conditions in the Main Theorem are satisfied. Suppose that the measures μ_1 and μ_2 is concentrated on the union of finite numbers of intervals of kind (2.12) with $n = k$, $z = re^{i\theta}$, $\nu = \mu_1 - \mu_2$. Then for every integer $p \neq 0$*

$$\Big|\int_{S(r_k, r_{k+1})} \operatorname{Re} \frac{\xi^p}{|p| z^p}\, d\nu(\xi)\Big| \le 8\pi A \frac{r_{k+1}^{\alpha}}{r^p} \max\{r_k^p, r_{k+1}^p\} \quad k \in \{1, 2, \dots\}.$$

Proof. Let $\xi = te^{i\varphi}$. By Lemma 1 with

$$f(t, \varphi) = \frac{1}{|p|} t^p \cos p(\varphi - \theta) = r^p \operatorname{Re} \frac{\xi^p}{|p| z^p}$$

and $\psi_2 = \psi_1 + 2\pi$ we obtain

$$\frac{1}{|p|}\Big|\int_{S(r_k,r_{k+1})} t^p \cos p(\varphi-\theta)\,d\nu(\xi)\Big| \le 2\pi A r_{k+1}^{\alpha}\Big(\frac{1}{|p|}\,r_{k+1}^p + \frac{1}{|p|}\,r_k^p + r_k^p + r_{k+1}^p +$$
$$+\frac{1}{|p|}\int_{r_k}^{r_{k+1}}\big(|p|^2 t^{p-1} + |p| t^{p-1}\big)\,dt\Big) = 2\pi A r_{k+1}^{\alpha}\Big(r_{k+1}^p\big(1+1/|p|+|p|/p+1/p\big) +$$
$$+r_k^p\big(1+1/|p|-|p|/p-1/p\big)\Big) = 4\pi\big(1+1/|p|\big)A r_{k+1}^{\alpha}\max\{r_k^p, r_{k+1}^p\},$$

and Lemma 2 is proved. □

§ 4. The estimate of the integral $\mathcal{I}_\nu$ from (2.18)

We set $z = re^{i\theta} \in S(r_n, r_{n+1})$ and $n > 2$, $r_n \ge 2$ everywhere in this paragraph as far as the proof of Corollary 1. Besides, everywhere below, numbers const. are generally speaking different constants which are independent of n, z, N, R.

Divide the integral $\mathcal{I}_\nu$ from (2.18) into a sum of six integrals

$$\mathcal{I}_\nu(z) = \int\limits_{|\xi|<r_{n-1}} \log|z-\xi|\,d\nu(\xi) - \int\limits_{|\xi|<r_{n+2}} \log|\xi|\,d\nu(\xi) +$$
$$+ \int\limits_{\substack{r_{n-1}\le|\xi|<r_{n+2}\\ \pi\ge|\arg\xi-\theta|>1}} \log|z-\xi|\,d\nu(\xi) + \int\limits_{|\xi|<r_{n+2}} \sum_{p=1}^{[\alpha]} \mathrm{Re}\,\frac{z^p}{p\xi^p}\,d\nu(\xi) +$$
$$+ \int\limits_{|\xi|\ge r_{n+2}} G\big(z/\xi,[\alpha]\big)\,d\nu(\xi) + \int\limits_{\substack{r_{n-1}\le|\xi|<r_{n+2}\\ |\arg\xi-\theta|\le 1}} \log|z-\xi|\,d\nu(\xi) = \sum_{k=1}^{6} I_k(z). \tag{4.1}$$

Estimate I_1. The expansion in the Taylor series with the center $\xi = 0$ of a holomorphic branch of the function $\log(z-\xi)$ in the disk $|\xi| < r_{n-1}$ implies

$$\log|z-\xi| = \log|z| - \sum_{p=1}^{\infty} \mathrm{Re}\,\frac{\xi^p}{pz^p}, \quad |\xi| < r_{n-1}. \tag{4.2}$$

It follows from (2.8) and (2.18) that

$$\Big|\int_{|\xi|<r_{n-1}} \log|z|\,d\nu(\xi)\Big| \le \text{const.}\,|z|^\alpha \log|z|. \tag{4.3}$$

By Lemma 2 and condition (2.2) we get

$$\sum_{k=1}^{n-2}\sum_{p=1}^{\infty}\Big|\int_{S(r_k,r_{k+1})} \mathrm{Re}\,\frac{\xi^p}{pz^p}\,d\nu(\xi)\Big| \le \text{const.}\sum_{k=1}^{n-2} r_{k+1}^{\alpha}\sum_{p=1}^{\infty}\Big(\frac{r_{k+1}}{r_n}\Big)^p \le$$
$$\le \text{const.}\,|z|^\alpha \sum_{k=1}^{n-2}\sum_{p=1}^{\infty}\Big(\frac{1}{(1+a)^{n-k-1}}\Big)^p = \text{const.}\,|z|^\alpha \sum_{m=1}^{n-2}\frac{1}{(1+a)^m - 1} \le$$
$$\le \text{const.}\,|z|^\alpha\Big(\frac{1}{a} + \sum_{m=2}^{n-2}\frac{2}{m(m-1)a}\Big) \le \text{const.}\,|z|^\alpha.$$

The last estimate together with (4.2) and (4.3) give the estimate

$$\big|I_1(z)\big| \le \text{const.}\,|z|^\alpha \log|z|. \tag{4.4}$$

Estimate I_2. Using the integration by parts, by (2.7), it is easy to obtain the estimate

$$\big|I_2(z)\big| \le \text{const.}\,|z|^\alpha \log|z|. \tag{4.5}$$

Estimate I_3. For every polar rectangle

$$S_k(\theta) = S(r_k, r_{k+1}) \cap \big\{te^{i\varphi} : \pi \ge |\varphi - \theta| > 1\big\}, \quad n-1 \le k \le n+1,$$

a direct calculations show that ($\xi = te^{i\varphi}$)

$$\max_{\xi\in S_k(\theta)}\Big|\frac{\partial}{\partial\varphi}\log|z - te^{i\varphi}|\Big| \le \text{const.},$$

$$\max_{\xi\in S_k(\theta)}\Big|\frac{\partial}{\partial t}\log|z - te^{i\varphi}|\Big| + \max_{\xi\in S_k(\theta)}\Big|\frac{\partial^2}{\partial\varphi\partial t}\log|z - te^{i\varphi}|\Big| \le \frac{\text{const.}}{|z|}$$

where const. is independent of z, k, θ. Hence, if we apply condition (2.5) and Lemma 1 with $f(t,\varphi) = \log|z - te^{i\varphi}|$ then we get

$$\big|I_3(z)\big| \le \sum_{k=n-1}^{n+1}\Big|\int_{S_k(\theta)}\log|z-\xi|\,d\nu(\xi)\Big| \le$$

$$\le \text{const.}\,r_{n+2}^\alpha\Big(\log|z| + \int_{r_{n-1}}^{r_{n+1}}\frac{dt}{|z|}\Big) \le \text{const.}\,|z|^\alpha\log|z|\,. \tag{4.6}$$

Estimate I_4. For $0 < p \le [\alpha]$, by the Lemma 2 (for $-p$ instead of p) and the conditions (2.2), (2.4), we obtain

$$\big|I_4(z)\big| \le \sum_{k=1}^{n+1}\sum_{p=1}^{[\alpha]}\Big|\int_{S(r_k,r_{k+1})}\operatorname{Re}\frac{z^p}{p\,\xi^p}\,d\nu(\xi)\Big| = \sum_{p=1}^{[\alpha]}\sum_{k=1}^{n+1}\Big|\int_{S(r_k,r_{k+1})}\operatorname{Re}\frac{\xi^{-p}}{|p|z^{-p}}\,d\nu(\xi)\Big| \le$$

$$\le 8\pi A\sum_{p=1}^{[\alpha]}\sum_{k=1}^{n+1}\frac{r_{k+1}^\alpha}{r^{-p}}\max\{r_k^{-p}, r_{k+1}^{-p}\} = \text{const.}\sum_{p=1}^{[\alpha]}\sum_{k=1}^{n+1} r^p\,\frac{r_{k+1}^\alpha}{r_k^p} \le$$

$$\le \text{const.}\sum_{p=1}^{[\alpha]}\sum_{k=1}^{n+1} a^{-\alpha}r^p r_k^{\alpha-p} \le \text{const.}[\alpha]r^\alpha(n+1) \le \text{const.}\,|z|^\alpha\log|z|. \tag{4.7}$$

Estimate I_5. Consider the expansion in the Taylor series with the center $\xi = \infty$ of a holomorphic branch of following function of ξ:

$$\mathcal{G}(z/\xi, [\alpha]) \overset{\text{def}}{=} \log\Big(1 - \frac{z}{\xi}\Big) + \sum_{p=1}^{[\alpha]}\frac{z^p}{p\xi^p} = \sum_{p=[\alpha]+1}^{\infty} -\frac{z^p}{p\xi^p}, \quad |\xi| \ge r_{n+2}.$$

Hence, for $G\big(z/\xi, [\alpha]\big) = \operatorname{Re}\mathcal{G}(z/\xi, [\alpha])$ we have for $k \ge n+2$ by Lemma 2

$$\Big|\int_{S(r_k,r_{k+1})} G(z/\xi, [\alpha])d\nu(\xi)\Big| \le \sum_{p=[\alpha]+1}^{\infty}\Big|\int_{S(r_k,r_{k+1})}\operatorname{Re}\frac{z^p}{p\xi^p}\,d\nu(\xi)\Big| \le$$

$$\le \text{const.}\,r_{k+1}^\alpha\sum_{p=[\alpha]+1}^{\infty}\frac{|z|^p}{r_k^p} = \text{const.}\,r_{k+1}^\alpha\Big(\frac{|z|}{r_k}\Big)^{[\alpha]+1}\frac{1}{1 - |z|/r_k} \le \text{const.}\,\frac{|z|^{[\alpha]+1}}{r_k^{1-\{\alpha\}}}$$

where $\{\alpha\}$ is the fractional part of α, and const. is independent of k. Summing over $k \geq n+2$, we get

$$\big|I_5(z)\big| \leq \text{const.}\,|z|^{[\alpha]+1}\frac{1}{r_{n+2}^{1-\{\alpha\}}}\sum_{k=n+2}^{\infty}\left(\frac{r_{n+2}}{r_k}\right)^{1-\{\alpha\}} \leq$$

$$\leq \text{const.}\,|z|^{\alpha}\sum_{k=0}^{\infty}\left(\frac{1}{(1+a)^k}\right)^{1-\{\alpha\}} = \text{const.}\,|z|^{\alpha}\frac{(1+a)^{1-\{\alpha\}}}{(1+a)^{1-\{\alpha\}}-1} \leq \text{const.}\,|z|^{\alpha}. \tag{4.8}$$

Estimate I_6. We set

$$Q_k^+(\theta) = \big\{\xi = te^{i\varphi} \in S(r_k, r_{k+1}) : 0 \leq \varphi - \theta \leq 1\big\}, \quad n-1 \leq k \leq n+1,$$
$$Q_k^-(\theta) = \big\{\xi = te^{i\varphi} \in S(r_k, r_{k+1}) : -1 \leq \varphi - \theta < 0\big\}, \quad n-1 \leq k \leq n+1.$$

We confine ourselves by an estimate of an integral over the polar rectangle Q_n^+. The integrals over Q_n^-, $Q_k^{\pm}$, $k \in \{n+1, n-1\}$, can be estimated similarly.

For the convenience we renumber all intervals of type (2.12) from Q_n^+ counterclockwise. More particularly, a set of different enumerated intervals $p_l = \big[r_n e^{i\varphi_l}, r_{n+1}e^{i\varphi_l}\big)$, $l \in \{1, 2, \dots, q\}$, should coincide with the set of all intervals (2.12) situated in Q_n^+, and $\varphi_l < \varphi_{l+1}$ for $l \in \{1, 2, \dots, q-1\}$. By (2.3), we have the condition

$$br_n^{-s} \leq \varphi_{l+1} - \varphi_l \leq b^{-1}r_n^{-s}, \quad l \in \{1, 2, \dots, q-1\}. \tag{4.9}$$

Therefore,

$$\frac{1}{2}\,lbr_n^{-s} \leq \varphi_l - \theta \leq 2lb^{-1}r_n^{-s}, \quad \text{for } 2 \leq l \leq q. \tag{4.10}$$

The left-hand side of (4.10) implies

$$q \leq \text{const.}\,r_n^s, \tag{4.11}$$

where const. depends only on a, b, s.

Set $\nu_l(t) = \nu\big(\,[r_n e^{i\varphi_l}, te^{i\varphi_l})\,\big)$. Then

$$\int_{Q_n^+(\theta)} \log|z-\xi|\,d\nu(\xi) =$$

$$= \int_{r_n}^{r_{n+1}} \log|z - te^{i\varphi_l}|\,d\nu_1(t) + \sum_{l=2}^{q}\int_{r_n}^{r_{n+1}} \log|z - te^{i\varphi_l}|\,d\nu_l(t) = J + \Sigma, \tag{4.12}$$

and, according to (2.5),

$$\sum_{l=k}^{m} \nu_l(t) \leq \text{const.}\,r_{n+1}^{\alpha} \ \text{ for all } t \text{ and } 1 \leq k \leq m \leq q. \tag{4.13}$$

Estimate J. In order to estimate the integral J over the interval p_1 from (4.12) we use the traditional method of normal points.

Suppose that the point z is $(N|z|^{\beta}, 1/2)$-normal with respect to the measure $\mu_1+\mu_2$, i. e.,

$$\mu_1(z,t) + \mu_2(z,t) \leq N|z|^{\beta}t \ \text{ for every } \quad t \leq \frac{1}{2}|z| \tag{4.14}$$

where $\beta = \alpha - 1 + s \geq \rho - 1 \geq -1$ because $s \geq \rho - \alpha$ and $\rho \geq 0$. It follows from [14, § 2, Normal Points Lemma] that the set of points that are not $(N|z|^\beta, 1/2)$-normal with respect to the measure $\mu_1 + \mu_2$ is contained in a countable set of disks $D(z_k, t_k)$, $t_k \leq |z_k|/2$, satisfying the condition

$$\sum_{R/2 \leq |z_k| < 2R} t_k \leq \frac{\text{const.}}{N} R^{\rho-\beta} = \frac{\text{const.}}{N} R^{1+\rho-\alpha-s} \tag{4.15}$$

for sufficiently large values $R \geq R_0$.

Let $\tilde{\mu}_1$ and $\tilde{\mu}_2$ be restrictions of two measures μ_1 and μ_2, respectively, on the set $D_\beta = D\big(z, |z|^{-\beta}/2\big) \cap p_1$. In particular, $|z|^{-\beta}/2 \leq |z|/2$ because $-\beta \leq 1$. Then, by (4.14), using the integration by parts, we get

$$\Big| \int_{D_\beta} \log|z - \xi| \, d(\tilde{\mu}_1 - \tilde{\mu}_2)(\xi) \Big| \leq \Big| \big(\tilde{\mu}_1\big(z, |z|^{-\beta}/2\big) - \tilde{\mu}_2\big(z, |z|^{-\beta}/2\big)\big) \log \frac{1}{2}|z|^{-\beta} \Big| + \\ + \int_0^{\frac{1}{2}|z|^{-\beta}} \frac{\mu_1(z,t) + \mu_2(z,t)}{t} \, dt \leq \text{const.} N \log|z|. \tag{4.16}$$

Let $\tilde{\nu}_1$ be the restriction of charge ν on $p_1 \setminus D_\beta$. By (4.13) we have $\big|\tilde{\nu}_1(t)\big| \leq \text{const.}\, r_{n+1}^\alpha$ for all $t \in [r_n, r_{n+1})$ and $|z - te^{i\varphi_1}| \geq |z|^{-\beta}/2$ for $te^{i\varphi_1} \in \operatorname{supp} \tilde{\nu}_1$. Hence, using integration by parts, we get the inequality

$$\Big| \int_{r_n}^{r_{n+1}} \log|z - te^{i\varphi_1}| \, d\tilde{\nu}_1(t) \Big| \leq \text{const.}\, r_{n+1}^\alpha \Big(\log|z| + \int_{\tilde{p}_1} \frac{\big|t - r\cos(\varphi_1 - \theta)\big|}{|z - te^{i\varphi_1}|^2} \, dt \Big) \tag{4.17}$$

where $\tilde{p}_1 = \big\{ t | \; te^{i\varphi_1} \in \operatorname{supp} \tilde{\nu}_1 \big\}$.

For $|\varphi - \theta| \leq 1$ we have the inequality

$$\frac{\big|t - r\cos(\varphi - \theta)\big|}{|z - te^{i\varphi}|^2} = \frac{\big|t - r + 2r\sin^2\frac{\varphi-\theta}{2}\big|}{|t-r|^2 + 4rt\sin^2\frac{\varphi-\theta}{2}} \leq \frac{|t-r|}{|t-r|^2 + crt|\varphi - \theta|^2} + \frac{1}{2t} \tag{4.18}$$

where c is a positive constant.

Assume $\varphi_1 - \theta > \frac{1}{2}|z|^{-\beta-1}$. Then inequality (4.18) implies

$$\int_{r_n}^{r_{n+1}} \frac{\big|t - r\cos(\varphi_1 - \theta)\big|}{|z - te^{i\varphi_1}|^2} \, dt \leq \int_{r_n}^{r_{n+1}} \frac{|t-r| \, dt}{|t-r|^2 + c_1|z|^{-2\beta}} + \frac{1}{2}\log\frac{r_{n+1}}{r_n} \leq \\ \leq 2\int_0^{r_{n+1}} \frac{dx}{x + c_1|z|^{-2\beta}} + \text{const.} \leq \text{const.} \log|z| \tag{4.19}$$

where a positive constant c_1 depends only on α and β.

In the case $\varphi_1 - \theta \leq \frac{1}{2}|z|^{-\beta-1}$ we have $\operatorname{supp}\, \tilde{\nu}_1 \subset e^{i\varphi_1} \cdot \big\{ [r_n, r - \gamma r^{-\beta}] \cup [r + \gamma r^{-\beta}, r_{n+1}] \big\}$ where γ is a positive constant. Consequently, in that case, if we use inequality (4.18) then the last integral from (4.17) can be estimated as

$$\int_{\tilde{p}_1} \frac{\big|t - r\cos(\varphi_1 - \theta)\big|}{|z - te^{i\varphi_1}|^2} \, dt = \Big(\int_{r_n}^{r-\gamma r^{-\beta}} + \int_{r+\gamma r^{-\beta}}^{r_{n+1}} \Big) \frac{\big|t - r\cos(\varphi_1 - \theta)\big|}{|z - te^{i\varphi}|^2} \, dt \leq \\ \leq \Big(\int_{r_n}^{r-\gamma r^{-\beta}} + \int_{r+\gamma r^{-\beta}}^{r_{n+1}} \Big) \frac{dt}{|t-r|} + \int_{r_n}^{r_{n+1}} \frac{1}{2t} \, dt \leq \int_{\gamma r^{-\beta}}^{r_{n+1}} \frac{dx}{x} + \text{const.} \leq \text{const.} \log|z|.$$

Hence, according to (4.16)–(4.19) the integral J over the interval p_1 from (4.12) can be estimated by const. $N\log|z|$ + const. $|z|^\alpha\log|z|$, when the point $z=re^{i\theta}\in S(r_n,r_{n+1})$ with $r_n\ge 2$ is not $(N|z|^\beta,1/2)$-normal with respect to the measure $\mu_1+\mu_2$, i. e., in view of (4.15) outside exceptional set (2.7).

Estimate Σ. Now we estimate the rest of the sum Σ of integrals over intervals p_l, $l\ge 2$, from (4.12). In this item we will use an improvement of technique of the proof of Lemma 1.

Using integration by parts to every integral from (4.12) for $l\ge 2$ we obtain

$$|\Sigma| = \Big|\sum_{l=2}^{q}\int_{r_n}^{r_{n+1}}\log|z-te^{i\varphi_l}|\,d\nu_l(t)\Big| \le \Big|\sum_{l=2}^{q}\nu_l(r_{n+1})\log|z-r_{n+1}e^{i\varphi_l}|\Big| +$$

$$+\Big|\sum_{l=2}^{q}\nu_l(r_n)\log|z-r_ne^{i\varphi_l}|\Big| + \Big|\int_{r_n}^{r_{n+1}}\sum_{l=2}^{q}\frac{t-r\cos(\varphi_l-\theta)}{|z-te^{i\varphi_l}|^2}\nu_l(t)\,dt\Big| \tag{4.20}$$

It follows from (4.9) and (4.10) that for $t\in[r_n,r_{n+1})$

$$\big|\log|z-te^{i\varphi_{l+1}}|-\log|z-te^{i\varphi_l}|\big| \le \log\Big(1+\frac{t|e^{i\varphi_l}-e^{i\varphi_{l+1}}|}{|z-te^{i\varphi_l}|}\Big) \le \text{const.}\,\frac{t|\varphi_{l+1}-\varphi_l|}{l\;r_n^{1-s}} \le \frac{\text{const.}}{l}$$

where const. depends only on a,b,s. Further we use this estimate and the Abel transform of sum in order to estimate the first sum in the right-hand side from (4.20):

$$\Big|\sum_{l=2}^{q}\nu(r_{n+1})\log|z-r_{n+1}e^{i\varphi_l}|\Big| \le \sum_{l=2}^{q-1}\big|\log|z-te^{i\varphi_{l+1}}|-\log|z-te^{i\varphi_l}|\big|\,\Big|\sum_{m=2}^{l}\nu_m(r_{n+1})\Big| +$$

$$+\,\text{const.}\log|z-te^{i\varphi_q}|\,\Big|\sum_{m=2}^{l}\nu_m(r_{n+1})\Big| \le \sum_{l=2}^{q-1}\frac{\text{const.}}{l}\,\Big|\sum_{m=2}^{l}\nu_m(r_{n+1})\Big| +$$

$$+\,\text{const.}\log|z|\,\Big|\sum_{m=2}^{l}\nu_m(r_{n+1})\Big| \le \text{const.}\,r_{n+1}^{\alpha}\big(\log q+\log|z|\big) \le \text{const.}\,|z|^\alpha\ln|z|$$

where we use (4.10), (4.13) and (4.11). The second sum in the right-hand side from (4.20) can be estimated just as the first one.

It follows from (4.9)–(4.10) that for $t\in[r_n,r_{n+1})$

$$\Big|\frac{t-r\cos(\varphi_l-\theta)}{|z-te^{i\varphi_l}|^2}-\frac{t-r\cos(\varphi_{l+1}-\theta)}{|z-te^{i\varphi_{l+1}}|^2}\Big| \le$$

$$\le 2\Big|\frac{r(r+t)|t-r|\sin\frac{\varphi_{l+1}-\varphi_l}{2}\sin\frac{\varphi_{l+1}+\varphi_l-2\theta}{2}}{|z-te^{i\varphi_{l+1}}|^2|z-te^{i\varphi_l}|^2}\Big| \le$$

$$\le \text{const.}\,\frac{r_n^2|t-r||\varphi_{l+1}-\varphi_l||\varphi_{l+1}-\theta|}{\big(|t-r|^2+\frac14 rt|\varphi_l-\theta|^2\big)^2} \le \text{const.}\,\frac{|t-r|lr_n^{2-2s}}{\big(|t-r|^2+c^2l^2r_n^{2-2s}\big)^2}$$

where the positive constant c depends only on a,b,s. Using the Abel transform, the last inequality and (4.13) we get

$$\Big| \int\limits_{r_n}^{r_{n+1}} \sum_{l=2}^{q} \frac{t - r\cos(\varphi_l - \theta)}{|z - te^{i\varphi_l}|^2} \nu_l(t)\,dt \Big| \le$$

$$\le \text{const.} \int\limits_{r_n}^{r_{n+1}} \sum_{l=2}^{q-1} \frac{|t-r|\; l r_n^{2-2s}}{(|t-r|^2 + c^2 l^2 r_n^{2-2s})^2} \Big| \sum_{m=2}^{l} \nu_m(t) \Big|\, dt + \text{const.}\, r_{n+1}^{\alpha} \int\limits_{r_n}^{r_{n+1}} \frac{|t - r\cos(\varphi_q - \theta)|}{|z - te^{i\varphi_q}|^2}\, dt.$$

The last integral can be estimated just as in (4.19). By (4.13), the first integral in the right-hand side can be estimated by

$$\text{const.}\, r_{n+1}^{\alpha} \sum_{l=2}^{q} \int\limits_{r}^{+\infty} \frac{c^2 l r_n^{2-2s}(t-r)\,dt}{\big((t-r)^2 + c^2 l^2 r_n^{2-2s}\big)^2}.$$

The change of variable $x = \left(\dfrac{t-r}{clr^{1-s}}\right)^2$ shows that the last sum is equal to

$$\sum_{l=2}^{q} \frac{1}{2l} \int\limits_{0}^{+\infty} \frac{dx}{(x+1)^2} \le \text{const.} \log q \le \text{const.} \log|z|$$

where for last step we use (4.11).

Thus, we have the estimate

$$\Bigg| \int\limits_{Q_n^{+}(\theta)} \log|z - \xi|\, d\nu(\xi) \Bigg| \le \text{const.}\, N \log|z| + \text{const.}\, |z|^{\alpha} \log|z| \le \text{const.}\, N|z|^{\alpha} \log|z|$$

when z lies outside exceptional set (2.7).

Main Theorem is proved.

§ 5. Proofs of Corollaries 1 and 3

Proof of Corollary 1. Choose $\gamma' > \gamma$ and set $s = 1 + \rho - \alpha + \gamma'$, $N = 1$. Then, according to conditions (1.6), it is easy to construct an s-narrow system (2.1) satisfying conditions (2.5) with $\mu_1 = \mu$ and $\mu_2 = \nu$. Further, it is enough to apply Main Theorem. □

Proof of Corollary 3. If we put $\lambda = \mu_u$ in Corollary 2 then there is a δ-subharmonic function v with the Riesz charge $\mu_u \ge 0$ satisfying (1.9) outside $E_\gamma = \bigcup_{k=1}^{\infty} D(z_k, t_k)$, $t_k \le |z_k|/2$, and (1.8) holds as well. Then $\Delta(u - v) = 0$ and by Weil lemma the function $H = u - v$ is harmonic. In view of (1.9) $|u(z) - H(z)| = |v(z)| = O\big(|z|^\rho \log|z|\big)$ as $z \to \infty$ outside E_γ. Hence

$$H(z) \le u(z) + O\big(|z|^\rho \log|z|\big) \text{ as } z \to \infty \text{ outside } E_\gamma. \tag{5.1}$$

In view of (1.8) there is an increasing sequence of positive numbers $r_k \to \infty$ such that $r_{k+1}/r_k = O(1)$, $k \to \infty$, and $z \notin E_\gamma$ if $|z| = r_k$. Therefore, by (5.1) we get $H(z) \le u(z) + O\Big(|z|^\rho \log|z|\Big)$ when $|z| = r_k \ge 1$, and H is the harmonic function of order ρ. Every harmonic function H of finite order ρ can be represented in the form $H = \operatorname{Re} p$ where p is a polynomial of the degree $\le \rho$. Therefore, $|H(z)| = O\big(|z|^\rho\big)$, $z \to \infty$. The last relation together with (5.1) complete the proof. □

Remark. Analogues of results of this paper can be proved for an arbitrary proximate order $\alpha(r) \longrightarrow \alpha$ $(r \to \infty)$ instead of a constant $\alpha \geq 0$ in (2.5) and (1.6). For example, in this case, the right-hand side of relation (1.7) need to be changed by

$$O\Big(r^{\alpha(r)} \log r + r^{[\alpha]} \int\limits_0^r t^{\alpha(t)-[\alpha]-1}\, dt + r^{[\alpha]+1} \int\limits_r^{+\infty} t^{\alpha(t)-[\alpha]-2}\, dt\Big), \quad r = |z| \to +\infty.$$

§ 6. Counterexamples

Let u_1 be a function in SH_ρ with the Riesz measure μ_1 in the following two counterexamples (see the introduction) for the implication ($\hat{\mathrm{m}}$)⇒($\hat{\mathrm{u}}$) of R. S. Yulmukhametov's criterion from [10, Theorem 1].

Note that relation (1.5) is informative only provided that $\gamma \leq 2$.

Counterexample (for $\sigma = 0$). Let $\rho > 0$, and δ_w is the Dirac measure, i. e., the unit mass at $w \in \mathbb{C}$. Consider $u_2(z) = u_1(z) + \log|z - w|$, $z \in \mathbb{C}$. Then $u_2 \in SH_\rho$, and its Riesz measure is $\mu_2 = \mu_1 + \delta_w$. For this case we obtain

$$N(z, R; \mu_1, \mu_2) = \int_0^R \frac{\delta_w(z,\tau)}{\tau}\, d\tau = \begin{cases} 0, & \text{for } R < |z-w|. \\ \displaystyle\int_{|z-w|}^R \frac{1}{\tau}\, d\tau & \text{for } R \geq |z-w|. \end{cases}$$

Hence, under the notation $\log^+ t \overset{\text{def}}{=} \max\{0, \log t\}$, if $R \leq |z|$ then

$$N(z, R; \mu_1, \mu_2) \leq \log^+ \frac{R}{|z-w|} \leq \log^+ \frac{|z|}{|z-w|} \leq \log\Big(1 + \frac{|w|}{|z-w|}\Big) \leq \log 2$$

when $|z-w| \geq |w|$. Thus, relation (1.4) is realized outside the exceptional set $E = D\big(w, |w|\big)$ for which the sum on the left-hand side of (1.5) with $z_1 = w$, $t_1 = |w|$ and $k = 1$ vanishes for every $R > 0$. In other words, relations (1.4) and (1.5) hold for each γ with $\sigma = 0$, $C_\gamma \equiv \log 2$ and $E_\gamma = D\big(w, |w|\big) = D(z_1, t_1)$, i. e. in this case assertion ($\hat{\mathrm{m}}$) is fulfilled for $\sigma = 0$.

Now let us suppose that, given $\gamma \leq 1$, there exists a harmonic function H such that there are a constant C'_γ and an exceptional set $E'_\gamma \subset \bigcup_{k=1}^\infty D(z_k, t_k)$ for which (1.5) holds and

$$\big|u_1(z) - u_2(z) + H(z)\big| = \big|-\log|z-w| + H(z)\big| \leq C'_\gamma, \quad z \notin E'_\gamma. \tag{6.1}$$

Then by (1.5) and $\gamma \leq 1$ there exists a sequence of positive numbers r_k, $r_k \to +\infty$, $k \to \infty$, such that $z \notin E'_\gamma$ when $|z| = r_k$. Hence by (6.1) we obtain $H(z) = O(\log r_k)$, $|z| = r_k \to +\infty$. But such harmonic functions in $\mathbb{C}$ is a constant. The last fact contradicts to (6.1).

Counterexample (for $\sigma = 1$). Let $\rho > 1$. Consider the subharmonic function $u_0(z) = \log\big|1/\Gamma(-z-1)\big|$ where Γ is the classical gamma function. Recall that $1/\Gamma(-z-1)$ is an entire function of order 1 with zero set $\mathbb{N}$ and all zeros of this function are simple [2, Ch. I, § 11]. The latter means that the Riesz measure of the subharmonic function u_0 is the sum of Dirac measures $\sum_{n=1}^\infty \delta_n = \mu_0 \in \mathcal{M}_1$. Set $u_2 = u_1 + u_0$. The measure $\mu_2 = \mu_1 + \mu_0$ is the Riesz measure of $u_2 \in SH_\rho$ because $\rho > 1$.

Given $\gamma \leq 2$, we construct the exceptional set $E_\gamma = \cup_{k=0}^\infty D(k, (k+1)^{\gamma-2})$ for ($\hat{\mathrm{m}}$). If $z \in E_\gamma$ then there is a number $k_z \in \mathbb{N} = \operatorname{supp} \mu_0$ such that $|z - k_z| < (k_z + 1)^{\gamma-2}$. It follows from the agreement $\gamma \le 2$ that $|z| > k_z - (k_z+1)^{\gamma-2} \ge k_z - 1$ and $|z-k_z| < (|z|+2)^{\gamma-2}$. Therefore, $|z-k| \ge (|z|+2)^{\gamma-2}$ for every $z \notin E_\gamma$ and $k \in \mathbb{N}$. Besides, $\mu_0(z,t) \le t+1$ for every $z \in \mathbb{C}$ and $t > 0$. Thus,

$$N(z,R;\mu_1,\mu_2) = \int_0^R \frac{\mu_0(z,\tau)}{\tau}\,d\tau = \begin{cases} 0, & \text{for } R < (|z|+2)^{\gamma-2}. \\ \displaystyle\int_{(|z|+2)^{\gamma-2}}^R \frac{\tau+1}{\tau}\,d\tau & \text{for } R \ge (|z|+2)^{\gamma-2}. \end{cases}$$

Hence, for $R \in (0,|z|)$ and $z \notin E_\gamma$, we obtain the inequalities

$$N(z,R;\mu_1,\mu_2) \le R + \log\frac{R}{(|z|+2)^{\gamma-2}} \le |z| + (3-\gamma)\log(2+|z|) \le (7-2\gamma)|z|, \quad z \notin E_\gamma,$$

which give (1.4) for $\sigma = 1$.

Set $z_k = k-1$ and $t_k = k^{\gamma-2}$, i. e. $E_\gamma = \cup_{k=1}^\infty D(z_k,t_k)$ by the construction of E_γ. In addition to (1.4) with $\sigma = 1$, we have

$$\sum_{R/2<|z_k|<2R} t_k = \sum_{R/2<k-1<2R} k^{\gamma-2} = O(R^{\gamma-1}), \quad R \to +\infty.$$

The last gives (1.5). Thus, the assertion ($\hat{\text{m}}$) is in this case fulfilled for $\sigma = 1$.

Suppose now that, given $\gamma \le 2$, there exists a harmonic function H such that there are a constant C'_γ and an exceptional set $E'_\gamma \subset \bigcup_{k=1}^\infty D(z_k,t_k)$ for which hold (1.5) and

$$\bigl|u_1(z) - u_2(z) + H(z)\bigr| = \bigl|-\log\bigl|1/\Gamma(-z-1)\bigr| + H(z)\bigr| \le C'_\gamma|z|, \quad z \notin E'_\gamma. \tag{6.2}$$

In particular, relation (1.5) for $\gamma \le 2$ implies that *for every angle* $S(\infty;\varphi,\psi) \overset{\text{def}}{=} \{z = te^{i\theta} : \varphi \le \theta < \psi\} \ne \varnothing$ there exists a sequence of points $\zeta_k \to \infty$, $k \in \mathbb{N}$, such that $\zeta_k \in S(\infty;\varphi,\psi) \setminus E'_\gamma$. Since the function $\log\bigl|1/\Gamma(-z-1)\bigr|$ is a subharmonic function of finite order 1, by (6.2), the function H is the same. Moreover, every harmonic function H of finite order 1 can be represented in the form $H = \operatorname{Re} p$ where p is a polynomial of degree ≤ 1. Therefore, $|H(z)| = O(|z|)$, $z \to \infty$. Hence, in view of (6.2), we obtain $\bigl|\log|\Gamma(-z-1)|\bigr| = O(|z|)$, $z \in \mathbb{C} \setminus E'_\gamma$, $z \to \infty$. In particular, $\bigl|\log|\Gamma(-\zeta_k-1)|\bigr| = O(|\zeta_k|)$, $k \to \infty$, where the sequence ζ_k is as above. This contradicts to well-known asymptotic behavior of the gamma function because for every $\varepsilon > 0$ there are constants $c_\varepsilon > 0$ and $r_\varepsilon > 1$ such that $\log|\Gamma(-z-1)| \le -c_\varepsilon|z|\log|z|$ as $z \in S(\infty;\pi/2+\varepsilon, 3\pi/2-\varepsilon) \setminus D(r_\varepsilon)$ [2, Ch. I, § 11].

Similar counterexamples can be constructed also for other values σ.

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