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I. E. Chyzhykov

AN ESTIMATE OF A SIZE OF THE SET WHERE THE MODULUS OF AN ANALYTIC FUNCTION IS GREATER THAN 1

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For an entire function f of arbitrary rapid growth we obtain a sharp estimate of a measure of the set $|f(z)|>1$ in terms of the growth of $M(r,f)=\max\{|f(z)|:|z|=r\}$. The corresponding example is constructed. We also consider the similar problem for analytic functions in the unit disc.

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Для целой функций f сколь угодно быстрого роста в терминах возрастания $M(r,f)=\max\{|f(z)|:|z|=r\}$ получена точная оценка меры множества, на котором $|f(z)|>1$. Построен соответствующий пример. Рассмотрена подобная задача для аналитических в единичном круге функций.

1. Introduction and main results. Let $f(z)$ be an analytic or meromorphic function in $\{z : |z| < R \le +\infty\}$, $M(r,f)=\max\{|f(z)| : |z|=r\}$ be its maximum modulus, $T(r,f)=m(r,f)+N(r,f)$ be the Nevanlinna characteristic of the function f ($m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\ln^+|f(re^{i\varphi})|\,d\varphi$, $N(r,f)$ is Nevanlinna's counting function of the poles).

The order ρ and the lower order μ of a meromorphic function in $\mathbb{C}$, $f\not\equiv$ const, are defined by the formulas

$$\rho[f]=\varlimsup_{r\to+\infty}\frac{\ln T(r,f)}{\ln r},\quad \mu[f]=\varliminf_{r\to+\infty}\frac{\ln T(r,f)}{\ln r}.$$

If f is an entire function, then one can take $T(r,f)$ in the definition of the (lower) order instead of $\ln^+ M(r,f)$. The Nevanlinna deficiency and the quantity of deviation in the sense of V. P. Petrenko, which characterize the rate of approaching f to ∞, is defined as

$$\delta(\infty,f)=\varliminf_{r\to+\infty}\frac{m(r,f)}{T(r,f)},\quad \beta(\infty,f)=\varliminf_{r\to+\infty}\frac{\ln^+ M(r,f)}{T(r,f)},$$

respectively.

Let $\Gamma_r=\{z:|z|=r\}$, $0<r<\infty$. For a function $c\colon\mathbb{R}_+\to\mathbb{R}$ we put

$$D_c=\{z\in\mathbb{C}:\ln|f(z)|>c(|z|)\}.$$

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What can be said on a size of the set D_c depending on $c(|z|)$? The question has been considered by many authors in different cases.

An approach consists in estimating of the angular measure of $D_c \cap \Gamma_r$ as $r \to +\infty$. In 1965 A. Edrei [1] introduced the notion of the *spread* of a meromorphic function

$$\sigma(\infty, f) = \varlimsup_{r\to+\infty} \operatorname{mes}\{\varphi : re^{i\varphi} \in D_0\},$$

where $D_0 = \{z : |f(z)| > 1\}$.

In 1973 A. Baernstein [2] with the aid of introduced by himself $*$-function obtained the sharp estimate (the so called *spread relation*) for this value via the lower order μ and the deficiency $\delta(\infty, f)$: $\sigma(\infty, f) \ge \min\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\delta(\infty, f)/2}\}$. It had been conjectured, independently, by Teichmuller and Edrei. The spread relation remains valid if in the definition of $\sigma(\infty, f)$ we take an arbitrary $c(|z|)$ satisfying $|c(r)| = o(T(r, f))$ $(r \to +\infty)$, instead of $c(|z|) \equiv 0$. Therefore, there are two principal cases $c(r) = c = \text{const}$ and $c(r) = \alpha T(r, f)$, $\alpha \in \mathbb{R}$. In the second case, sharp estimates were obtained in [3] in terms of α, $\alpha > 0$, and the lower order μ even in more general situation when $\ln|f(z)|$ is replaced by an arbitrary δ-subharmonic function $u(z)$ of the lower order μ. If $\alpha < 0$, the problem is still open (see [3]). I. I. Marchenko proved a sharp estimate for $\sigma(\infty, f)$ via quantity of Petrenko's deviation $\beta(\infty, f)$ instead of $\delta(\infty, f)$ [4].

Another, more old, approach consists in estimating the area of D_c. It is based on Carleman's method [5]. Let $c(r) \equiv c$, $E = \{r : \Gamma_r \cap D_c = \Gamma_r\}$, $E^* = [0, \infty) \setminus E$. If $r \in E^*$, then $\Gamma_r \cap D_c = \bigsqcup_{k=1}^{n(r)} A_k(r)$ where $A_k(r)$, $k \in \{1, \dots, n(r)\}$ are open arcs, the components of $\Gamma_r \cap D_c$. Let $r\theta_k(r)$ be their lengths. For $r \in E^*$ we define $\theta_f(r) = \max_k \theta_k(r)$. The value $\theta_f(r)$ is related to the growth of f in the following way [8] (see also [9]).

Theorem A. *Let f be an analytic function in $\{z : |z| < R\}$, $c(r) \equiv$ const, and $\theta_f(r)$ be defined as above. Then for an arbitrary $a = a(r)$, $0 < a(r) < 1$, we have*

$$\ln\ln M(r) > \pi \int_{E_r^*} \frac{dt}{t\theta_f(t)} + \frac{1}{2}\ln\ln\frac{1}{a(r)} + K(r_0), \quad r_0 < r < R, \tag{1.1}$$

where $E_r^* = E^* \cap [r_0, a(r)r]$, $K(r_0)$ is a constant depending on r_0 and the function f only.

Remark 1. In the case when $a(r) \equiv$ const, and the function f is entire, Theorem A is proved in [8, Theorem 1]. But the proof is valid for the formulated assertion as well.

Using this theorem, A. A. Gol'dberg in [10] proved the conjecture of W. Hayman stating that the condition

$$\int^{\infty} \frac{r\,dr}{\ln\ln M(r, f)} < +\infty \tag{1.2}$$

defines the minimal growth providing finiteness of the area of D_c, $c = \text{const}$. An example constructed in [10] shows that condition (1.2) cannot be improved, more precisely, for any continuous positive nondecreasing functions $\Phi(r)$ on $[0, \infty)$ satisfying $\int_1^\infty (\Phi(r))^{-1} r\,dr < \infty$, there exists an entire function f such that $\ln\ln M(r, f) = O(\Phi(r))$, $r \to \infty$, and the area of D_c is finite for an arbitrary constant $c \in \mathbb{R}$.

A result in another direction is due to A. Edrei and P. Erdős. Results from [11] imply the following assertion

Theorem B ([11, Corollary 1.1]). *Let f be an entire function. If there exists a constant $c \in \mathbb{R}$ such that the area of D_c is finite for $c(r) \equiv c$, then*

$$\underline{\lim}_{r\to\infty} \frac{\ln\ln\ln M(r,f)}{\ln r} \geq 2.$$

In [11] it is proved also that under the condition $\overline{\lim}_{r\to\infty} \ln\ln\ln M(r,f)/\ln r < 2$ the area of the set closer to $\{z : r < |z| < 2r, \ln|f(z)| > \frac{1}{2}T(r,f)\}$ is at least r^d for sufficiently small $d > 0$ and all large r.

It is not difficult to show that (1.2) implies $\underline{\lim}_{r\to+\infty} \ln\ln\ln M(r,f)/\ln r \geq 2$. Therefore Theorem B is a consequence of the mentioned A. Gol'dberg's result.

Using estimate (1.1) in the paper [12] the following theorem is established.

Theorem C. *Let f be an entire function of order ρ $(1 \leq \rho < \infty)$ and $A(r)$ denote the area of the region $D_0 \cap \{z : |z| < r\}$. Then*

$$\overline{\lim}_{r\to\infty} \frac{A(r)}{r^2} \geq \frac{\pi}{2\rho}.$$

Note that $A(r) \geq \int_0^r \theta_f(t)t\,dt$.

The estimate given by Theorem C is sharp. For the Mittag-Leffler function $E_\rho(z)$ of order $\rho \in [1/2, \infty)$ we have $A(r) = \dfrac{\pi}{2\rho} r^2(1 + o(1))$ $(r \to \infty)$.

The latter results imply that the set D_c can have zero plane density for entire functions of infinite order. The question arises: *what is the sharp below estimate for measure of the set $D_c \cap \{z : |z| < r\}$ as $r \to +\infty$, in terms of growth of $M(r,f)$ without restrictions on the growth?*

We generalize Theorem C onto the case of entire functions with arbitrary rapid growth using the method from [12].

Theorem 1. *Suppose that the function $\Lambda(r) = \displaystyle\int_1^r \frac{\lambda(t)dt}{t}$, where $\lambda(t)$ is nondecreasing continuous, and $\lambda(t) \geq 1$ for $t \geq 1$, satisfies*

$$\Lambda\Big(r\Big(1 - \frac{1}{\Lambda(r)}\Big)\Big) \sim \Lambda(r) \tag{1.3}$$

If $\overline{\lim}_{r\to\infty} \dfrac{\ln\ln M(r)}{\Lambda(r)} \leq 1$, *then* $\underline{\lim}_{r\to\infty} \dfrac{\int_1^r \lambda^2(t)\theta_f(t)t^{-1}dt}{\Lambda(r)} \geq \pi$.

If $\underline{\lim}_{r\to\infty} \dfrac{\ln\ln M(r)}{\Lambda(r)} \leq 1$, *then* $\overline{\lim}_{r\to\infty} \dfrac{\int_1^r \lambda^2(t)\theta_f(t)t^{-1}dt}{\Lambda(r)} \geq \pi$.

Remark 2. For $\lambda(t) = \lambda \in [1, \infty)$ it is easy to deduce Theorem C from Theorem 1 (see [12]).

Remark 3. Condition (1.3) does not restrict the growth of the function Λ in any manner. For example, the condition $r\Lambda'(r) = o(\Lambda^2(r))$ $(r \to \infty)$ is sufficient for (1.3).

Corollary 1. *If*

$$\varlimsup_{r\to\infty}\frac{\ln\ln M(r,f)}{r^\alpha}\le 1,\quad \alpha\in[1,2],$$

then

$$(\forall\varepsilon>0):\ A(r)\ge\frac{(\pi-\varepsilon)}{\alpha^2}r^{2-\alpha},\quad r\to\infty.$$

Usually, the order of an analytic function f in the unit disc is defined as

$$\rho_M[f]=\varlimsup_{r\to-1-0}\frac{\ln^+\ln^+M(r,f)}{-\ln(1-r)}\ \text{ or }\ \rho_T[f]=\varlimsup_{r\to-1-0}\frac{\ln^+T(r,f)}{-\ln(1-r)}. \tag{1.4}$$

Since for meromorphic functions in the disc and of finite order of growth $\sigma(\infty,f)$ can equal zero even when $\beta(\infty,f)>0$, in this case in [6] the following analogues of the spread was introduced by I. I. Marchenko and A. I. Shcherba in [6]:

$$\omega(t,f)=\varlimsup_{r\to1-}\operatorname{arctg}\frac{\operatorname{mes}\{\varphi:\ln|f(re^\varphi)|>tc(r)\}}{2(1-r)},$$

with $c(r)=T(r,f)$, $c(r)=T(r,f)/(1-r)$, $c(r)=\ln M(r,f)$, $0\le r<1$, $t\in[0,\infty)$. Estimates obtained in [6] for $\omega(t,f)$ in terms of Petrenko's deviations and Nevanlinna's deficiencies are sharp for $t=0$.

In this connection we should recall V. Ya. Eiderman's result on estimates outside of exceptional sets of δ-subharmonic functions in the unit ball in $\mathbb{R}^m$ [7].

Let $B(r)=\{x\in\mathbb{R}^m:\|x\|<1\}$, $r>0$, where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^m$, and $\operatorname{mes}_k E$ means k-dimensional Lebesgue measure of a set E.

Theorem D. *Let $u(x)$ be δ-subharmonic in $B(1)$, $P(t)$ be continuous function on $[0,1)$ such that $P(t)\uparrow+\infty$ $(t\uparrow 1)$. Then there exists a set $E\subset B_1$ and a constant A depending on m only with the following properties:*

a) $u(x)<P(|x|)T(|x|,u)$ when $x\in B_1\setminus E$;

b) $mes_{m-1}(E\cap\{x\in\mathbb{R}^m:\|x\|=r\})\le A/P(r)$, $0<r<1$;

c) $mes_m(E\cap\{x\in\mathbb{R}^m:r\le\|x\|<1\})\le A\int_r^1\frac{dt}{P(t)}$, $0<r<1$,

In particular, Theorem D implies that for a function f meromorphic in the unit disc we have $\operatorname{mes}\{\varphi:\ln|f(re^{i\varphi})|>T(r,f)/(1-r)\}=O(1-r)$ as $r\uparrow1$.

The class of all analytic (meromorphic) functions in the disc of positive finite order has not such good properties as the class of all entire (meromorphic in the plane) functions. In particular, $\rho_T[f]\le\rho_M[f]\le\rho_T[f]+1$, and equalities in both inequalities are possible.

The inconvenience can be dropped, if one accepts the following definition of the order (similarly, lower order) of an analytic function,

$$\rho[f]=\varlimsup_{r\to-1-0}\frac{\ln^+\ln^+\ln^+M(r,f)}{-\ln(1-r)}. \tag{1.5}$$

For analytic in the unit disc functions the following result by K. Arima [8] is known.

Theorem E. *Let f be analytic in $U=\{z:|z|<1\}$. If $\varlimsup_{r\to1-}\theta_f(r)/(1-r)<2\pi$, then either $|f(z)|<1$ in U, or* $\varliminf_{r\to1-}\dfrac{\ln\ln M(r,f)}{-\ln(1-r)}>0.$

Prof. O. B. Skaskiv indicated that the last theorem admits the following generalization. Let B be the class of all nonnegative nondecreasing on $[1,\infty)$ functions γ such that $\gamma(2t) = O(\gamma(t))$, $\ln t = o(\gamma(t))$ $(t \to +\infty)$.

Theorem F. *Let f be analytic in $U = \{z : |z| < 1\}$. If $\gamma \in B$ and*

$$\varliminf_{r\to 1-} \frac{-\ln \theta_f(r)}{\gamma\left(\frac{1}{1-r}\right)} > 1,$$

then

$$\varliminf_{r\to 1-} \frac{\ln\ln\ln M(r,f)}{\gamma\left(\frac{1}{1-r}\right)} > 1.$$

This result is sharp in the sense that there exists an analytic function f in U such that

$$\ln \theta_f(r) \sim -(1-r)^{-1}, \quad \ln\ln\ln M_f(r) \le (1+o(1))\frac{1}{1-r}, \quad (r \to 1-)$$

(see Example 3).

The method of the proof of Theorem 1 can be applied to analytic functions in the disc. But it does not yield a sharp estimate in a class of functions of finite order of the growth by definition (1.4), but yields that for functions of positive order by definition (1.5).

Theorem 2. *Suppose that the function $\Phi(r) = \int_0^r \varphi(t)dt$, where $\varphi(t)$ is nondecreasing, continuous for $t \in [0,1)$ and*

$$\varliminf_{r\to 1-} \varphi(r)(1-r) > 0,$$

satisfies

$$\Phi\Big(r\Big(1 + \frac{1-r}{\Phi(r)}\Big)\Big) \sim \Phi(r), \quad r \to 1- \tag{1.6}$$

If $\displaystyle\varlimsup_{r\to 1-} \frac{\ln\ln M(r)}{\Phi(r)} \le 1$, *then* $\displaystyle\varliminf_{r\to 1-} \frac{\int_0^r \varphi^2(t)\theta_f(t)dt}{\Phi(r)} \ge \frac{\pi}{1 + \frac{1}{2\rho_M[f]}}$.

If $\displaystyle\varliminf_{r\to 1-} \frac{\ln\ln M(r)}{\Phi(r)} \le 1$, *then* $\displaystyle\varlimsup_{r\to 1-} \frac{\int_0^r \varphi^2(t)\theta_f(t)dt}{\Phi(r)} \ge \frac{\pi}{1 + \frac{1}{2\rho_M[f]}}$.

Corollary 2. *Suppose that conditions of Theorem 2 are fulfilled, and $\rho_M[f] = +\infty$. Then*

$$\varliminf_{r\to 1-} \frac{\int_0^r \varphi^2(t)\theta_f(t)dt}{\Phi(r)} \ge \pi.$$

Corollary 3. *Suppose that conditions of Theorem 2 are fulfilled, and $\rho_M[f] = +\infty$. Then*

$$\varliminf_{r\to 1-} \varphi(t)\theta_f(t) \ge \pi.$$

The proof of Theorem 2 is similar to that of Theorem 1.

Remark 4. Condition (1.6) does not restrict the growth of $\Phi(r)$,

2. Proof of Theorem 1 and Corollary 1.

Proof of Theorem 1. First, suppose that $\varlimsup_{r\to\infty} \dfrac{\ln\ln M(r)}{\Lambda(r)} \le 1$. Choose $a(r) = 1 - 1/\Lambda(r)$. Using the condition on Λ, we obtain

$$\Lambda(a(r)r) \sim \Lambda(r), \quad r \to \infty. \tag{2.1}$$

For chosen $a(r)$ Theorem A yields

$$\ln\ln M(r) \ge \pi \int\limits_{E_r^*} \frac{dt}{t\theta(t)} - (\frac{1}{2} + o(1)) \ln \Lambda(r), \quad r \to +\infty. \tag{2.2}$$

where $E_r^* = E^* \cap [r_0, a(r)r]$. Without loss of generality we may assume that $r_0 > 1$. By the Cauchy-Bunyakovski inequality

$$\left(\int_{E_r^*} \frac{\lambda(t)}{t} dt\right)^2 \le \int_{E_r^*} \frac{\lambda^2(t)\theta(t)}{t} dt \int_{E_r^*} \frac{dt}{t\theta(t)}.$$

Therefore, applying (2.2), we obtain for $r \ge r_1$

$$\int_{E_r^*} \frac{\lambda^2(t)\theta(t)}{t} dt \ge \left(\int_{E_r^*} \frac{\lambda(t)}{t} dt\right)^2 \frac{1}{\displaystyle\int_{E_r^*} \frac{dt}{t\theta(t)}} \ge \left(\int_{E_r^*} \frac{\lambda(t)}{t} dt\right)^2 \frac{\pi}{\ln\ln M(r) + \ln\ln \Lambda(r)} \tag{2.3}$$

Define $\theta(r) = 2\pi$ for $r \in E$, and let $E_r = E \cap [r_0, a(r)r]$. Then

$$\int_{E_r^*} \frac{\theta(t)\lambda^2(t)}{t} dt = \int_{r_0}^{a(r)r} \frac{\lambda^2(t)\theta(t)}{t} dt - 2\pi \int\limits_{E_r} \frac{\lambda^2(t)}{t} dt.$$

From the last inequality and (2.3), taking into account the inequality $\lambda(t) \ge 1$, we deduce

$$\begin{aligned} \int_{r_0}^{ar} \frac{\lambda^2(t)\theta(t)}{t} dt &\ge 2\pi \int_{E_r} \frac{\lambda^2(t)}{t} dt + \frac{\pi \left(\int_{E_r^*} \frac{\lambda(t)}{t} dt\right)^2}{\ln\ln M(r) + \ln\ln \Lambda(r)} \ge \\ &\ge \pi \int_{E_r} \frac{\lambda(t)}{t} dt + \frac{\pi \left(\Lambda(ar) - \int_{E_r} \frac{\lambda(t)}{t} dt + O(1)\right)^2}{\ln\ln M(r) + o(\Lambda(r))}, \quad r \to \infty. \end{aligned} \tag{2.4}$$

For the set E we define

$$\mu_r(E) = \frac{1}{\Lambda(r)} \int_{E\cap[1,r]} \frac{\lambda(t)}{t} dt.$$

Dividing both sides of (2.4) on $\Lambda(ar)$, using (2.1), one can obtain ($R = a(r)r$)

$$\frac{\displaystyle\int_{r_0}^{R} \frac{\lambda^2(t)\theta(t)}{t} dt}{\Lambda(R)} \ge \pi\mu_R(E) + \frac{\pi\left(1 - \mu_R(E) + o(1)\right)^2}{1 + o(1)}, \quad R \to +\infty. \tag{2.5}$$

Since $\mu_R(E) \in [0,1]$, and $\min\limits_{\mu\in[0,1]}\{\pi\mu + \pi(1-\mu)^2\} = \pi$, we have

$$\varliminf_{r\to\infty} \frac{\displaystyle\int_1^r \frac{\lambda^2(t)\theta(t)}{t}dt}{\Lambda(r)} \ge \pi.$$

If $\displaystyle\varlimsup_{r\to\infty} \frac{\ln\ln M(r)}{\Lambda(r)} \le 1$, then (2.5) holds on the corresponding sequence and we are done.

□

Proof of Corollary 1. For $\lambda(r) = \alpha r^\alpha$ we have $\Lambda(r) = r^\alpha - 1$. For $A(r)$ being the area of $\{z : |z| \ge 1, |f(z)| \ge 1\}$ the inequality $A(r) \ge B(r) = \int_1^r \theta(t)tdt$ holds. On the other hand,

$$\int_1^r \frac{\lambda^2(t)\theta(t)}{t}dt = \alpha^2\int_1^r t^{2\alpha-1}\theta(t)dt = \alpha^2\int_1^r t^{2\alpha-2}dB(t) =$$
$$= \alpha^2 B(r)r^{2\alpha-2} - 2\alpha^2(\alpha-1)\int_1^r B(t)t^{2\alpha-3}dt \le \alpha^2 B(r)r^{2\alpha-2}$$

Applying Theorem 1, we obtain

$$\int_1^r \frac{\lambda^2(t)\theta(t)}{t}dt \ge (\pi-\varepsilon)\Lambda(r) = (\pi-\varepsilon)(r^\alpha - 1), \quad r\to+\infty.$$

Thus

$$A(r)r^{2\alpha-2} \ge (\pi-\varepsilon)\frac{r^\alpha}{\alpha^2}, \quad r\to+\infty.$$

For $\alpha = 1$ we have $A(r) \ge (\pi-\varepsilon)r$. □

3. Examples on sharpness.

Example 1. We construct an example showing that the constant π in the estimate given by Theorem 1 is the best possible. The example is a modification of that constructed in [10]. Similar methods were also used in [11].

Let $l(r) \nearrow +\infty$ $(r\to+\infty)$ be a positive differentiable function on $[1,\infty)$ such that for some $\varepsilon > 0$ there exists $r_2 > 0$ such that for all $r \ge r_2$

$$rl'(r)(l(r))^{\varepsilon-2} \le K_2, \quad l\Big(r + \frac{1}{l^2(r)}\Big) \sim l(r), \quad r\to+\infty. \tag{3.1}$$

Note that the imposed conditions provide some regularity of the growth only, and does not restrict the rate of the growth $l(r)$ as $r\to+\infty$. In particular, it follows from (3.1) that

$$\int_1^\infty \frac{r(l'(r))^2}{l^3(r)}dr \le K_3 + K_2\int_1^\infty \frac{l'(r)}{l^{1+\varepsilon}(r)}dr < \infty. \tag{3.2}$$

Theorem 3. *Let a nondecreasing unbounded positive function $\lambda(x)$ on $[1,\infty)$ satisfy conditions (3.1), (3.2) with $l(r) = \lambda(r)/2$ and $\Lambda(r) = \int_1^r \lambda(t)t^{-1}\,dt$. Then there exits an entire function f such that $\ln\ln M(r,f) = (1+o(1))\Lambda(r)$ $(r\to\infty)$ and*

$$\lim_{r\to\infty} \frac{\displaystyle\int_1^r \frac{\theta_f(t)\lambda^2(t)}{t}dt}{\Lambda(r)} = \pi,$$

where $\theta_f(r)$ is defined above.

Proof. In the z-plane ($z = x + iy$) we consider the curvilinear substrip

$$S(q) = \Big\{ z = re^{i\varphi} : |\varphi| \le \frac{q\pi}{2l(r)} \Big\}, \quad q > 0.$$

Let

$$\tilde{S}(q) = \ln S(q) = \Big\{ w = u + iv : -\infty < u < +\infty, |v| \le \frac{q\pi}{2l(e^u)} \Big\},$$

here with $u = \ln r$, $v = \varphi$. We need the following theorem of S. Warshawski [13].

Theorem G. *Let $\omega(x)$ be positive and continuously differentiable for all x and suppose that $\omega(x)$ satisfies the conditions $\omega'(x) = o(1)$ $(x \to +\infty)$ and $\int_0^\infty \frac{(\omega'(x))^2}{\omega(x)} dx < +\infty$. Let ζ be a conformal mapping of the curvilinear strip $\{w : |\operatorname{Im} w| \le \omega(\operatorname{Re} w)\}$ onto the strip $\mathcal{K} = \{\zeta : |\operatorname{Im} \zeta| \le \pi/2\}$ in such a way that $\zeta(z) \to \pm\infty$ as $z \to \pm\infty$. Then there exists a real constant k such that*

$$\zeta(u + iv) = k + \frac{\pi}{2} \int_0^u \frac{dt}{\omega(t)} + i \frac{\pi v}{2\omega(u)} + o(1), \quad u \to +\infty.$$

According to (3.1) and (3.2) the function $\omega(u) = \pi/(2l(e^u))$ satisfies the conditions of Theorem G. Let us apply it to the domain $\tilde{S}(1)$, and let $\zeta(z)$ maps $\tilde{S}(1)$ onto $\mathcal{K}$.

Consider the function $\mathcal{F}(z) = \exp\exp\{2\zeta(\ln z)\}$ in $S(1)$, here $\ln z$ is the principal branch of the $\operatorname{Ln} z$ in $S(1)$.

By the Warshawski theorem we have

$$\begin{aligned} \zeta(\ln r + i\varphi) &= k + \frac{\pi}{2} \int_0^{\ln r} \frac{dt}{\omega(t)} + i \frac{\pi\varphi}{2\omega(\ln r)} + o(1) = \\ &= \int_0^{\ln r} l(e^t)\, dt + k + i\varphi l(r) + o(1) = \\ &= \int_1^r \frac{l(t)}{t} dt + k + i\varphi l(r) + o(1), \quad |\varphi| \le \frac{\pi}{2l(r)}, \ r \to +\infty. \end{aligned} \tag{3.3}$$

Hence

$$\begin{aligned} \ln|\mathcal{F}(z)| &= \ln|\exp\exp\{2\zeta(\ln z)\}| = \operatorname{Re} \exp\Big\{2k + 2\int_1^r \frac{l(t)}{t} dt + i2\varphi l(r) + o(1)\Big\} = \\ &= \exp\Big\{2k + 2\int_1^r \frac{l(t)}{t} dt + o(1)\Big\} \cos\left(2\varphi l(r) + o(1)\right) = \\ &= \exp\{2L(r) + O(1)\}(\cos(2\varphi l(r)) + o(1)), \quad |\varphi| \le \frac{\pi}{2l(r)}, \ r \to +\infty. \end{aligned} \tag{3.4}$$

The following Cauchy's type integral determines in the domain $\mathbb{C} \setminus \overline{S(3/4)}$ an analytic function

$$f(z) = \frac{1}{2\pi i} \int\limits_{-\partial S(3/4)} \frac{\mathcal{F}(\tau)}{\tau - z} d\tau.$$

The integral is absolutely convergent, because for $\tau \to \infty$, $\tau \in \partial S(3/4)$ from (3.4) one can deduce

$$\ln|\mathcal{F}(\tau)| = -\frac{1}{\sqrt{2}} \exp\{2L(|\tau|) + O(1)\}. \tag{3.5}$$

Then, the arguments repeat those from [10, p. 516–517].

The function f admits an analytic continuation to an entire function, which we denote by the same symbol, and is equal to

$$f(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int\limits_{-\partial S(3/4)} \dfrac{\mathcal{F}(\tau)}{\tau - z} d\tau, & z \notin \overline{S(3/4)}, \\ \dfrac{1}{2\pi i} \displaystyle\int\limits_{-\partial S(3/4)} \dfrac{\mathcal{F}(\tau)}{\tau - z} d\tau + \exp\exp\{2\ln\zeta(z)\}, & z \in S(3/4). \end{cases} \tag{3.6}$$

In fact, let $S_t(\frac{3}{4}) = S(\frac{3}{4}) \setminus \{z : |z| < t\}$, $t > 0$. Then

$$f_t(z) = \frac{1}{2\pi i} \int\limits_{-\partial S_t(3/4)} \frac{\mathcal{F}(\tau)}{\tau - z} d\tau$$

is an analytic continuation of $f(z)$ to $\mathbb{C} \setminus S(\frac{3}{4}) \cup \{z : |z| < t\}$. For $|z| < t$ we put $\tilde{f}(z) = f_t(z)$, $\tilde{f}$ is entire. Moreover, for $|z| < t$ by Cauchy's residues theorem,

$$\tilde{f}(z) = \frac{1}{2\pi i} \int\limits_{-\partial S_t(3/4)} \frac{\mathcal{F}(\tau)}{\tau - z} d\tau = \frac{1}{2\pi i} \int\limits_{-\partial S(3/4)} \frac{\mathcal{F}(\tau)}{\tau - z} d\tau + \exp\exp\{2\zeta(\ln z)\}.$$

We write

$$\frac{1}{2\pi i} \int\limits_{-\partial S(3/4)} \frac{\mathcal{F}(\tau)}{\tau - z} d\tau = -\frac{1}{2\pi i z} \int\limits_{-\partial S(3/4)} \mathcal{F}(\tau) d\tau + \frac{1}{2\pi i z} \int\limits_{-\partial S(3/4)} \frac{\tau \mathcal{F}(\tau)}{\tau - z} d\tau. \tag{3.7}$$

Since $\lambda(t) \nearrow +\infty$, we have $\ln t = o(L(t))$ $(t \to +\infty)$. This together with (3.5) yields

$$\int\limits_{-\partial S(3/4)} |\mathcal{F}(\tau)||d\tau| < +\infty, \qquad \int\limits_{-\partial S(3/4)} |\tau||\mathcal{F}(\tau)||d\tau| < +\infty. \tag{3.8}$$

If the distance from z to $\partial S(3/4)$ is greater than 1, then by (3.5)

$$\left| \frac{1}{2\pi i} \int\limits_{-\partial S(3/4)} \frac{\tau \mathcal{F}(\tau)}{\tau - z} d\tau \right| = O(1). \tag{3.9}$$

If $z \notin \overline{S(3/4)}$, $z = re^{i\varphi}$, $\varphi \in (0, \pi/2)$, $r > 1$, and the distance from z to $\partial S(3/4)$ is not greater than 1, then we denote by $K(z)$ the boundary of the domain

$$\left\{ \tau : ||\tau| - r| < 1, \frac{5\pi}{16 l(|\tau|)} < \arg\varphi < \frac{3\pi}{8 l(|\tau|)} \right\}.$$

If $|\zeta - re^{i\varphi}| \le l^{-2}(r)$, then $|\arg\zeta - \varphi| \le K_5 r^{-1} l^{-2}(r)$. Therefore, $\{\zeta : |\zeta - z| \le l^{-2}(r)\}$ does not intersect $\partial S(5/8)$, because for $||\tau| - r| \le l^{-2}(r)$, $\tau \in \partial S(5/8)$ the second condition (3.1) provides $\arg\tau \sim 5\pi/(16 l(r))$ as $r \to +\infty$. Thus, we have $|\tau - z| \ge l^{-2}(r)$ for $\tau \in K(z) \setminus \partial S(3/4)$.

Using Cauchy's theorem, (3.9), (3.4) and the last estimate we obtain

$$\left|\frac{1}{2\pi i}\int\limits_{-\partial S(3/4)}\frac{\tau\mathcal{F}(\tau)}{\tau-z}d\tau\right|=\left|\frac{1}{2\pi i}\int\limits_{-\partial S(3/4)}\frac{\tau\mathcal{F}(\tau)}{\tau-z}d\tau+\frac{1}{2\pi i}\int\limits_{K(z)}\frac{\tau\mathcal{F}(\tau)}{\tau-z}d\tau\right|\le$$
$$\le O(1)+\frac{1}{2\pi}\int_{K(z)\setminus\partial S(3/4)}\frac{|\tau||\mathcal{F}(\tau)|}{|\tau-z|}d\tau\le O(1)+K_6l^2(r)\max_{\tau\in K(z)}\{|\tau||\mathcal{F}(\tau)|\}r= \tag{3.10}$$
$$=O(1)+K_6r^2l^2(r)\exp\Big\{\cos\frac{5}{8}\pi\exp(2L(r)+O(1))\Big\}=$$
$$=O(1)+o(1)=O(1),\quad r\to+\infty.$$

Similarly we deduce

$$\left|\frac{1}{2\pi i}\int_{-\partial S(3/4)}\frac{\tau\mathcal{F}(\tau)}{\tau-z}d\tau\right|=O(1),\quad z\to\infty \tag{3.11}$$

in the other cases. In the case when $z\in S(3/4)$ we deform the integration contour such that it coincides with a part of $S(1)$ instead of $S(5/8)$ in the annulus $\{\tau:||\tau|-r|<1\}$. Estimates (3.6)–(3.11) imply

$$f(z)=\begin{cases}O\left(\frac{1}{z}\right), & z\notin S(3/4),\\ O\left(\frac{1}{z}\right)+\exp\exp\{2\zeta(\ln z)\}, & z\in S(3/4),\end{cases}\quad z\to\infty. \tag{3.12}$$

Hence, similarly to (3.4), we arrive to

$$\ln\ln M(r,f)=2L(r)+O(1),\quad r\to\infty. \tag{3.13}$$

Relation (3.12) implies that

$$\theta_f(t)\sim\frac{\pi}{2l(t)}=\frac{\pi}{\lambda(t)},\quad t\to+\infty.$$

Together with (3.13) it yields the assertion of Theorem 3 with $\Lambda(r)=2L(r)$, $\lambda(r)=2l(r)$. □

Remark 5. For $\alpha=1$ the conclusion of Corollary 1 is also sharp, because in this case we have

$$\int_1^r\frac{\lambda^2(t)\theta(t)}{t}dt=\int_1^r\theta(t)t\,dt=A(r)+O(1).$$

Now we construct an example which shows that the constant π from Corollary 2 is the best possible.

Example 2. Let f be the entire function constructed in Example 1, $f_1(w)=f\big(\frac{1}{1-w}\big)$; $w\in\mathbb{D}=\{w:|w|<1\}$, and $\mathcal{F}_1(w)=\mathcal{F}(1/(1-w))$. Suppose that $l(r)$ satisfies the additional condition $l\big((1+o(1))r\big)\sim l(r)$ $(r\to+\infty)$.

The function $z=\dfrac{1}{1-w}$ maps $\{w:|w|=d\}$, $0<d<1$ onto the circle

$$C_d=\Big\{z:\ |z-\frac{1}{1-d^2}|=\frac{d}{1-d^2}\Big\}\subset\Big\{z:|z|<\frac{1}{1-d}\Big\}.$$

Therefore,

$$\ln\ln M(d,f_1)\le \ln\ln M\Big(\frac{1}{1-d},f\Big)=2\int_1^{\frac{1}{1-d}}\frac{\lambda(t)}{t}\,dt+O(1).$$

Indeed, the inequality can be replaced by the equality. In view of the asymptotics for $\mathcal{F}$ in order to find an asymptotic for $\theta_{f_1}(t)$ $(t\to 1-)$ it is sufficient to estimate the intersection $\{w:|w|=d\}$ with the preimage of the domain $S(\frac12)$ under the mapping $z=1/(1-w)$. Let W be the part of the preimage of $S(\frac12)$ under the mapping that is contained in $\mathbb{D}$. W is a curvilinear angle with the vertex $w=1$ symmetric with respect to the real axis.

The points of the boundary $\partial S(\frac12)$ have the form $re^{i\varphi_\pm(r)}$, where $\varphi_\pm(r)=\pm\frac{\pi}{4l(r)}$.

Let $w_+=w_+(d)$, $w_-=w_-(d)$ be the complex conjugated points from $\partial W\cap\Gamma_d$, for which $\operatorname{Re}w_\pm=\max\{\operatorname{Re}w: w\in\partial W\cap C_d\}$. Let z_+, z_- be their preimages. Since $z_\pm\in\partial S(\frac12)$, $z_\pm=re^{i\varphi_\pm(r)}$, where $\varphi_\pm(r)=\pm\frac{\pi}{4l(r)}$. Moreover, the points $z_\pm$ belong to the circle C_d, i.e. $\Big|re^{i\frac{\pi}{4l(r)}}-\frac{1}{1-d^2}\Big|=\frac{d}{1-d^2}$. It implies that $r=\frac{1+o(1)}{1-d}$ $(d\to 1-)$. But $w_\pm=1-\frac{1}{z_\pm}$, hence

$$\operatorname{Im}w_+-\operatorname{Im}w_-=\operatorname{Im}\Big(\frac{1}{z_-}-\frac{1}{z_+}\Big)=\frac1r\sin\frac{\pi}{2l(r)}=(1+o(1))\frac{\pi(1-d)}{2l\big(\frac{1+o(1)}{1-d}\big)}. \tag{3.14}$$

Since $l(x(1+o(1)))\sim l(x)$ $(x\to+\infty)$, and the radian measure of the smallest arc of the circle C_d with the ends at w_+ and w_- is equivalent to $\operatorname{Im}w_+-\operatorname{Im}w_-$ as $d\to 1-$, from (3.14) we deduce that

$$\theta_{f_1}(s)\sim\frac{\pi(1-s)}{2l(\frac{1}{1-s})},\quad s\to 1-.$$

For

$$\Phi(s)=2\int_1^{\frac{1}{1-s}}\frac{l(t)}{t}\,dt=2\int_0^s\frac{l\big(\frac{1}{1-t}\big)}{1-t}\,dt,\quad \varphi(s)=2\frac{l\big(\frac{1}{1-s}\big)}{1-s}$$

we have the equality in the inequality of Corollary 2.

Example 3. For $\lambda(t)=e^t$ from Example 2 we obtain sharpness of Theorem F.

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Faculty of Mechanics and Mathematics, Lviv Ivan Franko National University

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