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ON THE MAXIMUM MODULUS POINTS OF ENTIRE AND MEROMORPHIC FUNCTIONS

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We prove upper estimates for the number of separated maximum modulus points on the circle $|z| = r$ of both entire and meromorphic functions of finite lower order.

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Доказаны оценки сверху числа различных точек максимума модуля целых и мероморфных функций конечного нижнего порядка.

Let $\nu(r)$ denote the number of maximum modulus points of an entire function $g(z)$ on the circle $|z| = r$. In 1964 P. Erdős formulated the following question: is it possible to build an entire function $g(z) \neq cz^m$ such that $\nu(r)$ is unbounded? In 1968 F. Herzog and G. Piranian [6] presented an example of an entire function with $\nu(r) \to \infty$ as $r \to \infty$.

In this paper we provide an upper estimate of the number of separated maximum modulus points on the circle $|z| = r$ for both entire and meromorphic functions of finite lower order.

We shall use the standard notations of value distribution theory of meromorphic functions: $N(r, a, f)$ and $T(r, f)$ [4, 8]. In 1969 V.P. Petrenko constructed his own theory of growth of meromorphic functions. Let us remind the basic terms of this theory.

For each $a \in \mathbb{C}$ we put :

$$\mathcal{L}(r, \infty, f) = \max_{|z|=r} \log^+ |f(z)| \ , \ \mathcal{L}(r, a, f) = \mathcal{L}\Big(r, \infty, \frac{1}{f-a}\Big) \ .$$

The quantity

$$\beta(a, f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r, a, f)}{T(r, f)}$$

is called *Petrenko's magnitude of the deviation of the meromorphic function $f(z)$ at number a.*

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Theorem A [9]. *If $f(z)$ is a meromorphic function of finite lower order λ, then for each $a \in \overline{\mathbb{C}}$*

$$\beta(a,f) \le B(\lambda) := \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & \textit{if} \;\; \lambda \le 0.5 \;, \\ \pi\lambda & \textit{if} \;\; \lambda > 0.5 \;. \end{cases}$$

Let $f(z)$ be a meromorphic function. For $0 < \eta \le 1$ and $r > 0$ we denote by $p_\eta(r,\infty,f)$ the number of component intervals of the set

$$\{\theta : \log|f(re^{i\theta})| > (1-\eta)T(r,f)\}$$

possessing at least one maximum modulus point of the function $f(z)$. We set:
$p_\eta(\infty,f) = \liminf_{r\to\infty} p_\eta(r,\infty,f)$ and $p(\infty,f) = \sup_{\{\eta\}} p_\eta(\infty,f)$.

Theorem 1. *For a meromorphic function $f(z)$ of finite lower order λ,*

$$p(\infty,f) \le \max\left(\left[2\frac{\pi\lambda}{\beta(\infty,f)}\right],1\right),$$

where $[x]$ is the integer part of x.

Corollary 1. *For an entire function $g(z)$ of finite lower order λ, we have*

$$p(\infty,g) \le \max([2\pi\lambda],1)\;.$$

Let now $g(z)$ be an entire function and let $M(r,g) = \max_{|z|=r}|g(z)|$. For $0 < \eta \le 1$ and $r > 0$ we denote by $q_\eta(r,\infty,f)$ the number of component intervals of the set

$$\{\theta : \log|g(re^{i\theta})| > (1-\eta)\log M(r,g)\}$$

possessing at least one maximum modulus point of the function $g(z)$. We set
$q_\eta(\infty,g) = \liminf_{r\to\infty} q_\eta(r,\infty,g)$ and $q(\infty,g) = \sup_{\{\eta\}} q_\eta(\infty,g)$.

Theorem 2. *For an entire function $g(z)$ of finite lower order λ and for $0 < \eta \le 1$,*

$$q_\eta(\infty,g) \le \max\left(\left[\frac{(2-\eta)}{\eta}\pi\lambda\right],1\right),$$

where $[x]$ is the integer part of x.

1. Auxiliary results. For $0 < \eta \le 1$ we consider the function

$$u_\eta(z) = \max(\log|f(z)|, (1-\eta)T(|z|,f)),$$

where $f(z)$ is a meromorphic function in $\mathbb{C}$.

Lemma 1. *The function $u_\eta(z)$ is a δ-subharmonic function in $\mathbb{C}$.*

Proof. Let $g_1(z)$ and $g_2(z)$ be entire functions without common zeros such that $f(z) = \frac{g_1(z)}{g_2(z)}$. Then we can write

$$u_\eta(z) = \max(\log|g_1(z)| - \log|g_2(z)|, (1-\eta)T(|z|, f)) =$$

$$= \max(\log|g_1(z)|, (1-\eta)T(|z|) + \log|g_2(z)|) - \log|g_2(z)|.$$

The characteristic function $T(r, f)$ is a nondecreasing and convex function of $\log r$ for $r > 0$, hence the function $T(|z|, f)$ is a subharmonic function in $\mathbb{C}$ [10]. Therefore, $u_\eta(z)$ is a difference of two subharmonic functions,

$$U_1(z) = \max(\log|g_1(z)|, (1-\eta)T(|z|) + \log|g_2(z)|)$$

and $U_2(z) = \log|g_2(z)|$. □

As $\log M(|z|, g)$ is a convex function of $\log r$ for entire functions, it is also a subharmonic function in $\mathbb{C}$. Therefore, we have the following lemma.

Lemma 2. *Let $g(z)$ be an entire function. For $0 < \eta \le 1$ the function*

$$v_\eta(z) := \max(\log|g(z)|, (1-\eta)\log M(|z|, g))$$

is a subharmonic function in $\mathbb{C}$.

For a complex number $z = re^{i\theta}$, put [1]

$$m^*(r, \theta, u_\eta) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\eta(re^{i\varphi}) d\varphi,$$

$$T^*(r, \theta, u_\eta) = T^*(re^{i\theta}) = m^*(r, \theta, u_\eta) + N(r, \infty, f),$$

where $r \in (0, \infty)$, $\theta \in [0, \pi]$, $|E|$ is the Lebesgue measure of the set E and $N(r, \infty, f)$ is the Nevanlinna counting function. Denote by $\tilde{u}_\eta(z)$ for the circular symmetrization of the function $u_\eta(z)$. The function $\tilde{u}_\eta(re^{i\varphi}) = \tilde{u}_\eta(r, \varphi)$ is non-negative and non-increasing on the interval $[0, \pi]$, even in φ and for each fixed r equimeasurable with $u_\eta(re^{i\varphi})$. Moreover, it satisfies the relations:

$$\tilde{u}_\eta(r, 0) = \max(\log M(r, f), (1-\eta)T(r, f)),$$

$$\tilde{u}_\eta(r, \pi) = \max\left(\log \min_{|z|=r}|f(z)|, (1-\eta)T(r, f)\right),$$

$$m^*(r, \theta, u_\eta) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\eta(re^{i\varphi})\, d\varphi = \frac{1}{\pi} \int_0^\theta \tilde{u}_\eta(re^{i\varphi}) d\varphi.$$

By Baernstein's theorem [1], the function $T^*(r, \theta, u_\eta)$ is subharmonic on

$$D = \{re^{i\theta} : 0 < r < \infty, 0 < \theta < \pi\},$$

continuous on $D \cup (-\infty, 0) \cup (0, +\infty)$ and logarithmically convex in $r > 0$ for each fixed $\theta \in [0, \pi]$. Moreover,

$$T^*(r, 0, u_\eta) = N(r, \infty, f), \quad T^*(r, \pi, u_\eta) \le (2-\eta)T(r, f), \quad \frac{\partial}{\partial\theta} T^*(r, \theta, u_\eta) = \frac{\tilde{u}_\eta(re^{i\theta})}{\pi}$$

for $0 < \theta < \pi$, where $T(r, f)$ is the Nevanlinna characteristic function of $f(z)$.

If for an entire function $g(z)$ we consider the properties of the function $v_\eta(z)$ and its symmetric rearrangement $\tilde{v}_\eta(re^{i\varphi}) = \tilde{v}_\eta(r, \varphi)$ in the same way as above we obtain the following relations:

$$T^*(r, \theta, v_\eta) = m^*(r, \theta, v_\eta) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E v_\eta(re^{i\varphi}) d\varphi,$$
$$\tilde{v}_\eta(r, 0) = \log M(r, g), \quad \tilde{v}_\eta(r, \pi) \geq (1 - \eta) \log M(r, g),$$
$$m^*(r, \theta, v_\eta) = \frac{1}{\pi} \int_0^\theta \tilde{v}_\eta(re^{i\varphi}) d\varphi, \quad T^*(r, 0, v_\eta) = 0,$$
$$T^*(r, \pi, v_\eta) \leq (1 - \eta) \log M(r, g) + T(r, g).$$

For a real-valued function $\alpha(r)$ of a real variable r let

$$L\alpha(r) = \liminf_{h \to 0} \frac{\alpha(re^h) + \alpha(re^{-h}) - 2\alpha(r)}{h^2}.$$

If $\alpha(r)$ is twice differentiable, then

$$L\alpha(r) = r \frac{d}{dr} r \frac{d}{dr} \alpha(r).$$

Lemma 3. *For all $0 < \eta \leq 1$, for almost all $\theta \in [0, \pi]$ and for all $r > 0$ such that on the set $\{z : |z| = r\}$ the meromorphic function $f(z)$ has neither zeros nor poles we have*

$$LT^*(r, \theta, u_\eta) \geq -\frac{p_\eta^2(r, \infty, f)}{\pi} \frac{\partial \tilde{u}_\eta(re^{i\theta})}{\partial \theta}.$$

Proof. Assume that r_0 is a number satisfying the assumption. Since $\tilde{u}_\eta(r_0, \theta)$ is a non-increasing function of θ, the derivative $\frac{\partial \tilde{u}_\eta(r_0, \theta)}{\partial \theta}$ exists for almost all $\theta \in [0, 2\pi]$. Choose $\theta \in (0, \pi)$ such that $\frac{\partial \tilde{u}_\eta(r_0, \theta)}{\partial \theta}$ exists. If $\tilde{u}_\eta(r_0, \theta) = (1 - \eta)T(r_0, f)$, then $\tilde{u}_\eta(r_0, x) = (1 - \eta)T(r_0, f)$ for all $x > \theta$, hence $\frac{\partial \tilde{u}_\eta(r_0, \theta)}{\partial \theta} = 0$. As $T^*(r, \theta, u_\eta)$ is a convex function of $\log r$, we obtain $LT^*(r_0, \theta, u_\eta) \geq 0$. Therefore, the lemma is proved in the case when $\frac{\partial \tilde{u}_\eta(r_0, \theta)}{\partial \theta} = 0$ or when $\tilde{u}_\eta(r_0, \theta) = (1 - \eta)T(r_0, f)$.

Assume now that $\frac{\partial \tilde{u}_\eta(r_0, \theta)}{\partial \theta} < 0$ and $\tilde{u}_\eta(r_0, \theta) > (1 - \eta)T(r_0, f)$. There exists a set $E(r_0, \theta)$ [1] such that

$$m^*(r_0, \theta, u_\eta) = \frac{1}{2\pi} \int_{E(r_0, \theta)} u_\eta(r_0, \varphi) d\varphi.$$

Moreover,

$$\{\varphi : u_\eta(r_0, \varphi) > \tilde{u}_\eta(r_0, \theta)\} \subset E(r_0, \theta) \subset \{\varphi : u_\eta(r_0, \varphi) \geq \tilde{u}_\eta(r_0, \theta)\}.$$

Let us now consider the function $F(\varphi) = \log |f(r_0 e^{i\varphi})|$. The set $\{\varphi : F(\varphi) = \tilde{u}_\eta(r_0, \theta)\}$ is finite. Otherwise, there would exist a convergent sequence $\{\varphi_k\}$ such that $F(\varphi_k) = \tilde{u}_\eta(r_0, \theta)$. As r_0 is chosen so that there are neither zeros nor poles of $f(z)$ on the circle $|z| = r_0$, the function $F(\varphi)$ is an analytic function of φ for $\varphi \in [0, 2\pi]$. Applying the uniqueness theorem we can state that if $F(\varphi_k) = \tilde{u}_\eta(r_0, \theta)$ then $F(\varphi) = \tilde{u}_\eta(r_0, \theta)$ for all $\varphi \in [0, 2\pi]$. This would mean that $u_\eta(r_0,\varphi) = \tilde{u}_\eta(r_0,\theta)$ for all $\varphi \in [0,2\pi]$. As a result $\frac{\partial \tilde{u}_\eta(r_0,\theta)}{\partial\theta} = 0$, which is a contradiction. Therefore, the set $\{\varphi : F(\varphi) = \tilde{u}_\eta(r_0,\theta)\}$ must be finite. This, together with our assumption that $\tilde{u}_\eta(r_0,\theta) > (1-\eta)T(r_0,f)$, leads us to the conclusion that also the set $\{\varphi : u_\eta(r_0,\varphi) = \tilde{u}_\eta(r_0,\theta)\}$ is finite. As a result,

$$m^*(r_0,\theta,u_\eta) = \frac{1}{2\pi}\int_{E_1(r_0,\theta)} u_\eta(r_0,\varphi)d\varphi,$$

where $E_1(r_0,\theta) = \{\varphi : u_\eta(r_0,\varphi) > \tilde{u}_\eta(r_0,\theta)\}$.

Let us now consider for $r > 0$ the function [3]

$$\Psi(r) = \frac{1}{2\pi}\int_{E_1(r_0,\theta)} u_\eta(r,\varphi)d\varphi.$$

We have $\Psi(r_0) = m^*(r_0,\theta,u_\eta)$ and $\Psi(r) \le m^*(r,\theta,u_\eta)$ for all $r > 0$. Hence

$$Lm^*(r_0,\theta,u_\eta) \ge L\Psi(r_0).$$

Since the set $E_1(r_0,\theta)$ is an open subset of the circle $|z| = r_0$, it implies that $E_1(r_0,\theta) = \bigcup_k(\alpha_k,\beta_k)$. As $F(\alpha_k) = F(\beta_k) = \tilde{u}_\eta(r_0,\theta)$, it follows again from the uniqueness theorem that the family of intervals (α_k,β_k) is finite. Let m_0 denote the number of those intervals.

The function $\log|f(z)|$ is harmonic on a certain neighborhood of the circle $|z| = r_0$ as $f(z)$ has neither zeros nor poles on this circle. Therefore,

$$L\Psi(r_0) = \frac{1}{2\pi}\sum_{k=1}^{m_0}\int_{\alpha_k}^{\beta_k} r_0\frac{d}{dr}r\frac{d}{dr}u_\eta(re^{i\varphi})\Big|_{r=r_0} d\varphi =$$

$$= \frac{1}{2\pi}\sum_{k=1}^{m_0}\int_{\alpha_k}^{\beta_k} r_0\frac{d}{dr}r\frac{d}{dr}\log|f(re^{i\varphi})|\Big|_{r=r_0} d\varphi = \frac{1}{2\pi}\sum_{k=1}^{m_0}\int_{\alpha_k}^{\beta_k}\left(-\frac{\partial^2 u_\eta(r_0,\varphi)}{\partial\varphi^2}\right)d\varphi =$$

$$= -\frac{1}{2\pi}\sum_{k=1}^{m_0}\left[\frac{\partial u_\eta(r_0,\varphi)}{\partial\varphi}\right]_{\alpha_k}^{\beta_k}.$$

Finally, it follows from our previous considerations that

$$Lm^*(r_0,\theta,u_\eta) \ge L\Psi(r_0) \ge -\frac{1}{2\pi}\sum_{k=1}^{m_0}\left[\frac{\partial u_\eta(r_0,\varphi)}{\partial\varphi}\right]_{\alpha_k}^{\beta_k}.$$

Following the same lines as in the proof of Lemma 1 in [7], we arrive at the following conclusion

$$Lm^*(r_0,\theta,u_\eta) \ge -\frac{m_0^2}{\pi}\frac{\partial\tilde{u}_\eta(r_0,\theta)}{\partial\theta}.$$

By definition, $p_\eta(r_0,\infty,f)$ is the number of component intervals of the set $\{\theta : |f(r_0e^{i\theta})| > (1-\eta)T(r_0,f)\}$ possessing at least one maximum modulus point of $f(z)$. On the other hand, m_0 is the number of component intervals of the set $E_1(r_0,\theta) = \{\varphi : u_\eta(r_0,\varphi) > \tilde{u}_\eta(r_0,\theta)\}$ and $\tilde{u}_\eta(r_0,\theta) \ge (1-\eta)T(r_0,f)$. Therefore, we have $m_0 \ge p_\eta(r_0,\infty,f)$. Also $LT^*(r,\theta,u_\eta) \ge Lm^*(r,\theta,u_\eta)$, so we finally obtain

$$LT^*(r_0,\theta,u_\eta) \ge -\frac{p_\eta^2(r_0,\infty,f)}{\pi}\frac{\partial\tilde{u}_\eta(r_0,\theta)}{\partial\theta}.$$

□

This lemma for $\eta = 1$ was proved by one of the authors in [7].

If in the proof of Lemma 3 we consider the function $\tilde{v}_\eta(re^{i\theta})$ instead of $\tilde{u}(re^{i\theta})$ we shall obtain the following result for entire functions.

Lemma 4. *For $0 < \eta \le 1$, for almost all $\theta \in [0, \pi]$ and for all $r > 0$ such that on the set $\{z : |z| = r\}$ the entire function $g(z)$ has no zeros we have*

$$Lm^*(r, \theta, v_\eta) \ge -\frac{q_\eta^2(r, \infty, g)}{\pi} \frac{\partial \tilde{v}_\eta(re^{i\theta})}{\partial \theta}.$$

In order to prove Theorems 1 and 2 we need two more lemmas.

Lemma 5 ([9]). *Let $f(z)$ be a meromorphic function of lower order λ. There exist sequences S_k, R_k tending to infinity such that $\lim_{k\to\infty} S_k/R_k = 0$ and for each $\varepsilon > 0$, for all $k \ge k_0(\varepsilon)$ we have*

$$\frac{T(2R_k, f)}{R_k^\lambda} + \frac{T(2S_k, f)}{S_k^\lambda} < \varepsilon \int_{2S_k}^{R_k} \frac{T(r, f)}{r^{\lambda+1}} dr.$$

If in the proof of Lemma 5 instead of the characteristic function $T(r, g)$ we put $\log M(r, g)$ for an entire function $g(z)$, we obtain the following lemma.

Lemma 6. *Let $g(z)$ be an entire function of finite lower order λ. There exist sequences $\tilde{S}_k$, $\tilde{R}_k$ tending to infinity such that $\lim_{k\to\infty} \tilde{S}_k/\tilde{R}_k = 0$ and for each $\varepsilon > 0$, for all $k \ge k_0(\varepsilon)$ we have*

$$\frac{\log M(2\tilde{R}_k, g)}{\tilde{R}_k^\lambda} + \frac{\log M(2\tilde{S}_k, g)}{\tilde{S}_k^\lambda} < \varepsilon \int_{2\tilde{S}_k}^{\tilde{R}_k} \frac{\log M(r, g)}{r^{\lambda+1}} dr.$$

2. Main results.

Proof of Theorem 1. If $\beta(\infty, f) = 0$ or $p_\eta(\infty, f) = 0$ then the theorem is obvious. Therefore, assume that $\beta(\infty, f) > 0$. Then also $p(\infty, f) > 0$. We shall first consider the case $\lambda > 0$. Let us take $p_\eta(\infty, f) > 0$.

Let α and ψ be the numbers satisfying the inequalities

$$0 < \alpha \le \min\left(\pi, \frac{\pi p_\eta(\infty, f)}{2\lambda}\right), \quad -\frac{\pi p_\eta(\infty, f)}{2\lambda} \le \psi \le \frac{\pi p_\eta(\infty, f)}{2\lambda} - \alpha.$$

Moreover, define [2]

$$\sigma(r) = \int_0^\alpha T^*(r, \varphi, u_\eta) \cos \frac{\lambda(\varphi + \psi)}{p_\eta(\infty, f)} d\varphi.$$

Applying Fatou's lemma we obtain

$$\begin{aligned} L\sigma(r) = L \int_0^\alpha T^*(r, \varphi, u_\eta) \cos \frac{\lambda(\varphi + \psi)}{p_\eta(\infty, f)} d\varphi \ge \\ \ge \int_0^\alpha LT^*(r, \varphi, u_\eta) \cos \frac{\lambda(\varphi + \psi)}{p_\eta(\infty, f)} d\varphi \ge 0. \end{aligned} \tag{1}$$

It follows from this inequality that $\sigma(r)$ is a convex function of $\log r$, thus, $r\sigma'_-(r)$ is an increasing function on $(0,\infty)$. Therefore, for almost all $r>0$,

$$L\sigma(r) = r\frac{d}{dr}r\sigma'_-(r),$$

where $\sigma'_-(r)$ is the left derivative of $\sigma(r)$ at the point r. From inequality (1) and Lemma 3 it follows that for almost all $r>0$,

$$L\sigma(r) = r\frac{d}{dr}r\sigma'_-(r) \geq -\int_0^\alpha \frac{p_\eta^2(r,\infty,f)}{\pi}\frac{\partial\tilde{u}_\eta(r,\theta)}{\partial\theta}\cos\frac{\lambda(\theta+\psi)}{p_\eta(\infty,f)}d\theta. \tag{2}$$

By definition $p_\eta(r,\infty,f)$ takes only integral values. Thus, for $r\geq r_0$ there is $p_\eta(\infty,f)\leq p_\eta(r,\infty,f)$. From this and from (2) it follows that for almost all $r\geq r_0$,

$$r\frac{d}{dr}r\sigma'_-(r) \geq -\int_0^\alpha \frac{p_\eta^2(\infty,f)}{\pi}\frac{\partial\tilde{u}_\eta(r,\theta)}{\partial\theta}\cos\frac{\lambda(\theta+\psi)}{p_\eta(\infty,f)}d\theta. \tag{3}$$

If for $r>0$ there are neither zeros nor poles of $f(z)$ on the circle $|z|=r$ the function $u_\eta(r,\theta)$ fulfills the Lipschitz condition in θ. Therefore, $\tilde{u}_\eta(r,\theta)$ also fulfills the Lipschitz condition on $[0,\pi]$ [5]. This implies that the function $\tilde{u}_\eta(r,\theta)$ is absolutely continuous on $[0,\pi]$. Integrating twice by parts we obtain for $r\geq r_0$:

$$\int_0^\alpha \frac{p_\eta^2(\infty,f)}{\pi}\frac{\partial\tilde{u}_\eta(r,\theta)}{\partial\theta}\cos\frac{\lambda(\theta+\psi)}{p_\eta(\infty,f)}d\theta = \frac{p_\eta^2(\infty,f)}{\pi}\tilde{u}_\eta(r,\alpha)\cos\frac{\lambda(\alpha+\psi)}{p_\eta(\infty,f)} -$$
$$-\frac{p_\eta^2(\infty,f)}{\pi}\tilde{u}_\eta(r,0)\cos\frac{\lambda\psi}{p_\eta(\infty,f)} + \lambda p_\eta(\infty,f)T^*(r,\alpha,u_\eta)\sin\frac{\lambda(\alpha+\psi)}{p_\eta\infty,f)} -$$
$$-\lambda p_\eta(\infty,f)N(r,\infty,f)\sin\frac{\lambda\psi}{p_\eta(\infty,f)} - \lambda^2\sigma(r) := -h_\eta(r,\lambda)-\lambda^2\sigma(r).$$

In this way we obtain the inequality

$$r\frac{d}{dr}r\sigma'_-(r) \geq h_\eta(r,\lambda)+\lambda^2\sigma(r). \tag{4}$$

Dividing this inequality by $r^{\lambda+1}$, integrating it by parts over the intervals $[2S_k,R_k]$ defined in Lemma 5 and then applying suitable estimates we obtain that for $k\geq k_0(\varepsilon)$

$$\int_{2S_k}^{R_k}\frac{h_\eta(r,\lambda)}{r^{\lambda+1}}dr < \varepsilon\int_{2S_k}^{R_k}\frac{T(r,f)}{r^{\lambda+1}}dr.$$

Therefore, for all $k\geq k_0(\varepsilon)$ there exists $r_k\in[2S_k,R_k]$ such that $h_\eta(r_k,\lambda)<\varepsilon T(r_k,f)$. From the definition of $h_\eta(r,\lambda)$ and considering the fact that $p_\eta(\infty,f)>0$ we obtain that there is a sequence $r_k\to\infty$ such that for $k\geq k_0$

$$-\frac{p_\eta^2(\infty,f)}{\pi}\tilde{u}_\eta(r_k,\alpha)\cos\frac{\lambda(\alpha+\psi)}{p_\eta(\infty,f)} + \frac{p_\eta^2(\infty,f)}{\pi}\log M(r_k,f)\ \cos\frac{\lambda\psi}{p_\eta(\infty,f)} +$$
$$+\lambda p_\eta(\infty,f)N(r_k,\infty,f)\sin\frac{\lambda\psi}{p_\eta(\infty,f)} -$$
$$-\lambda p_\eta(\infty,f)T^*(r_k,\alpha,u_\eta)\sin\frac{\lambda(\alpha+\psi)}{p_\eta(\infty,f)} < \varepsilon T(r_k,f). \tag{5}$$

Let us first assume that $\frac{\lambda}{p_\eta(\infty,f)} > \frac{1}{2}$. Then $\frac{\pi p_\eta}{2\lambda} < \pi$. We can take $\alpha = \frac{\pi p_\eta(\infty,f)}{2\lambda}$ and $\psi = 0$. In this way we have

$$\frac{p_\eta^2(\infty,f)}{\pi}\log M(r_k,f) - \lambda p_\eta(\infty,f)(2-\eta)T(r_k,f) < \varepsilon T(r_k,f).$$

Hence

$$p_\eta(\infty,f)\frac{\log M(r_k,f)}{T(r_k,f)} < (2-\eta)\pi\lambda + \varepsilon.$$

Passing to the limit with $k\to\infty$ and $\varepsilon\to 0$ we obtain

$$p_\eta(\infty,f)\beta(\infty,f) \le (2-\eta)\pi\lambda.$$

This leads us to the conclusion as $p_\eta(\infty,f)$ takes only integral values.

Let us now assume that $\frac{\lambda}{p_\eta(\infty,f)} \le \frac{1}{2}$. Then $\pi \le \frac{\pi p_\eta(\infty,f)}{2\lambda}$. We consider $p_\eta(\infty,f)\ge 2$. Take $\alpha=\pi$ and $\psi = 0$ in (5). This means that for $k\ge k_0$ there is $\tilde{u}_\eta(r_k,\alpha) = (1-\eta)T(r_k,f)$. Thus, we obtain

$$-\frac{p_\eta^2(\infty,f)}{\pi}(1-\eta)T(r_k,f)\cos\frac{\pi\lambda}{p_\eta(\infty,f)} + \frac{p_\eta^2(\infty,f)}{\pi}\log M(r_k,f)$$

$$-\lambda p_\eta(\infty,f)(2-\eta)T(r_k,f)\sin\frac{\pi\lambda}{p_\eta(\infty,f)} < \varepsilon T(r_k,f)\ .$$

Therefore,

$$p_\eta(\infty,f)(\frac{\log M(r_k,f)}{T(r_k,f)} - (1-\eta)\cos\frac{\pi\lambda}{p_\eta(\infty,f)}) < (2-\eta)\pi\lambda\sin\frac{\pi\lambda}{p_\eta(\infty,f)} + \varepsilon.$$

We have not used the fact that λ is the lower order of the function $f(z)$. Therefore, the inequality above is true for any positive number λ such that $\frac{\lambda}{p_\eta(\infty,f)} \le \frac{1}{2}$. Thus, we obtain

$$p_\eta(\infty,f)\frac{\log M(r_k,f)}{T(r_k,f)} \le (2-\eta)\pi\lambda + \varepsilon.$$

Passing to the limit as $k\to\infty$ and $\varepsilon\to 0$ we receive the statement in this case.

The proof for $\lambda = 0$ or in the case when $p(\infty,f)=\infty$ can be obtained similarly [7]. □

Proof of Theorem 2. We first consider the case $\lambda > 0$. The definition of $q_\eta(\infty,g)$ implies that

$q_\eta(\infty,g)\ge 1$. Let α and ψ be the numbers satisfying the inequalities

$$0<\alpha\le\min\left(\pi,\frac{\pi q_\eta(\infty,g)}{2\lambda}\right),\qquad -\frac{\pi q_\eta(\infty,g)}{2\lambda}\le\psi\le\frac{\pi q_\eta(\infty,g)}{2\lambda}-\alpha.$$

We put [2]

$$\sigma(r) = \int_0^\alpha T^*(r,\varphi,v_\eta)\cos\frac{\lambda(\varphi+\psi)}{q_\eta(\infty,g)}d\varphi,$$

and

$$\tilde{h}_\eta(r,\lambda) := -\frac{q_\eta^2(\infty,g)}{\pi}\tilde{v}_\eta(r,\alpha)\cos\frac{\lambda(\alpha+\psi)}{q_\eta(\infty,g)} +$$
$$+\frac{q_\eta^2(\infty,g)}{\pi}\tilde{v}_\eta(r,0)\cos\frac{\lambda\psi}{q_\eta(\infty,g)} - \lambda q_\eta(\infty,g)T^*(r,\alpha,v_\eta)\sin\frac{\lambda(\alpha+\psi)}{q_\eta(\infty,g)}.$$

As Lemma 4 and all the considerations leading to (4) are equally true for an entire function $g(z)$ and the subharmonic function $v_\eta(z)$, we obtain

$$r\frac{d}{dr}r\sigma'_-(r) \geq \tilde{h}_\eta(r,\lambda) + \lambda^2\sigma(r). \tag{6}$$

Dividing this inequality by $r^{\lambda+1}$, integrating it by parts over the intervals $[2\tilde{S}_k, \tilde{R}_k]$ defined in Lemma 6 and then applying suitable estimates we obtain that there is a sequence $r_k \to \infty$ such that for $k \geq k_0$

$$-\frac{q_\eta^2(\infty,g)}{\pi}\tilde{v}_\eta(r_k,\alpha)\cos\frac{\lambda(\alpha+\psi)}{q_\eta(\infty,g)} + \frac{q_\eta^2(\infty,g)}{\pi}\tilde{v}_\eta(r_k,0)\ \cos\frac{\lambda\psi}{q_\eta(\infty,g)}-$$

$$-\lambda q_\eta(\infty,g)T^*(r_k,\alpha,v_\eta)\sin\frac{\lambda(\alpha+\psi)}{q_\eta(\infty,g)} < \varepsilon\log M(r_k,g). \tag{7}$$

We first put $\frac{\lambda}{q_\eta(\infty,g)} > \frac{1}{2}$. Then we have $q_\eta(\infty,g) < 2\lambda < \frac{2-\eta}{\eta}\pi\lambda$ for $0 < \eta \leq 1$.

Next we put $\frac{\lambda}{q_\eta(\infty,g)} \leq \frac{1}{2}$. Then $\pi \leq \frac{\pi q_\eta(\infty,g)}{2\lambda}$. We consider $q_\eta(\infty,g) \geq 2$. Let us take $\alpha = \pi$ and $\psi = 0$ in (7). This means that for $k \geq k_0$ there is $\tilde{v}_\eta(r_k,\alpha) = (1-\eta)\log M(r_k,g)$. Thus, we obtain

$$-\frac{q_\eta^2(\infty,g)}{\pi}(1-\eta)\log M(r_k,g)\cos\frac{\pi\lambda}{q_\eta(\infty,g)} + \frac{q_\eta^2(\infty,g)}{\pi}\log M(r_k,g)-$$

$$-\lambda q_\eta(\infty,g)\{(1-\eta)\log M(r_k,g) + T(r_k,g)\}\sin\frac{\pi\lambda}{q_\eta(\infty,g)} < \varepsilon\log M(r_k,g).$$

Therefore, applying suitable estimates, and passing to the limit as $k \to \infty$ and $\varepsilon \to 0$ we obtain

$$q_\eta(\infty,f) \leq \frac{\pi\lambda}{\eta}(2-\eta)\ . \tag{8}$$

□

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