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ON COARSE ANTI-LAWSON SEMILATTICES

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The category of the anti-Lawson semilattices is isomorphic to that of the algebras for the finite hyperspace monad in the coarse category. A notion of anti-Lawson coarse semilattice is introduced. We construct an example of a coarse semilattice which is not an anti-Lawson coarse semilattice. It is proved also that every coarse semilattice of asymptotic dimension (in the sense of Gromov) zero is an anti-Lawson coarse semilattice.

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Категория антилоусоновских полурешеток изоморфна категории алгебр для монады конечного гиперпространства в грубой категории. Вводится понятие антилоусоновской грубой полурешетки. Ми строим пример грубой полурешетки, которая не является антилоусоновской грубой полурешеткой. Доказано, что каждая грубая полурешетка асимптотической размерности (в смысле Громова) ноль является антилоусоновской грубой полурешеткой.

1. Introduction. The notion of Lawson semilattice is one of the most important in the theory of compact semilattices. The Lawson semilattices are defined to be the compact semilattices possessing the base of topology consisting of subsemilattices, i.e. the compact semilattices with small subsemilattices. In the categorical setting, the coarse semilattices are characterized as the algebras of the hyperspace monad (see [1] for the case of compact Hausdorff spaces and [2] for the case of uniform spaces).

In different applications, one need to know macroscopic properties of (metric) spaces rather than their local structure. A formalization of this lead to notions of coarse spaces, coarse topological spaces, etc. In the category of coarse spaces (see, e.g. [3]) the notion of coarse semilattice can be naturally defined. Connections between the coarse semilattices and the algebras of the hyperspace monad in the category of coarse spaces are established in [4].

In this note we introduce a counterpart of the notion of Lawson semilattices in the category of coarse semilattices. We call the obtained class of coarse semilattices the anti-Lawson semilattices.

Examples of compact semilattices that are not Lawson semilattices were constructed by Lawson [5] and Gierz [6]. In Section 3 we construct two examples of metric coarse semilattices that are not anti-Lawson semilattices. Ideas of our examples are based on modifications of the examples from [5, 6].

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Lawson [7] has proved that all zero-dimensional compact semilattices are Lawson semilattices. In Section 4 we prove an asymptotic counterpart of this result.

2. Preliminaries.

2.1. Coarse structures. For the convenience of reader we recall some definitions of the coarse topology; see, e.g. [3, 8-11] for details.

Let X be a set and $M, M \subset X \times X$. The *composition* of M and N is the set

$$MN = \{(x, y) \in X \times X \mid \text{ there exists } z \in X \text{ such that } (x, z) \in M,\ (z, y) \in N\},$$

the *inverse* of M is the set $M^{-1} = \{(x, y) \in X \times X \mid (y, x) \in M\}$.

A *coarse structure* on a set X is a family $\mathcal{E}$ of subsets, which are called the *entourages*, in the product $X \times X$ that satisfies the following properties:

1. any finite union of entourages is contained in an entourage;
2. for every entourage M, its inverse M^{-1} is contained in an entourage;
3. for every entourages M, N, their composition MN is contained in an entourage;
4. $\bigcup \mathcal{E} = X \times X$.

A coarse structure on X is called *unital* if the diagonal Δ_X is contained in an entourage. A coarse structure on X is called *anti-discrete* if $X \times X$ is an entourage.

If $\mathcal{E}_1$, $\mathcal{E}_2$ are coarse structures on X, then $\mathcal{E}_1 \leq \mathcal{E}_2$ means that for every $M \in \mathcal{E}_1$ there is $N \in \mathcal{E}_2$ such that $M \subset N$.

Two coarse structures, $\mathcal{E}_1$ and $\mathcal{E}_2$, are said to be *equivalent* if $\mathcal{E}_1 \leq \mathcal{E}_2$ and $\mathcal{E}_2 \leq \mathcal{E}_1$. We usually identify coarse spaces with equivalent coarse structures.

If $\mathcal{E}$ is a coarse structure on a set X, then, obviously, the coarse structure $\mathcal{E}_1 = \{M \cup M^{-1} \mid M \in \mathcal{E}\}$ is equivalent to $\mathcal{E}$ and is *symmetric* in the sense that $N^{-1} \in \mathcal{E}_1$ for every $N \in \mathcal{E}_1$.

Given $M \in \mathcal{E}$ and $A \subset X$, we define the *M-neighborhood* $M(A)$ of A as follows: $M(A) = \{x \in X \mid (a, x) \in M \text{ for some } a \in A\}$. We use the notation $M(a)$ instead of $M(\{a\})$. A set $A \subset X$ is *bounded* if there exists $x \in X$ such that $A \subset M(x)$. Given $M \in \mathcal{E}$, we say that two sets $A, B \subset X$ are *M-close* if $A \subset M(B)$ and $B \subset M(A)$.

Let $(X_i, \mathcal{E}_i)$, $i = 1, 2$, be coarse spaces. A map $f\colon X_1 \to X_2$ is called *coarse* if the following two conditions hold:

1. for every $M \in \mathcal{E}_1$ there exists $N \in \mathcal{E}_2$ such that $(f \times f)(M) \subset N$;
2. for any bounded subset A of X_2 the set $f^{-1}(A)$ is bounded.

Let (X, d) be a metric space. The family

$$\mathcal{E}_d = \{\{(x, y) \in X \times X \mid d(x, y) < n\} \mid n \in \mathbb{N}\}$$

forms a *metric coarse structure* on X.

It is easy to see that the coarse spaces and coarse maps form a category. We denote it by CS.

The *product* of coarse spaces $(X_1, \mathcal{E}_1)$, $(X_2, \mathcal{E}_2)$ is the coarse space $(X_1 \times X_2, \mathcal{E}_1 \times \mathcal{E}_2)$, where $\mathcal{E}_1 \times \mathcal{E}_2 = \{U_1 \times U_2 \mid U_1 \in \mathcal{E}_1,\ U_2 \in \mathcal{E}_2\}$.

Let $(X_1, \mathcal{E}_1)$, $(X_2, \mathcal{E}_2)$ be coarse spaces. A coarse map $f\colon X_1 \to X_2$ is said to be *strongly open* such that for every $V \in \mathcal{E}_2$ there exists $U \in \mathcal{E}_1$ with the following property: for every $x, y \in V$ the sets $f_1(x)$ and $f^{-1}(y)$ are nonempty and U-close.

Note that the restriction $f\colon X \to \mathbb{R}$ onto the set $X = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x\}$ of the projection map onto the first coordinate is an example of a strongly open coarse map.

2.2. Finite hyperspace monad. Define the finite hyperspace functor $\exp_\omega\colon \mathrm{CS} \to \mathrm{CS}$ as follows: Given a coarse space $(X, \mathcal{E})$, endow the set

$$\exp_\omega X = \{A \subset X \mid A \neq \varnothing,\ A \text{ is finite}\}$$

with the so called Vietoris coarse structure, $\mathcal{E}_V$, consistion of the sets

$$\langle U \rangle = \{(A,B) \mid A \subset U(B),\ B \subset U(A)\},$$

where $U \in \mathcal{E}$. For a coarse map $f\colon X \to Y$, it is easy to see that the map $\exp_\omega f\colon \exp_\omega X \to \exp_\omega Y$ defined by the formula $\exp_\omega(A) = f(A)$, is coarse (see, e.g., [4]).

Recall that a monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ consisting of an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta\colon 1_{\mathcal{C}} \to T$ (unit), $\mu\colon T^2 \to T$ (multiplication) making the diagrams

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & \searrow^{1_T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

commutative (see [4] for details).

Given a monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$, we say that a pair (X, ξ) is a $\mathbb{T}$-algebra if $\xi\colon TX \to X$ is a morphism in $\mathcal{C}$ such that $\xi \circ \eta_X = 1_X$ and $\xi \circ \mu_X = \xi \circ T\xi$. The $\mathbb{T}$-algebras form a category, $\mathcal{C}^{\mathbb{T}}$; a morphism of a $\mathbb{T}$-algebra (X, ξ) into a $\mathbb{T}$-algebra (X', ξ') in $\mathcal{C}^{\mathbb{T}}$ is a morphism $f\colon X \to X'$ in $\mathcal{C}$ such that $f \circ \xi = \xi' \circ T\xi$.

Similarly as in [4], it can be proved that the functor $\exp_\omega$ generates a monad, $\mathbb{H}_\omega = (\exp_\omega, s, u)$, on the category CS. The natural transformations $s\colon 1_{\mathrm{CS}} \to \exp_\omega$ and $u\colon \exp_\omega^2 \to \exp_\omega$ are defined as follows: $s_X(x) = \{x\}$, $u_X(\mathcal{A}) = \bigcup \mathcal{A}$.

2.3. Coarse semilattices. Let $(X, \mathcal{E})$ be a coarse space and $(X, \vee)$ a semilattice. We say that $(X, \vee, \mathcal{E})$ is a *coarse semilattice* whenever $\vee\colon (X, \mathcal{E}) \times (X, \mathcal{E}) \to (X, \mathcal{E})$ is a coarse map. If no ambiguity arises, one abbreviates $(X, \vee, \mathcal{E})$ to $(X, \vee)$ or even X.

A coarse semilattice $(X, \vee, \mathcal{E})$ is said to be *anti-Lawson* if for every $U \in \mathcal{E}$ there exists $V \in \mathcal{E}$ such that for every nonempty finite subsets $A, B \subset X$ with $A \subset U(B)$ and $B \subset U(A)$ we have $(\sup A, \sup B) \in V$. (As usual, for any $C = \{c_1, \dots, c_k\} \in \exp_\omega X$ we denote by $\sup C$ the element $c_1 \vee \dots \vee c_k \in X$.)

In the case of metric coarse structures we obtain the following definition. A coarse semilattice $(X, \vee)$ is *anti-Lawson* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every nonempty finite subsets $A, B \subset X$ with $A \subset O_\varepsilon(B)$ and $B \subset O_\varepsilon(A)$ we have $d(\sup A, \sup B) < \delta$.

A typical example of a coarse semilattice is

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \mid x_i \ge 0 \text{ for every } i\}$$

endowed with the coarse structure induced by the Euclidean metric and the coordinatewise maximum semilattice operation. Note that $\mathbb{R}^n$ is an anti-Lawson semilattice.

Proposition 2.1. *Let $f\colon X_1 \to X_2$ be a strongly open homomorphism of a coarse semilattices. If X_1 is an anti-Lawson semilattice, then so is X_2.*

Proof. We denote by $\mathcal{E}_i$ the coarse structure on X_i, $i = 1, 2$. Given $U \in \mathcal{E}_2$, find $V \in \mathcal{E}_1$ such that, for any $x, y \in U$, the sets $f^{-1}(x)$ and $f^{-1}(y)$ are V-close. Since X_1 is an anti-Lawson semilattice, there exists $W \in \mathcal{E}_1$ such that, for any nonempty finite V-close subsets $A, B \subset X_1$, we have $(\sup A, \sup B) \in W$.

Since the map f is coarse, there exists $M \in \mathcal{E}_2$ such that $(f \times f)(W) \subset M$. Now, if A, B are nonempty finite U-close subsets $A, B \subset X_2$, using strong openness of the map f, one can find finite $A', B' \subset X_1$ such that $f(A') = A$, $f(B') = B$ and A', B' are V-close. Then $(\sup A', \sup B') \in W$ and

$$(\sup A, \sup B) = (f \times f)(\sup A', \sup B') \subset (f \times f)(W) \subset M.$$

□

The following result gives a characterization of the category of $\mathbb{H}_f$-algebras.

Theorem 2.2. *The category of $\mathbb{H}_\omega$-algebras is isomorphic to the category of anti-Lawson semilattices and their coarse homomorphisms.*

Proof. First, we show that every anti-Lawson semilattice, $(X, \vee)$, possesses a natural structure of $\mathbb{H}_\omega$-algebra. Given $A \in \exp_\omega X$, define $\xi(A) = \sup A$. It can be easily verified that (X, ξ) is an $\mathbb{H}_\omega$-algebra.

Given an $\mathbb{H}_\omega$-algebra (X, ξ), define a map $\vee \colon X \times X \to X$ by the formula $x \vee y = \xi(\{x, y\})$, $x, y \in X$. It is easy to see that $\vee$ is a semilattice operation. The map $\vee \colon X \times X \to X$, being a composition of two coarse maps, $g \colon X \times X \to \exp_\omega X$, $g(x, y) = \{x, y\}$, and ξ, is also a coarse map.

Note that, under this definition of the semilattice operation, we have $\xi(A) = \sup A$, for every $A \in \exp_\omega X$. It follows from the definition that $(X, \vee)$ is an anti-Lawson coarse semilattice. □

3. Examples. We provide two examples of coarse semilattices that are not anti-Lawson coarse semilattices.

3.1. In the space ℓ^∞ define the semilattice operation $\vee$ as follows: $(x_i) \vee (y_i) = (\max\{x_i, y_i\})$.

Let

$$X = \Big\{(n, (x_i), t) \in \mathbb{N} \times \ell^\infty \times \mathbb{R} \mid x_i \in \{0, 1\} \text{ for every } i,$$
$$x_i = 0 \text{ for every } i > n,\ t \geq \ln\Big(1 + \sum x_i\Big)\Big\}.$$

Given $(n, (x_i), t), (m, (y_i), s) \in X$, we let

$$(n, (x_i), t) * (m, (y_i), s) = \Big(\max\{n, m\}, (x_i) \vee (y_i), \max\Big\{t, s, \ln\Big(1 + \sum \max\{x_i, y_i\}\Big)\Big\}\Big).$$

It is easy to verify that $(X, *)$ is a semilattice.

We are going to show that $(X, *)$ is a coarse semilattice with respect to the coarse structure induced by the natural metric on X.

Fix $\varepsilon > 1$ and suppose that

$$d((n, (x_i), t), (n', (x'_i), t')) < \varepsilon,\ d((m, (y_i), s), (m', (y'_i), s')) < \varepsilon.$$

This implies that

$$|n-n'|<\varepsilon,\ \ |m-m'|<\varepsilon,\ \ |t-t'|<\varepsilon,\ \ |s-s'|<\varepsilon. \tag{3.1}$$

We estimate

$$d((n,(x_i),t)*(n',(x_i'),t'),(m,(y_i),s)*(m',(y_i'),s')).$$

Since $|\max\{n,m\},\max\{n',m'\}|<\varepsilon$ and $d((x_i)\vee(y_i),(x_i')\vee(y_i'))\le 1$, it is enough to estimate

$$D=d\left(\max\left\{t,s,\ln\left(1+\sum\max\{x_i,y_i\}\right)\right\},\max\left\{t',s',\ln\left(1+\sum\max\{x_i',y_i'\}\right)\right\}\right).$$

The rest of the proof splits into few cases. In what follows, Case (a,b) means

$$\max\left\{t,s,\ln\left(1+\sum\max\{x_i,y_i\}\right)\right\}=a,\quad \max\left\{t',s',\ln\left(1+\sum\max\{x_i',y_i'\}\right)\right\}=b.$$

Case (t,t'). Then $D<\varepsilon$, by (3.1).
Case (t,s'). Then $s\le t$, $t'\le s'$ and it follows from the inequalities

$$t-\varepsilon<t'\le s'<s+\varepsilon\le t+\varepsilon$$

that $D<\varepsilon$.
Case $(t,\ln(1+\sum\max\{x_i',y_i'\}))$. First suppose that $\max\{s',t'\}=t'$, then

$$t'\le\ln\left(1+\sum\max\{x_i',y_i'\}\right)\le\max\left\{\ln\left(1+\sum x_i'\right),\ln\left(1+\sum y_i'\right)\right\}+\ln 2\le$$
$$\le\max\{s',t'\}+\ln 2=t'+\ln 2$$

and therefore $D<\varepsilon+\ln 2$.

Now suppose that $\max\{s',t'\}=s'$, then

$$s'\le\ln\left(1+\sum\max\{x_i',y_i'\}\right)\le\max\left\{\ln\left(1+\sum x_i'\right),\ln\left(1+\sum y_i'\right)\right\}+\ln 2\le s'+\ln 2$$

and it follows from the inequalities

$$s'-\varepsilon<s\le t<t'+\varepsilon\le s'+\varepsilon$$

that $D<\ln 2+\varepsilon$.
Case $(\ln(1+\sum\max\{x_i,y_i\}),\ln(1+\sum\max\{x_i',y_i'\}))$. Then

$$\max\{t,s\}\le\ln\left(1+\sum\max\{x_i,y_i\}\right)\le\max\{t,s\}+\ln 2,$$
$$\max\{t',s'\}\le\ln\left(1+\sum\max\{x_i',y_i'\}\right)\le\max\{t',s'\}+\ln 2$$

and from the obvious inequality $|\max\{t,s\}-\max\{t',s'\}|<\varepsilon$ it follows that $D<\varepsilon+2\ln 2$.

The remaining cases are treated similarly.

It can be easily shown that the preimage under the map $\vee$ of every bounded subset in X is bounded. Therefore $\vee$ is a coarse map and $(X,\vee)$ is a coarse semilattice.

Now we show that X is not an anti-Lawson semilattice. For every $n \in \mathbb{N}$ let

$$A_n = \left\{(n, (x_i), \ln 2) \in X \mid \sum x_i = 1\right\}, \ B_n = \{(n, (1, 0, \dots, 0), \ln 2\},$$

then $A_n \subset O_2(B_n)$ and $B_n \subset O_2(A_n)$. However,

$$d(\sup A_n, \sup B_n) = d((n, (1, \dots, 1), \ln(n+1)), (n, (1, 0, \dots, 0), \ln 2)) = \ln(n+1) - \ln 2$$

and there is no $\delta > 0$ with $d(\sup A_n, \sup B_n) < \delta$.

3.2. The *hyperspace* $\exp X$ of X is the space of all nonempty compact subsets in X endowed with the so-called *Vietoris* topology.

For a metric space (X, ρ) the Vietoris topology on $\exp(X)$ is induced by the Hausdorff metric ρ_{H}:

$$\rho_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

If X is a subset of a metric linear space, we can consider the subspace

$$\mathrm{cc}(X) = \{A \in \exp(X) \mid A \text{ is convex}\}$$

(the hyperspace of compact convex subsets).

Given a subset A of a metric linear space, we denote by $\mathrm{conv}(A)$ its closed convex hull. It is well-known that for any $A, B \in \mathrm{cc}(X)$, we have

$$\mathrm{conv}(A \cup B) = \{ta + (1-t)b \mid a \in A,\ b \in B,\ t \in [0, 1]\}$$

Let K denote a Roberts compactum, i.e. a compact convex subset in a linear metric space which does not possess extreme points [12]. We assume, without loss of generality, that K is endowed with an F-norm $\|\cdot\|$ (i.e. $\|x + y\| \le \|x\| + \|y\|$ and $\|tx\| \le \|x\|$ for every t with $|t| \le 1$) and the diameter of K is ≤ 1.

Let $X = \mathbb{N} \times \mathrm{cc}(K)$. We endow X with the metric d defined as follows:

$$d((m, A), (n, B)) = \begin{cases} \max\{m, n\} & \text{if } m \ne n, \\ m d_H(A, B) & \text{if } m = n. \end{cases}$$

We leave to the reader an easy verification that d is a metric.

Define a binary operation $\vee$ on X as follows:

$$(m, A) \vee (n, B) = \begin{cases} (m, A) & \text{if } m > n, \\ (n, B) & \text{if } n > m, \\ (m, \mathrm{conv}(A \cup B)) & \text{if } n = m. \end{cases}$$

It is easy to see that $(X, \vee)$ is a semilattice.

We are going to verify that $(X, \vee)$ is a coarse semilattice. Let (m, A), (m', A'), (n, B), (n', B') be elements of X with

$$d((m, A), (m', A')) \le \varepsilon,\ d((n, B), (n', B')) \le \varepsilon.$$

We are going to estimate $D = d((m, A) \vee (n, B), (m', A') \vee (n', B'))$.

Case 1). $m \neq m'$, $n \neq n'$. Then $\max\{m, m'\} \leq \varepsilon$, $\max\{n, n'\} \leq \varepsilon$ and

$$(m, A) \vee (n, B), (m', A') \vee (n', B') \in (\mathbb{N} \cap [0, \varepsilon]) \times K.$$

We conclude that $D \leq \operatorname{diam}((\mathbb{N} \cap [0, \varepsilon]) \times K) \leq 3\varepsilon$.
Case 2). $m \neq m'$, $n = n'$. Without loss of generality, we may assume that $m' < m$. Then $m \leq \varepsilon$ and $d_H(B, B') \leq \varepsilon$.

First, suppose that $n < m$. Then $(m, A) \vee (n, B) = (m, A) \in (\mathbb{N} \cap [0, \varepsilon]) \times K$ and also $(m', A') \vee (n', B') \in (\mathbb{N} \cap [0, \varepsilon]) \times K$. Therefore, $D \leq 3\varepsilon$.

Next, suppose that $n' = n = m$. Then

$$(m, A) \vee (n, B) = (m, \operatorname{conv}(A \cup B)),\ (m', A') \vee (n', B') = (m, B').$$

Since $m \leq \varepsilon$, we have $\operatorname{diam}(\{m\} \times K) \leq \varepsilon$ and

$$D = d_H(\{m\} \times \operatorname{conv}(A \cup B)),\ \{m\} \times B') \leq \varepsilon.$$

Finally, if $m' < m < n = n'$, then

$$(m, A) \vee (n, B) = (n, B),\ (m', A') \vee (n', B') = (n', B')$$

and therefore $D \leq \varepsilon$.
Case 3). $m = m'$, $n = n'$. Without loss of generality, we may assume that $m \leq n$. The case $m < n$ is trivial. Therefore, we have to show that, if $d_H(A, A') \leq \varepsilon$, $d_H(B, B') \leq \varepsilon$, then $d_H(\operatorname{conv}(A \cup B), \operatorname{conv}(A' \cup B')) \leq \varepsilon$.

Let $c \in \operatorname{conv}(A \cup B)$, then $c = ta + (1-t)b$, where $a \in A$, $b \in B$, and $t \in [0, 1]$. There exist $a' \in A'$, $b' \in B'$, such that $d(a, a') < \varepsilon$ and $d(b, b') < \varepsilon$. Let $c' = ta' + (1-t)b'$, then

$$d(c, c') = \|t(a - a') + (1-t)(b - b')\| \leq \|a - a'\| + \|b - b'\| < 2\varepsilon,$$

by the properties of the F-norm in K.

It is also easy to see that the preimage of every bounded subset in X under the map $\vee$ is bounded in $X \times X$. Therefore $(X, \vee)$ is a coarse semilattice.

We are going to show that $(X, \vee)$ is not an anti-Lawson semilattice. Suppose the opposite and show that in this case the map $\sup\colon \exp(\operatorname{cc}(K)) \to \operatorname{cc}(K)$ is continuous.

First, we show that sup is uniformly continuous. By the anti-Lawson property, there exists $\delta > 0$ such that, for any finite subsets $Y, Z \subset X$ with $d_H(Y, Z) \leq 1$, we have $d(\sup Y, \sup Z) < \delta$.

Now suppose that $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $\delta/n < \varepsilon$.

For every subset $C \subset \operatorname{cc}(K)$ and $n \in \mathbb{N}$, let $C_n = \{\{n\} \times A \mid A \in C\}$.

Consider finite subsets $Y, Z \subset K$ with $d_H(Y, Z) \leq 1/n$, then $d_H(Y_n, Z_n) \leq 1$ and, by the choice of δ, $d_H(\sup Y_n, \sup Z_n) < \delta$. By the definition of metric in X, we conclude that $d_H(\sup Y, \sup Z) < \delta/n = \varepsilon$. Since the set $\{Y \in \exp(\operatorname{cc}(K)) \mid Y \text{ is finite}\}$ is dense in $\exp(\operatorname{cc}(K))$, the map $\sup\colon \exp(\operatorname{cc}(K)) \to \operatorname{cc}(K)$ is well-defined and continuous.

It follows from [13, Proposition VI-3.9] that $\operatorname{cc}(K)$ is a continuous lattice (see [13] for the definition). Then, as it is proved in [5], the space K can be affinely embedded into a locally convex space, a contradiction.

4. Asymptotically zero-dimensional anti-Lawson semilattices. Let $(X, \mathcal{E})$ be a coarse space and $\mathcal{U}$ a cover of X. We say that $\mathcal{U}$ is *uniformly bounded* if there exists $U \in \mathcal{E}$ such that for every $V \in \mathcal{U}$ there exists $x \in X$ such that $V \subset U(x)$.

Given $U \in \mathcal{E}$, we say that a family $\mathcal{A}$ of subsets in X is *U-discrete* provided that for every $x \in X$ there exists at most one element $A \in \mathcal{A}$ such that $A \cap U(x) \neq \varnothing$.

A coarse space $(X, \mathcal{E})$ has the *asymptotic dimension zero* (written $\operatorname{asdim}(X, \mathcal{E}) = 0$) if for every $U \in \mathcal{E}$ there exists a uniformly bounded U-discrete cover $\mathcal{U}$ of X.

The definition above is a natural modification of the definition given by Gromov [14] for the case of metric spaces.

Let X be a coarse $\vee$-semilattice, $\mathcal{E}$ the coarse structure on X. Given $U \in \mathcal{E}$, find $V \in \mathcal{E}$ such that

$$(x, x') \in U,\ (y, y') \in U \Rightarrow (x \vee x', y \vee y') \in V$$

for every $x, x', y, y' \in X$.

Lemma 4.1. *If A is a nonempty finite subset of X and $b \in U(A)$, then*

$$(\sup A, \sup(A \cup \{b\})) \in V.$$

Proof. Consider $a \in A$ with $(a, b) \in U$. Then

$$(\sup A, \sup(A \cup \{b\})) = ((\sup A) \vee a, (\sup A) \vee b) \in V.$$

□

Theorem 4.2. *Let X be a coarse $\vee$-semilattice with* $\operatorname{asdim} X = 0$. *Then X is an anti-Lawson semilattice.*

Proof. Let $U, V \in \mathcal{E}$ be as above.

There exists a uniformly bounded V-discrete cover $\mathcal{U}$ of X.

Now let A, B be nonempty finite subsets of X with $A \subset U(B), B \subset U(A)$. We are going to consider $(\sup A, \sup(A \cup B))$. Write $B = \{b_1, \dots, b_n\}$, then by Lemma 4

$$\begin{aligned}
&(\sup A, \sup(A \cup \{b_1\})) \in V,\\
&(\sup A \cup \{b_1\}, \sup(A \cup \{b_1, b_2\})) \in V,\\
&\dots\\
&(\sup A \cup \{b_1, \dots, b_{n-1}\}, \sup(A \cup B)) \in V.
\end{aligned} \tag{4.1}$$

Since the cover $\mathcal{U}$ is V-discrete, all the elements

$$\sup A,\ \sup(A \cup \{b_1\}),\ \dots,\ \sup A \cup \{b_1, \dots, b_{n-1}\},\ \sup(A \cup B)$$

belong to some element W of $\mathcal{U}$. Hence $(\sup A, \sup(A \cup B)) \in W$. Similarly, $(\sup B, \sup(A \cup B)) \in W$ and therefore $(\sup A, \sup B) \in WW^{-1} \subset \widehat{W}$, for some $\widehat{W} \in \mathcal{U}$, i.e. X is anti-Lawson.

□

5. Remarks and open questions. Let X be a metric space of diameter ≤ 1. The *open cone* of X is the set $\mathcal{O}X = (X \times \mathbb{R}_+)/(X \times \{0\})$ endowed with the metric (by $[x, t]$ we denote the equivalence class of $(x, t) \in X \times \mathbb{R}_+$):

$$d([x_1, t_1], [x_2, t_2]) = |t_1 - t_2| + \min\{t_1, t_2\} d(x_1, x_2).$$

If X is a $\vee$-semilattice, then one can define a binary operation $\tilde{\vee}$ on $\mathcal{O}X$ as follows: $[x, s]\tilde{\vee}[y, t] = [x \vee t, \max\{s, t\}]$.

Question 5.1. If X is compact, is there a compatible metric on X such that $(\mathcal{O}X, \tilde{\vee})$ with the metric defined above is a coarse semilattice?

Question 5.2. If X is a compact Lawson semilattice, is there a compatible metric on X such that $(\mathcal{O}X, \tilde{\vee})$ with the metric defined above is anti-Lawson semilattice?

In the theory of compact Hausdorff semilattices, it is proved by J. Lawson [7] that every semilattice whose underlying space is a finite-dimensional Peano continuum is a Lawson semilattice.

Question 5.3. Find a counterpart of this result in the theory of coarse semilattices.

I. V. Protasov conjectured that a result similar to Theorem 2.2 can be obtained for the monad on the category CS generated by the *bounded hyperspace functor* $\exp_b$: $\exp_b X = \{A \subset X \mid A \neq \varnothing \text{ is bounded}\}$.

I. V. Protasov [15] introduced the notion of *ball structure*, which turned out to be closely related to that of coarse structure. One can formulate and prove counterparts of the results of this note in the category of sets endowed with ball structures (the ball category). Remark that the metric spaces carry the natural ball structure; therefore our examples fit also for the ball category.

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