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THE BAIRE CATEGORIES AND WIMAN'S INEQUALITY FOR ENTIRE FUNCTIONS

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By the classical Wiman-Valiron theorem, for every entire function f there exists an increasing to $+\infty$ sequence (r_n) such that $\ln M(r_n,f)-\ln\mu(r_n,f)\le\left(\frac12+o(1)\right)\ln\ln\mu(r_n,f)$, $n\to\infty$, where $M(r,f)$ is the maximum modulus and $\mu(r,f)$ is the maximal term of f. It is known that in this theorem the constant $\frac12$ cannot be replaced by a smaller number. Using the concept of the Baire category, it is shown in the presented paper that for the majority of entire functions the constant $\frac12$ may be replaced by $\frac14$.

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Согласно классической теореме Вимана-Валирона, для любой целой функции f существует возрастающая к $+\infty$ последовательность (r_n) такая, что $\ln M(r_n,f)-\ln\mu(r_n,f)\le\left(\frac12+o(1)\right)\ln\ln\mu(r_n,f)$, $n\to\infty$, где $M(r,f)$ — максимум модуля, а $\mu(r,f)$ — максимальный член f. Известно, что в этой теореме постоянную $\frac12$ нельзя заменить меньшим числом. Используя понятие бэровской категории, в настоящей работе показано, что для большинства целых функций постоянную $\frac12$ можно заменить на $\frac14$.

1. Introduction. For numbers $r\ge 0$, $\eta>0$ and an entire function

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \tag{1}$$

let $M(r,f)=\max\{|f(z)|:|z|=r\}$ be the maximum modulus, $\mu(r,f)=\max\{|a_n|r^n:n\ge 0\}$ be the maximal term, $G_2(r,f)=\left(\sum_{n=0}^{\infty}|a_n|^2r^{2n}\right)^{1/2}$,

$$\Delta(r,f)=\frac{\ln M(r,f)-\ln\mu(r,f)}{\ln\ln\mu(r,f)},\qquad E(\eta,f)=\{r>1:M(r,f)>\mu(r,f)\ln^{\eta}\mu(r,f)\}.$$

By the classical Wiman-Valiron theorem, for every entire function f and each $\eta>\frac12$ the set $E(\eta,f)$ has the finite logarithmic measure, i.e.

$$\int_{E(\eta,f)}\frac{dr}{r}<\infty.$$

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It is easy to check that this implies (see also [1]) that for every entire function f there exists a set $E(f) \subset (1, +\infty)$ of finite depending only on f logarithmic measure such that

$$\varlimsup_{\substack{r\to+\infty\\ r\notin E(f)}} \Delta(r, f) \le \frac{1}{2}.$$

Therefore, for each $\eta > \frac{1}{2}$ there exists a number $r_0(\eta, f) > 1$ such that $E(\eta, f) \subset (1, r_0(\eta, f)) \cup E(f)$.

Note that in the Wiman–Valiron theorem the constant $\frac{1}{2}$ cannot be replaced by a smaller number. Indeed, for the entire function $f(z) = e^z = \sum_{n=0}^{\infty} z^n/n!$ we have $M(r, f) = e^r$ and [2]

$$\mu(r, f) = \frac{r^{[r]}}{[r]!} \sim \frac{e^r}{\sqrt{2\pi r}}, \qquad r \to +\infty, \tag{2}$$

and hence

$$\lim_{r\to+\infty} \frac{M(r, f)}{\mu(r, f) \ln^{\frac{1}{2}} \mu(r, f)} = \sqrt{2\pi}. \tag{3}$$

On the other hand, from the articles [3–6] it follows that for majority of entire functions (in the sense of the Lebesgue measure or the probability) the constant $\frac{1}{2}$ may be replaced by $\frac{1}{4}$. We confine only by results from [4] and [5].

Below always $\theta = (\theta_n)_{n=0}^{\infty}$ is a sequence of nonnegative integers. Consider entire function (1) and put

$$f_t(z) = \sum_{n=0}^{\infty} e^{i\theta_n t} a_n z^n, \qquad t \in \mathbb{R}.$$

Note that the function f_t is also entire and completely determined by the function f and the sequence θ. Furthermore, $\mu(r, f_t) = \mu(r, f)$ for all $t \in \mathbb{R}$ and $r \ge 0$.

Theorem A [4]. *If a sequence θ has the Hadamard gaps, i.e.*

$$\varliminf_{n\to\infty} \frac{\theta_{n+1}}{\theta_n} > 1,$$

then for every entire function f there exists a set $E(f) \subset (1, +\infty)$ of finite logarithmic measure such that

$$\varlimsup_{\substack{r\to+\infty\\ r\notin E(f)}} \Delta(r, f_t) \le \frac{1}{4}$$

for almost all $t \in \mathbb{R}$.

It is known [7, p.45] that for the function $f(z) = \sum_{n=0}^{\infty} z^n/n!$ the inequality

$$G_2(r, f) > \frac{e^r}{4r^{1/4}}, \qquad r > r_0, \tag{4}$$

is valid. Since, by the Parseval equality,

$$M(r, f_t) \ge \left(\frac{1}{2\pi}\int_0^{2\pi} |f_t(re^{i\varphi})|^2 \delta\varphi\right)^{\frac{1}{2}} = G_2(r, f),$$

from (2) and (4) we obtain

$$\varlimsup_{r\to+\infty} \frac{M(r,f_t)}{\mu(r,f)\ln^{\frac{1}{4}}\mu(r,f)} \geq \sqrt{\frac{\pi}{8}}.$$

Hence, in Theorem A the constant $\frac{1}{4}$ cannot be replaced by a smaller number.

Theorem A is generalized and refined in [5]. In particular, from results obtained in [5], the next assertion follows.

Theorem B [5]. *If a sequence θ satisfies*

$$\delta(\theta) := \varlimsup_{n\to\infty} \frac{\ln\frac{\theta_n}{\theta_{n+1}-\theta_n}}{\ln n} \leq \delta \in \left[0, \frac{1}{2}\right), \tag{5}$$

then for every entire function f there exists a set $E(f) \subset (1,+\infty)$ of finite logarithmic measure such that

$$\varlimsup_{\substack{r\to+\infty\\ r\notin E(f)}} \Delta(r,f_t) \leq \frac{1+3\delta}{4+2\delta}$$

for almost every $t \in \mathbb{R}$.

We note that $\delta(\theta) \leq 0$ for each sequence θ with the Hadamard gaps. Moreover,

$$\frac{1}{4} \leq \frac{1+3\delta}{4+2\delta} < \frac{1}{2}, \qquad \delta \in \left[0, \frac{1}{2}\right).$$

Next, we assume that a sequence θ satisfies condition (5). Let $E \subset (1,+\infty)$ and

$$F_1(f,\theta,E) = \left\{t \in \mathbb{R} : \varlimsup_{\substack{r\to+\infty\\ r\notin E}} \Delta(r,f_t) \leq \frac{1+3\delta}{4+2\delta}\right\},$$

$$F_2(f,\theta) = \left\{t \in \mathbb{R} : \int_{E(\eta,f_t)} \frac{dr}{r} < \infty \text{ for all } \eta > \frac{1+3\delta}{4+2\delta}\right\},$$

$$F_3(f,\theta) = \left\{t \in \mathbb{R} : \varliminf_{r\to+\infty} \Delta(r,f_t) \leq \frac{1+3\delta}{4+2\delta}\right\}.$$

Then from Theorem B we see that for every entire function f there exists a set $E(f)$ of finite logarithmic measure such that the set $F_1(f,\theta,E(f))$ is large in the sense of the Lebesgue measure. From this it follows that $F_2(f,\theta)$ and $F_3(f,\theta))$ are large in the sense of the Lebesgue measure.

Like the Lebesgue measure, the Baire categories are also important concepts that are useful proving that some properties are typical for a subclass of analytic functions. In particular, these concepts were applied in the value distribution theory [10, 11] as well as for investigation of the radial growth of entire functions [9]. In this connection Prof. M. Sheremeta and Prof. O. Skaskiv posed the question: Is it possible to obtain an estimate for a number of entire functions for which the Wiman-Valiron inequality can be improved in terms of Baire's categories.

In this paper we consider the following specific question: *Is it true that for every entire function f there exists a set $E(f)$ of finite logarithmic measure such that the set $F_1(f,\theta,E(f))$*

is large in the sense of the Baire categories, i.e. $F_1(f,\theta,E(f))$ *is a residual set in* $\mathbb{R}$*?* (We recall that a set $F\subset\mathbb{R}$ is called [8] *residual in* $\mathbb{R}$, if its complement $F':=\mathbb{R}\backslash F$ is a set of the first Baire category in $\mathbb{R}$.)

It is clear, that if the answer to the question is affirmative, then the sets $F_2(f,\theta)$ and $F_3(f,\theta))$ are residual in $\mathbb{R}$. However, in reality, the situation is rather complicated. Shortly speaking, below we obtain the negative answer to the question, but we show that for every entire function f the set $F_3(f,\theta)$ is residual in $\mathbb{R}$.

Theorem 1. *Let a sequence* θ *have the Hadamard gaps and* $f(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$*. There exists a constant* $C=C(\theta)>0$ *such that for each sequence* (r_n) *increasing to* $+\infty$ *the set*

$$F_4=\left\{t\in\mathbb{R}:\ \varlimsup_{n\to+\infty}\frac{M(r_n,f_t)}{\mu(r_n,f)\ln^{\frac{1}{2}}\mu(r_n,f)}\geq C\right\}$$

is residual in $\mathbb{R}$.

For every set $E\subset(1,+\infty)$ we introduce the notation $E^*:=(1,+\infty)\backslash E$. By Theorem 1, if $\sup E^*=+\infty$ (for example, if E is a set of finite logarithmic measure) and $f(z)=e^z$, then $F_1(f,\theta,E)$ is a set of the first Baire category in $\mathbb{R}$. Hence, from Theorem 1 the negative answer to the question formulated above follows.

We note, that the condition of the Hadamard's lacunarity of a sequence θ in Theorem 1 appears as a consequence of the method applied in proof of this theorem. Certainly, this condition is redundant. We can make such a conclusion considering in some sense the extreme case $\theta_n=n$ for all $n\geq 0$. In this case, for $f(z)=e^z$ and each $t\in\mathbb{R}$ we have

$$M(r,f_t)=\max_{\varphi}\left|\sum_{n=0}^{\infty}\frac{r^n e^{in(\varphi+t)}}{n!}\right|=\sum_{n=0}^{\infty}\frac{r^n}{n!}=e^r.$$

Therefore, from equality (3), which is valid for $f(z)=e^z$, we obtain the analogous equality for f_t, $t\in\mathbb{R}$, i.e. in this case Theorem 1 is also valid.

Theorem 2. *If for a sequence* θ *condition (5) is fulfilled, then for every entire function* f *the set* $F_3(f,\theta)$ *is residual in* $\mathbb{R}$.

The following question is open.

Question 1. *Is it true that for every entire function* f *the set* $F_2(f,\theta)$ *is residual in* $\mathbb{R}$*?*

The answer to the following simpler question is also unknown.

Question 2. *Is it true that there exist a number* $\eta<\frac{1}{2}$ *and a sequence* θ *such that for every entire function* f *the set of those* $t\in\mathbb{R}$ *for which the set* $E(\eta,f_t)$ *has the finite logarithmic measure (or at least the zero logarithmic density) is residual in* $\mathbb{R}$*?*

2. Auxiliary results. We apply the following lemmas in the proofs of Theorems 1 and 2. The first of them is elementary.

Lemma 1. *Let a function* $\alpha=\alpha(x,y)$ *be continuous in* $[a,b]\times[c,d]$. *Then the function* $\beta(x)=\max\{\alpha(x,y):y\in[c,d]\}$ *is continuous in* $[a,b]$.

Lemma 2. **[12, 13]** *For every $q>1$ there exist positive constants $A=A(q)$ and $B=B(q)$ such that for each interval $I\subset\mathbb{R}$ and every trigonometrical polynomial $P(t)=\sum_{n=1}^{N}c_ne^{i\lambda_nt}$, for which $|I|\geq\frac{B}{\lambda_1}>0$ and $\frac{\lambda_{n+1}}{\lambda_n}\geq q$, $1\leq n\leq N-1$, there exists a point $t_0\in I$ such that*

$$\operatorname{Re}P(t_0)\geq A\sum_{n=1}^{N}|c_n|.$$

3. Proof of Theorem 1. Let (r_n) be an arbitrary sequence increasing to $+\infty$, $f(z)=e^z=\sum_{n=0}^{\infty}z^n/n!$, θ be a sequence with the Hadamard gaps, and $q=\inf\{\frac{\theta_{n+1}}{\theta_n}:n\geq 0\}$. It is clear that $q>1$.

Let $A=A(q)$ and $B=B(q)$ be the positive constants from Lemma 2, $C=A\sqrt{2\pi}>0$. Then the constant C depends only on the sequence θ.

Consider a positive sequence (C_n), increasing to C, and put, for fixed integers $k\geq 0$ and $m\geq 0$,

$$F_{m,k}=\left\{t\in\mathbb{R}:\frac{M(r_l,f_t)}{\mu(r_l,f_t)\ln^{\frac{1}{2}}\mu(r_l,f_t)}\leq C_m\text{ for all }l\geq k\right\}.$$

Since for every fixed $r\geq 0$ the function

$$\alpha(t,\varphi)=\left|\sum_{n=0}^{\infty}e^{i\theta_nt}\frac{r^ne^{in\varphi}}{n!}\right|$$

is continuous in $\mathbb{R}^2$ and periodic in the variables t and φ, the function $\beta(t)=\max\limits_{\varphi}\alpha(t,\varphi)=M(r,f_t)$ is continuous at every point $t\in\mathbb{R}$, by Lemma 1. Consequently, the set $F_{m,k}$ is closed in $\mathbb{R}$.

Now we show that the set $F'_{m,k}$ is everywhere dense. In order to do this we consider an arbitrary interval $I\subset\mathbb{R}$, $|I|>0$, and show that it contains some point $t_0\in F'_{m,k}$.

We choose an integer $p\geq 1$ and a number $\varepsilon>0$ so that the inequalities

$$|I|\geq\frac{B}{\theta_p},\qquad 1-2\varepsilon>\sqrt{\frac{c_m}{c}}\tag{6}$$

be fulfilled. (Recall that k and m are fixed.)

Next, by (3) and the relation $r^p=o(e^r)$, $r\to+\infty$, we can define the numbers

$$x_1=\inf\left\{r\geq 0:e^r\geq\sqrt{2\pi}(1-2\varepsilon)\mu(r,f)\ln^{\frac{1}{2}}\mu(r,f)\right\},$$
$$x_2=\inf\left\{r\geq 0:\sum_{n=0}^{p}\frac{r^n}{n!}\leq\frac{A}{A+1}\varepsilon e^r\right\},$$

respectively.

Choose integers $l\geq k$ and $s>p$ so that the inequalities

$$r_l>\max\{x_1,x_2\},\qquad\sum_{n=s+1}^{\infty}\frac{r_l^n}{n!}\leq\frac{A}{A+1}\varepsilon e_l^r\tag{7}$$

be fulfilled. By Lemma 2, in the interval I there exists a point t_0 such that

$$\operatorname{Re}\sum_{n=p}^{s} e^{i\theta_n t_0}\frac{r_l^n}{n!} \ge A\sum_{n=p}^{s}\frac{r_l^n}{n!}. \tag{8}$$

Using the definitions of x_1 and x_2, by (8), (7) and (6), we obtain

$$M(r_l, f_{t_0}) = \max_{\varphi}|f_{t_0}(r_l e^{i\varphi})| \ge |f_{t_0}(r_l)| \ge \operatorname{Re} f_{t_0}(r_l) \ge \operatorname{Re}\sum_{n=p}^{s} e^{i\theta_n t_0}\frac{r_l^n}{n!} - \sum_{n\notin[p,s]}\frac{r_l^n}{n!} \ge$$
$$\ge A\sum_{n=p}^{s}\frac{r_l^n}{n!} - \sum_{n\notin[p,s]}\frac{r_l^n}{n!} = A\sum_{n=0}^{\infty}\frac{r_l^n}{n!} - (1+A)\sum_{n\notin[p,s]}\frac{r_l^n}{n!} \ge Ae^{r_l} - (1+A)\frac{2A}{A+1}\varepsilon e^r \ge$$
$$\ge \frac{C}{\sqrt{2\pi}}(1-2\varepsilon)^2\sqrt{2\pi}\mu_f(r_l,f)\ln^{\frac12}\mu_f(r_l,f) > c_m\mu_f(r_l,f)\ln^{\frac12}\mu_f(r_l,f),$$

i.e. $t_0 \in F'_{m,k}$.

Since the set $F_{m,k}$ is closed in $\mathbb{R}$ and its complement $F'_{m,k}$ is everywhere dense, $F_{m,k}$ is nowhere dense. Hence, $F = \bigcup_{m=0}^{\infty}\bigcup_{k=0}^{\infty} F_{m,k}$ is a set of the first category. But $F_4 = F'$. Therefore, F_4 is a residual set. Thus, the proof of Theorem 1 is complete.

4. Proof of Theorem 2. Let f be an arbitrary entire function, and θ be a sequence, satisfying (5). Show that the set $F_3(f,\theta)$ is residual in $\mathbb{R}$.

Consider a sequence (c_n) such that

$$c_n \downarrow \frac{1+3\delta}{4+2\delta}, \qquad n\to\infty.$$

Fix integers $m \ge 0$, $k \ge 0$, and put

$$F_{m,k} = \{t\in\mathbb{R} : M(r,f_t) \ge \mu(r,f)\ln^{c_m}\mu(r,f) \text{ for all } r>k\}.$$

As it has been stated above, for every fixed $r \ge 0$ the function $\beta(t) = M(r,f_t)$ is continuous at every point $t\in\mathbb{R}$. Hence, the set $F_{m,k}$ is closed in $\mathbb{R}$. Furthermore, it is easy to see from Theorem B, the set $F'_{m,k}$ is everywhere dense. Therefore, $F_{m,k}$ is nowhere dense and $F = \bigcup_{m=0}^{\infty}\bigcup_{k=0}^{\infty} F_{m,k}$ is a set of the first category. Since $F_3(f,\theta) = F'$, the set $F_3(f,\theta)$ is residual in $\mathbb{R}$. Theorem 2 is proved.

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