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# MODIFICATION OF THE LIE-ALGEBRAIC SCHEME AND APPROXIMATION ERROR ESTIMATIONS

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A new modification of the Lie-algebraic scheme for solving partial differential equations with initial and boundary conditions based on constructing quasirepresentations of the Heisenberg-Weyl algebra operators involving boundary conditions is proposed. Approximation errors for the modified scheme are evaluated.

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Предложена новая модификация Ли-алгебраической схемы решения уравнений в частных производных с начальными и граничными условиями, основанная на построении квазипредставлений содержащих граничные условия. Получена оценка погрешностей аппроксимации для модифицированной схемы.

**Introduction.** The Lie-algebraic method of discrete approximations was investigated in several papers which appeared during last decades [1, 3, 4, 6, 7]. Boundary conditions in the standard Lie-algebraic method are not taken into account. This makes essential inconvenience if performing appropriate change of variables giving rise to a new problem with in general much more complicated expression for the differential operator. In this paper I intend to construct appropriate quasirepresentations of the Heisenberg-Weyl algebra operators in a finite-dimensional space involving boundary condition of the Dirichlet type. This will give rise to a new modification of the Lie-algebraic scheme and will make it possible to simplify the procedure of solving Cauchy problems with boundary conditions. Approximation error for the modified scheme will be evaluated.

**1. Preliminaries.** Henceforth the following notions will be useful. The Heisenberg-Weyl algebra of differential operations is defined as $\mathcal{G} = \bigoplus_{j=1}^{q} \mathcal{G}_j = \bigoplus_{j=1}^{q} \{x_j, \partial_{x_j}, 1\}$ [6], where $x_j$ is the operator of multiplication by the variable $x_j$, $\partial_{x_j}$ is the operator of differentiation, $\mathbf{1}$ is the identity operator, $\bigoplus$ denotes the direct sum. Let the space of algebraic polynomials of degree not exceeding $n_i - 1$ with respect to the variable $x_i$, $i \in \{1, \dots, q\}$, be denoted by $\mathcal{P}_{(n)}$. Consider the lattice $\Theta$ of the $q$-dimensional cube $D = \prod_{i=1}^{q} [a_j, b_j]$ with nodes

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$\left(x_k^{(1)}, x_k^{(2)}, \ldots x_k^{(n_k)}\right)$, $k \in \{1, \ldots, q\}$, being the mesh points with respect to each variable and hence $\Theta = \left\{x_{(i)} = \left(x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_q^{(i_q)}\right) \in D,\ 1 \leq i_k \leq n_k,\ k \in \{1, \ldots, q\}\right\}$. Then the Lagrangian projectors $P_{(n)}$ on the space $\mathcal{P}_{(n)}$ acting on the functions $u \in C(D)$ are defined as follows:

$$P_{(n)}\,(x;\, u) = \sum_{(i)\in \mathrm{Im}} u(x_{(i)}) L_{(i)}(x),$$

where $\mathrm{Im} = \{(i) = (i_1,\, i_2, \ldots, i_q) :\ 1 \leq i_k \leq n_k,\ k \in \{1, \ldots, q\}\}$ is the set of multi-indices determining the nodes of the lattice $\Theta$, $L_{(i)}(x) = \prod_{k=1}^{q} l_{k,\,i_k}(x_k)$ and $l_{k,\,i_k}(x_k)$, $1 \leq i_k \leq n_k$, are fundamental Lagrangian polynomials with respect to the variable $x_k$, $k \in \{1, \ldots, q\}$. The usual Lebesgue constant $\Lambda_{(n)} = \max_{x\in D} \sum_{(i)\in\mathrm{Im}} \left|L_{(i)}\,(x)\right|$.

The space $W_\infty^{(r)}(\mathbf{M};\, D) \overset{df}{=} \overline{Q_\infty^{(r)}(\mathbf{M};\, D)}$, $(r) = (r_1, \ldots,\, r_q)$, $\mathbf{M} = (M_1, \ldots,\, M_q)$, where $Q_\infty^{(r)}(\mathbf{M};\, D) = \left\{ v \in C^{(r)}(D;\, \mathbb{R}) : \left|\dfrac{\partial^{r_i} v}{\partial x_i^{r_i}}\right|_\infty \leq M_i,\ i \in \{1, \ldots, q\} \right\}$ with $|f|_\infty = \max_{x\in D} |f(x)|$ for any $f \in C(D)$.

**2. Quasirepresentations involving boundary conditions.** In this section I intend to construct appropriate quasirepresentations of the Heisenberg-Weyl algebra operators with the domain $D(S) = W_{2,0}^r(\Omega) = \left\{u \in W_2^r(\Omega) : u|_{\partial\Omega} = 0\right\}$ $\forall S \in \mathcal{G}$. Let $\Gamma = \partial\Omega := \left\{x \in \mathbb{R}^q : \phi(x) = 0\right\}$ with a function $\phi(x)$ being given in a neighbourhood of $\Gamma$. In accordance with the Sobolev theorem on function traces [5] for any $\phi \in W_2^{r-1/2}(\partial\Omega)$ there exists some function (more precisely a class of equivalent equal almost everywhere functions) $\tilde{\phi} \in W_2^r(\Omega)$ such that $\tilde{\phi}|_{\partial\Omega} = \phi$. Let us denote $\tilde{\phi}$ again by $\phi$. Let us now consider the space $\phi W_2^r(\Omega) = \left\{u : u = \phi g, g \in W_2^r(\Omega)\right\}$. It is obvious that $\phi W_2^r(\Omega) \subset W_{2,0}^r(\Omega)$. Let us consider the following Cauchy problem

$$\frac{d}{dt}u(t) = A\big(t, x, \partial/\partial x; u\big)u(t) + f(t; u),\ 0 \leq t \leq T, \tag{1}$$

$$u(0) = \psi \in L_2(\Omega) \tag{2}$$

with the quasilinear differential operator

$$A\big(t, x, \partial/\partial x; u\big) = \sum_{|\alpha|\leq r} a_\alpha(t, x; u)\frac{\partial^{|\alpha|}}{\partial x_\alpha}, \quad x \in D, \quad \alpha \in \mathbb{Z}_+^q,$$

where $r$ is its order, $\Omega \subseteq D$. Let us seek for solutions to problem (1), (2) in the functional space $B_0 = \phi W_2^r(\Omega)$, thereby problem (1), (2) will be solved with the boundary condition

$$u|_\Gamma = 0. \tag{3}$$

Similarly to the case with no boundary condition [6] let us define mappings $\Phi_{(n),0}\colon B_0 \to B_{(n),0}$ as the composition $\Phi_{(n),0} = \pi_{(n)} \circ R_{(n)}$, where $R_{(n)}\colon B_0 \to \tilde{B}_{(n),0}$ is the projector onto the space $\tilde{B}_{(n),0} = \left\{u :\ u = \phi g,\ g \in \mathcal{P}_{(n)}\right\}$, $R_{(n)}u = \phi P_{(n)}g\ \forall u \in B_0$. Let us denote $\mathrm{Im}_\Gamma = \left\{(i) \in \mathrm{Im} :\ x_{(i)} \in \Gamma\right\}$, where $x_{(i)} \in \Theta$. The isomorphism $\pi_{(n)}$ acts onto the space

$B_{(n),0} = \{v \in \otimes_{j=1}^{q}\mathbb{R}^{n_j} : v_{(i)} = 0\ \forall (i) \in \mathrm{Im}_\Gamma\}$. Thus $R_{(n)}u = \sum_{(i)\in\mathrm{Im}} u_{(i)}e_{(i)}(x)$, where $u_{(n)} = \{u_{(i)}\}_{(i)\in\mathrm{Im}} = \{u(x_{(i)})\}_{(i)\in\mathrm{Im}} \in B_{(n),0}$ and $e_{(i)}(x) = \frac{\phi(x)}{\phi_{(i)}}L_{(i)}(x)$, $(i) \in \mathrm{Im}$ are the corresponding interpolating functions. Indeed, $e_{(i)}(x_{(s)}) = \delta_{(i),(s)}$ where $\delta_{(i),(s)} = \begin{cases} 1, & (i) = (s) \\ 0, & (i) \neq (s). \end{cases}$

Let $\hat{X}_j^{(n)}$, $\hat{Z}_j^{(n)}$, $\hat{I}^{(n)} \in \mathbb{R}^{N\times N}$, $N = n_1 \cdot n_2 \cdot \ldots \cdot n_q$, be appropriate approximations of the operators $x_j$, $\partial_{x_j}$, 1 with the above mentioned domain respectively. One can calculate a quasirepresentation $S^{(n)}$ of the operator $S \in \mathcal{G}$ using the formulae $R_{(n)}(SR_{(n)}u) = \langle S^{(n)}u_{(n)}, e_{(n)}\rangle$. For the sake of convenience, in further calculations let us define a mapping $* : \mathrm{Im} \to \aleph$ which acts from the set of multi-indices $\mathrm{Im} = \{(k) = (k_1, k_2, \ldots, k_q) : 1 \leq k_j \leq n_j, j \in \{1, \ldots, q\}\}$ onto the set of usual indices $\aleph = \{1, 2, \ldots, N\}$:

$$*(k) = n_2 \cdot \ldots \cdot n_q(k_1 - 1) + n_3 \cdot \ldots \cdot n_q(k_2 - 1) + \ldots + n_q(k_q - 1) + k_q.$$

Using the latter mapping one can use multi-indices when performing operations with matrices from the space $\mathbb{R}^{N\times N} = \otimes_{j=1}^{q}\mathbb{R}^{n_j\times n_j}$ and vectors from $\mathbb{R}^N = \otimes_{j=1}^{q}\mathbb{R}^{n_j}$ in the same way as one uses usual indices. For instance, $Zv = \left\{\sum_{(s)\in\mathrm{Im}} Z_{*(k),*(s)}v_{*(s)}\right\}_{*(k)\in\{1,\ldots,N\}}$, $\langle v, w\rangle = \sum_{(i)\in\mathrm{Im}} v_{*(i)}w_{*(i)}\ \forall Z \in \mathbb{R}^{N\times N}\,\forall v, w \in \mathbb{R}^N$.

Now let us proceed to immediate calculation of quasirepresentations:

$$R_{(n)}(x_jR_{(n)}u) = R_{(n)}\Big(x_j\sum_{(i)\in\mathrm{Im}} u_{(i)}e_{(i)}(x)\Big) = \sum_{(i)\in\mathrm{Im}} u_{(i)}\frac{\phi(x)}{\phi_{(i)}}P_{(n)}(x_jL_{(i)}(x)) =$$
$$= \sum_{(i)\in\mathrm{Im}} u_{(i)}\frac{\phi(x)}{\phi_{(i)}}\sum_{(s)\in\mathrm{Im}}(X_j^{(n)})_{*(s),*(i)}L_{(s)}(x) = \sum_{(s)\in\mathrm{Im}}\Big[\sum_{(i)\in\mathrm{Im}}(X_j^{(n)})_{*(s),*(i)}u_{*(i)}\Big]e_{(s)}(x).$$

Hence $\hat{X}_j^{(n)} = X_j^{(n)}$, namely, quasirepresentations of the operators $x_j$, $j \in \{1, \ldots, q\}$, corresponding to the mappings $\Phi_{(n),0}$ coincide with the quasirepresentations corresponding to the mappings $\Phi_{(n)}$ and basis $\{L_{(i)}(x)\}_{(i)\in\mathrm{Im}} \subset \mathcal{P}_{(n)}$. In the same way one can verify that $\hat{I}^{(n)} = I^{(n)}$. Let us calculate $\hat{Z}_j^{(n)}$:

$$R_{(n)}(\partial_{x_j}R_{(n)}u) = \sum_{(i)\in\mathrm{Im}} u_{(i)}R_{(n)}\big[\partial_{x_j}\,(\phi(x)/\phi_{(i)}L_{(i)}(x))\big] =$$
$$= \sum_{(i)\in\mathrm{Im}} u_{(i)}\phi(x)/\phi_{(i)}P_{(n)}\left[\frac{\phi_j'}{\phi}(x)L_{(i)}(x) + \partial_{x_j}L_{(i)}(x)\right] =$$
$$= \sum_{(i)\in\mathrm{Im}} u_{(i)}\phi(x)/\phi_{(i)}\sum_{(s)\in\mathrm{Im}}\left[\frac{\phi_j'}{\phi}(x_{(s)})\delta_{(s),(i)} + (Z_j^{(n)})_{*(s),*(i)}\right]L_{(s)}(x) =$$
$$= \sum_{(s)\in\mathrm{Im}}\left[\sum_{(i)\in\mathrm{Im}}\left(\frac{\phi_j'}{\phi}(x_{(s)})\delta_{(s),(i)} + (Z_j^{(n)})_{*(s),*(i)}\right)u_{(i)}\right]e_{(s)}(x),$$

where $\phi_j' = \partial\phi/\partial x_j$.

Therefore $\hat{Z}_j^{(n)} = \mathrm{diag}\left\{\left(\frac{\phi_j'}{\phi}\right)_{(n)}\right\} + Z_j^{(n)}$ where $Z_j^{(n)}$ is the quasirepresentation of the operator $\partial_{x_j}$ corresponding to the mapping $\Phi_{(n)}$ and basis $\{L_{(i)}(x)\}_{(i)\in\mathrm{Im}} \subset \mathcal{P}_{(n)}$.

However $u_{(n)} = \Phi_{(n),0}u \in B_{(n),0}$ and hence $u_{(i)} = 0\,\forall\,(i) \in \mathrm{Im}_\Gamma$. Due to this fact in the following the reduced vector $u_{(n),0} = \{u_{(i)}\}_{(i)\in\mathrm{Im}\setminus\mathrm{Im}_\Gamma} \in \mathbb{R}^{N-|\mathrm{Im}_\Gamma|}$ will be considered putting the remained coordinates $u_{(i)} = 0$, $(i) \in \mathrm{Im}_\Gamma$. This modification causes appropriate changes in quasirepresentations: one obtains matrices $\hat{X}_j^{(n),0}$, $\hat{Z}_j^{(n),0}$ and $\hat{I}^{(n),0}$ from matrices $\hat{X}_j^{(n)}$,

$\hat{Z}_j^{(n)}$ and $\hat{I}^{(n)}$ respectively by means of getting out rows and columns with numbers $*(i)$ where $(i) \in \mathrm{Im}_\Gamma$.

Therefore the resulting sequence of the approximate problems has the following form:

$$\begin{cases} \dfrac{du_{(n),0}}{dt} = A_{(n),0}u_{(n),0} + f_{(n),0}(t;\, u_{(n),0}),\ 0 \le t \le T, \\ u_{(n),0}(0) = \psi_{(n),0}, \end{cases}$$

with the operator $A_{(n),0} = \sum\limits_{|\alpha|\le r} (a_\alpha)_{(n),0}\left(t, x;\, u_{(n),0}\right) \prod\limits_{j=1}^{q} \left(\hat{Z}_j^{(n),0}\right)^{\alpha_j}$, where

$$f_{(n),0}(t;\, u_{(n),0}) = f_{(n),0}\big(t;\, \langle u_{(n),0},\, e_{(n),0}(x)\rangle\big)$$

and $e_{(n),0}(x) = \left\{e_{(i)}(x)\right\}_{(i)\in \mathrm{Im} \setminus \mathrm{Im}_\Gamma}$. The coefficients $(a_\alpha)_{(n),0}\left(t, x;\, u_{(n),0}\right)$ are defined analogously. One can calculate an approximate solution $\tilde{u}(t,x)$ of problem (1), (2), (3) by means of the formula

$$\tilde{u}(t,x) = \left\langle u_{(n),0},\, e_{(n),0}(x)\right\rangle. \tag{4}$$

One can avoid inconvenience connected with getting out elements of matrices $\hat{X}_j^{(n)}$, $\hat{Z}_j^{(n)}$ and $\hat{I}^{(n)}$ by means of constructing an appropriate lattice $\Theta$ such that $\Theta \cap \Gamma = \varnothing$. For instance, if after the equispaced distribution it occurs that $x_{(k)} \in \Theta \cap \Gamma$ than one performs the shift of all nodes $x_{(s)}$, $s_m = k_m$, $m \in \{1, 2, \dots, q\}$ by a distance $h_m$ with respect to the variable $x_m$. Making use of the Taylor formula one obtains $\phi(\tilde{x}_{(k)}) = \phi'_m(x_{(k)})h_m + o(h_m)$, where $\tilde{x}_{(k)}$ is the new value of the node $x_{(k)}$. Therefore the choice of a shift distance depends on the value of the derivative $\phi'_m(x_{(k)})$ at the boundary point $x_{(k)} \in \Theta \cap \Gamma$. It is expedient to choose the number $m$ of the shift coordinate from the condition $m = \arg \max\limits_{1\le j\le q} \left\{\phi'_j(x_{(k)})\right\}$.

**3. Approximation error estimations.** Now let us proceed to construct analogous to stated in [2] estimations for the suggested here modified quasirepresentations. Suppose that $g \in W_\infty^{(n)}(\mathbf{M}, D)$, $\phi \in C^1(D)$ are such that $u = g\phi$, then using Theorem 1 proved in [2] one obtains

$$\begin{aligned} &\left|\left\langle \left(\tfrac{\partial u}{\partial x_j}\right)_{(n)} - \hat{Z}_j^{(n)} u_{(n)},\, e_{(n)} \right\rangle\right|_\infty = \left|R_{(n)}\left(\tfrac{\partial u}{\partial x_j}\right) - R_{(n)}\left(\tfrac{\partial}{\partial x_j} R_{(n)} u\right)\right|_\infty = \\ &= \left|\phi\left(P_{(n)}\left(\tfrac{\partial g}{\partial x_j}\right) - P_{(n)}\left(\tfrac{\partial}{\partial x_j}P_{(n)}g\right)\right) + \phi\left(P_{(n)}\left(\tfrac{\phi'_j}{\phi}g\right) - P_{(n)}\left(\tfrac{\phi'_j}{\phi}P_{(n)}g\right)\right)\right|_\infty \le \\ &\le |\phi|_\infty \left|P_{(n)}\left(\tfrac{\partial g}{\partial x_j}\right) - P_{(n)}\left(\tfrac{\partial}{\partial x_j}P_{(n)}g\right)\right|_\infty + 0 \le |\phi|_\infty\ M_j\,(b_j - a_j)^{n_j - 1}\, \tfrac{\Lambda_{(n)}}{n_j!}. \end{aligned}$$

Similarly one verifies that

$$\left|\left\langle (x_j u)_{(n)} - \hat{X}_j^{(n)} u_{(n)},\, e_{(n)} \right\rangle\right|_\infty = 0.$$

One can now evaluate the error of approximation of the structure identity $[\partial_{x_j}, x_j] = 1$ by means of the following equalities:

$$\begin{aligned} \left\langle \left(\hat{I}^{(n)} - [\hat{Z}_j^{(n)}, \hat{X}_j^{(n)}]\right) u_{(n)},\, e_{(n)} \right\rangle &= R_{(n)}\left[u - \tfrac{\partial}{\partial x_j}\left(x_j R_{(n)} u\right) + x_j \tfrac{\partial}{\partial x_j} R_{(n)} u\right] = \\ &= \phi P_{(n)}\left[g - \tfrac{\partial}{\partial x_j}\left(x_j P_{(n)} g\right) + x_j \tfrac{\partial}{\partial x_j} P_{(n)} g\right]. \end{aligned}$$

From the latter equality using Theorem 1 stated in [2] one obtains the estimation

$$\left|\left\langle\left(\hat{I}^{(n)} - [\hat{Z}_j^{(n)}, \hat{X}_j^{(n)}]\right)u_{(n)}, e(n)\right\rangle\right|_\infty \leq |\phi|_\infty \hat{M}\,(b_j - a_j)^{n_j-1}\,\frac{\Lambda_{(n)}}{(n_j-1)!}.$$

Thereby one can formulate the following theorem.

**Theorem 1.** *Let $u \in W^r_{2,0}(\Omega)$ with $\partial\Omega = \{x \in \mathbb{R}^q : \bar{\phi}(x) = 0\}$ where $\bar{\phi} \in W_2^{l-1/2}(\partial\Omega)$ and $l > q/2 + 1$. Suppose also that $u = g\phi$ with $g \in W^{(n)}_\infty(\mathbf{M};\, D)$, where $\phi \in C^1(D)$, $\phi|_{\partial\Omega} = \bar{\phi}$. Then for quasirepresentations $\hat{X}_j^{(n)}$, $\hat{Z}_j^{(n)}$ and $\hat{I}^{(n)}$ of the basic differential operators $x_j$, $\partial_{x_j}$ and 1 respectively the following estimations*

$$\left|\left\langle \hat{Z}_i^{(n)}\, u_{(n)} - \left(\frac{\partial u}{\partial x_i}\right)_{(n)},\, e_{(n)}\right\rangle\right|_\infty \leq |\phi|_\infty M_i (b_i - a_i)^{n_i-1}\frac{\Lambda_{(n)}}{n_i!},$$

$$\left|\left\langle \hat{X}_i^{(n)}\, u_{(n)} - \left(x_i\, u\right)_{(n)},\, e_{(n)}\right\rangle\right|_\infty = 0,$$

$$\left|\left\langle \left(\hat{I}^{(n)} - \left[\hat{Z}_i^{(n)}, \hat{X}_i^{(n)}\right]\right) u_{(n)},\, e_{(n)}\right\rangle\right|_\infty \leq |\phi|_\infty \hat{M}_i\left(b_i - a_i\right)^{n_i-1}\frac{\Lambda_{(n)}}{(n_i-1)!}$$

*hold, where $i \in \{1, \dots, q\}$, $\hat{M}_i = \text{const} > 0$.*

**Remark 1.** Using the Sobolev theorem on the functions traces and the theorem on imbedding [5] and taking into account that $\bar{\phi} \in W_2^{l-1/2}(\partial\Omega)$, $l > q/2+1$ one finds that there exists a function $\hat{\phi} \in C^1(\bar{\Omega})$, $\hat{\phi}|_{\partial\Omega} = \bar{\phi}$. Subject to $\phi \in C^1(D)$ we take a continuously differentiable extension of the function $\hat{\phi}$ onto the cube $D$.

**4. An application of the modified scheme.** I finish this paper with an application of the suggested here modified Lie-algebraic scheme to solving one Cauchy problem with boundary condition.

Let us examine the problem which had been considered in [6] and construct a hierarchy of the corresponding approximate problems. The above mentioned problem has the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\right)u + t^2\sin(x_1 + x_2), \\ u|_{t=0} = \psi(x_1, x_2), \\ u|_\Gamma = 0, \end{cases}$$

where $(x_1, x_2) \in \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, $\Gamma = \partial\Omega$. Similarly as in [6] consider the mesh $\Theta = \left\{x_{(k)} = \left(x_1^{(k_1)}, x_2^{(k_2)}\right) \in D = [-1, 1]^2, (k) \in \mathrm{Im}\right\}$ with the set of multi-indices $\mathrm{Im} = \left\{(k) = (k_1, k_2) : 1 \leq k_j \leq n_j, j = \{1, 2\}\right\}$ and the set of boundary indices $\mathrm{Im}_\Gamma = \left\{(k) \in \mathrm{Im} : x_{(k)} \in \Gamma\right\}$. Making use of the described in Section 2 modified qausirepresentations one obtains the following sequence of approximate problems:

$$\begin{cases} \dfrac{du_{(n),0}}{dt} = \left[\left(\hat{Z}_1^{(n),0}\right)^2 + \left(\hat{Z}_2^{(n),0}\right)^2\right]u_{(n),0} + t^2\sin\left(\hat{X}_1^{(n),0} + \hat{X}_2^{(n),0}\right)\bar{e}, \\ u_{(n),0}|_{t=0} = \psi\left(\hat{X}_1^{(n),0}, \hat{X}_2^{(n),0}\right)\bar{e}, \end{cases} \tag{5}$$

where

$$\hat{Z}_k^{(n),0} = \left\{ \frac{2x_k^{(i_k)}}{(x_1^{(i_1)})^2 + (x_2^{(i_2)})^2 - 1} \delta_{(i),(j)} + \left(Z_k^{(n)}\right)_{*(i),*(j)} \right\}_{(i),(j)\in \mathrm{Im} \setminus \mathrm{Im}_\Gamma},$$

$$\hat{X}_k^{(n),0} = \left\{ \left(X_k^{(n)}\right)_{*(i),*(j)} \right\}_{(i),(j)\in \mathrm{Im} \setminus \mathrm{Im}_\Gamma},$$

$k = \{1, 2\}$, with $\bar{e} = (1, \dots, 1) \in \mathbb{R}^{n_1 \cdot n_2 - |\mathrm{Im}_\Gamma|}$. One calculates an approximate solution $\tilde{u}(t, x_1, x_2)$ by means of (4) taking $\phi(x_1, x_2) = x_1^2 + x_2^2 - 1$.

It is obvious that approximate problem (5) is of considerably simpler form and is much more easier for solving it by means of standard numerical methods.

**Conclusions.** The standard Lie-algebraic scheme is suitable for solving Cauchy problems with no boundary conditions. Many examples given in [1, 3] have shown how much inconvenience makes this peculiarity of the scheme if solving some problems of mathematical physics with boundary conditions. One should perform an appropriate change of variables in order to obtain a new problem without boundary conditions. However this new problem is usually of a much more complicated form. The main idea of the suggested in this paper modification is to perform the change of variables in terms of the space in which we are looking for solution and to find the appropriate quasirepresentations of the main differential operators defined on this space involving boundary conditions. The analysis performed in Section 3 has shown that constructed in Section 2 quasirepresentations do not lose approximation convergence proved for standard quasirepresentations in [2]. Therefore the modification of the standard Lie-algebraic scheme introduced in this paper takes into account the Dirichlet boundary condition in a natural way without loss of the approximation convergence.

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