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GENERALIZED DERIVATIONS AND FOURIER TRANSFORM OF POLYNOMIAL ULTRADISTRIBUTIONS

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Algebras of symmetric ultradistributions on the spaces of functions of infinitely many variables are investigated. Linear representation of such algebras as convolution algebras of symmetric tensor products are constructed. Generalized derivations and Fourier transform of such algebras are described.

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Исследованы алгебры симметрических ультрараспределений на пространствах функций бесконечного числа переменных. Построены линейные представления таких алгебр как алгебр сверток симметрических тензорных произведений. Описаны обобщенные производные и преобразование Фурье таких алгебр.

1. Introduction. In this paper we research direct sums of symmetric ultradistributions

$$\mathcal{G}'_\infty = \sum_{n\in\mathbb{N}} \mathcal{G}'(\mathbb{R}^n),$$

where summation is carried out with respect to the number of independent variables. Therefore, the considered ultradistributions are given on a space of functions of infinitely many variables. Such spaces are dense in the respective Fock spaces and used in various quantum models (see [1, 2]). Above, $\mathcal{G}'(\mathbb{R}^n)$ is a space of ultradistributions on a subspace of ultradifferentiable in the sense of Gevrey functions, symmetric with respect to permutations of variables.

By using methods of the theory of analytic functions on locally convex spaces (see [3]) it is shown that the spaces of symmetric ultradistributions $\mathcal{G}'_\infty$ are represented by the space of polynomials on the space $G := \mathcal{G}(\mathbb{R}^1)$ of functions of one variable. Such a representation introduces in the space of ultradistributions $\mathcal{G}'_\infty$ a natural operation of multiplication with respect to which it becomes an algebra. The tensor structure of such algebras is essentially used. The presence of tensor structure is connected with the nuclearity property of considered spaces. This fact is based on classical results of Grothendieck [4], (also [9]) on tensor products of nuclear spaces and known results of Mityagin [7] about nuclearity of spaces of type $\mathcal{G}'(\mathbb{R}^n)$.

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In this paper one class of derivations in algebras of ultradistributions $\mathcal{G}'_\infty$ is studied. The specificity of these derivations is that they are generated by continuous linear operators given in the spaces G of functions of one variable. In the case of linear ultradistributions they coincides with known derivation of generalized functions, therefore we call them by generalized derivations.

The Fourier transformation of the algebra $\mathcal{G}'_\infty$ is also described. Such a transformation realizes algebraic and topological isomorphism on the algebra of polynomials $\mathcal{E}'_\infty$ for some subspace of entire analytic functions of exponential type.

2. Spaces of linear ultradistributions. For given number $\aleph > 1$ and arbitrarily chosen vector

$$\nu = (\nu_1, \dots, \nu_n) \in \operatorname{int} \mathbb{R}^n_+$$

and vectors

$$a = (a_1, \dots, a_n), \quad b = (b_1, \dots, b_n) \in \mathbb{R}^n$$

such that $b \succ a$, which means that $b_j > a_j$ $\big(\forall\, j \in \{1, \dots n\}\big)$, we define the space of functions ultradifferentiable in sense of Gevrey

$$G_{\nu,[a,b]}(\mathbb{R}^n) = \Big\{\varphi \in C^\infty(\mathbb{R}^n) : \operatorname{supp} \varphi \subset [a,b],\ \|\varphi\|_{G_{\nu,[a,b]}} < \infty\Big\}$$

with supports contained in an n-dimensional cube

$$[a,b] := \big\{\tau = (t_1, \dots, t_n) \in \mathbb{R}^n : t_j \in [a_j, b_j], \forall\, j \in \{1, \dots, n\}\big\}$$

and the norm

$$\|\varphi\|_{G_{\nu,[a,b]}} = \sup_{k \in \mathbb{Z}^n_+} \sup_{\tau \in [a,b]} \frac{|D^k \varphi(\tau)|}{\nu^k k^{k\aleph}},$$

where

$$D^k = D_1^{k_1} \dots D_n^{k_n}, \quad D_j^{k_j} = (-\mathrm{i})^{k_j} \frac{\partial^{k_j}}{\partial {t_j}^{k_j}}, \quad \nu^k = {\nu_1}^{k_1} \dots {\nu_n}^{k_n},$$

$$k^{k\aleph} = {k_1}^{k_1 \aleph} \dots {k_n}^{k_n \aleph}, \quad k = (k_1, \dots, k_n) \in \mathbb{Z}^n_+.$$

It is easy to prove that $G_{\nu,[a,b]}(\mathbb{R}^n)$ is a Banach space.

Now let us consider the inductive limit of spaces

$$G(\mathbb{R}^n) = \bigcup_{b \succ a} \bigcup_{\nu \succ 0} G_{\nu,[a,b]}(\mathbb{R}^n) = \liminf_{b \succ a, |a|, |b|, |\nu| \to \infty} G_{\nu,[a,b]}(\mathbb{R}^n).$$

where all injections

$$G_{\nu,[a,b]}(\mathbb{R}^n) \hookrightarrow G_{\nu',[a',b']}(\mathbb{R}^n) \ \big(\nu' \succ \nu;\ [a,b] \subset [a',b']\big)$$

are continuous and $|a| := |a_1| + \ldots + |a_n|$. By $G'(\mathbb{R}^n)$ we denote the dual space of linear continuous functionals on $G(\mathbb{R}^n)$. On $G'(\mathbb{R}^n)$ we have the strong topology $\beta(G'(\mathbb{R}^n), G(\mathbb{R}^n))$ with respect to the duality $\langle G'(\mathbb{R}^n) \mid G(\mathbb{R}^n)\rangle$.

Lemma 1. *The spaces $G(\mathbb{R}^n)$ and $G'(\mathbb{R}^n)$ are nontrivial, nuclear, reflexive, locally convex spaces. Moreover, $G(\mathbb{R}^n)$ is an (LN^*)-space and $G'(\mathbb{R}^n)$ is an (M^*)-space in the sense of Silva.*

Proof. From Carleman-Denjoy Theorem [6, Theorem 1.3.8] it follows that there exists a function $\rho(t) \geq 0$ such that

$$|\rho^{(k)}(t)| \leq 2^{k+1} \zeta^k(\aleph) k^{\aleph k}, \quad \operatorname{supp} \rho \subset [-1, 1] \quad \int_{-\infty}^{\infty} \rho(t)\, dt = 1,$$

where $\zeta(\aleph) = \sum_{m=1}^{\infty} m^{-\aleph}$ is the Riemann function. For the function $\eta(\tau) = \rho(t_1) \dots \rho(t_n)$ we have

$$|D^k \eta(\tau)| \leq 2^{|k|+n} \zeta^{|k|}(\aleph) k^{\aleph k}, \ (\forall\, k \in \mathbb{Z}_+^n)$$

and $\eta \in G_{\nu,[-1,1]}(\mathbb{R}^n)$ at $\nu_1 = \dots = \nu_n = 2\zeta(\aleph)$, i.e. the space $G(\mathbb{R}^n)$ is nontrivial.

For fixed $\mu \succ 0$ we consider a family of Banach spaces of ultradifferentiable functions

$$S_\mu(\mathbb{R}^n) := \{\varphi(\tau) \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{S_\mu} < \infty\},$$

$$\|\varphi\|_{S_\mu} = \sup_{\alpha,\beta \in \mathbb{Z}_+^n} \sup_{\tau \in \mathbb{R}^n} \frac{|\tau^\beta D^\alpha \varphi(\tau)|}{\mu^{\alpha+\beta} (\alpha+\beta)^{\aleph(\alpha+\beta)}}.$$

For every $\nu \succ \mu$ the inequality $\|\varphi\|_{S_\nu} \leq \|\varphi\|_{S_\mu}$ holds for all $\varphi \in S_\mu(\mathbb{R}^n)$. Therefore, the inductive limit

$$S(\mathbb{R}^n) = \liminf_{|\mu| \to \infty} S_\mu(\mathbb{R}^n)$$

makes sense. The sequence $M_{\alpha,\beta}^{(\mu)} = \mu^{\alpha+\beta}(\alpha+\beta)^{\aleph(\alpha+\beta)}$ is growing with respect to every variable. For all $\mu \succ 0$ and $j, k \in \mathbb{Z}_+^n$ there are a constant $C(\mu, j, k)$ and a number $\nu \succ \mu$ such that

$$\alpha^k M_{\alpha+j,\beta+k}^{(\mu)} \leq C(\mu, j, k) M_{\alpha,\beta}^{(\nu)}, \qquad \alpha, \beta \in \mathbb{Z}_+^n.$$

Moreover, as $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \to \infty$, we have

$$\frac{\alpha^k M_{\alpha+j,\beta+k}^{(\nu)}}{M_{\alpha,\beta}^{(\mu)}} \leq \left[(\alpha+\beta)\left(\frac{\mu}{\nu}\right)^{\frac{(\alpha+\beta)}{3k}}\right]^k \left[(\alpha+\beta+j+k)\left(\frac{\mu}{\nu}\right)^{\frac{(\alpha+\beta)}{3k}}\right]^{j+k} \times$$

$$\times \left[\frac{\alpha+\beta+j+k}{\alpha+\beta}\right]^{\aleph(\alpha+\beta)} \to 0,$$

where $\frac{\alpha}{\beta} := \left(\frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_n}{\beta_n}\right)$. For all $\nu \succ \mu$ the series

$$\sum_{\alpha,\beta \in \mathbb{Z}_+^n} \frac{M_{\alpha,\beta}^{(\mu)}}{M_{\alpha,\beta}^{(\nu)}} = \sum_{\alpha,\beta \in \mathbb{Z}_+^n} \left(\frac{\mu}{\nu}\right)^{\alpha+\beta}$$

converges. Thus, the space $S(\mathbb{R}^n)$ satisfies known conditions of nuclearity from [7].

The subspace of functions

$$S[a, b] = \left\{\varphi \in S(\mathbb{R}^n) : D^\alpha \varphi(\tau) = 0, \ \alpha \in \mathbb{Z}_+^n, \ (\tau \succeq b) \vee (a \succeq \tau)\right\}$$

is closed in $S(\mathbb{R}^n)$. Above, $b \succeq a$ means that $b_j \geq a_j$ $(\forall\, j \in \{1, \dots n\})$. Thus, $S[a, b]$ is also nuclear. We see also that

$$S[a, b] = \bigcup_{\nu \succ 0} G_{\nu,[a,b]}(\mathbb{R}^n) = \liminf_{|\nu| \to \infty} G_{\nu,[a,b]}(\mathbb{R}^n). \tag{1}$$

Indeed, if $\varphi \in \bigcup_{\nu \succ 0} G_{\nu,[a,b]}(\mathbb{R}^n)$, then $\operatorname{supp}\varphi \subset [a,b]$ and for some $\nu \succ 0$ we have

$$|D^k \varphi(\tau)| \le \nu^k k^{k\aleph} \|\varphi\|_{G_{\nu,[a,b]}}$$

for all $\tau \in [a,b]$ and $k \in \mathbb{Z}_+^n$. Whence it follows that

$$|\tau^j D^k \varphi(\tau)| \le \mu^{k+j} k^{k\aleph} \|\varphi\|_{G_{\nu,[a,b]}} \le \mu^{k+j} (k+j)^{\aleph(k+j)} \|\varphi\|_{G_{\nu,[a,b]}}$$

for all $\tau \in [a,b]$ and $k, j \in \mathbb{Z}_+^n$, where

$$\mu_j = \max\{|a_j|, |b_j|, \nu_j\} \quad \forall\, j \in \{1, \dots, n\}.$$

Hence, $\varphi \in S[a,b]$ and $\|\varphi\|_{S_\mu} \le \|\varphi\|_{G_{\nu,[a,b]}}$, i.e. the continuous embedding

$$G_{\nu,[a,b]}(\mathbb{R}^n) \hookrightarrow S[a,b]$$

is obtained for all $\nu \succ 0$. Therefore the continuous embedding

$$\liminf_{|\nu| \to \infty} G_{\nu,[a,b]}(\mathbb{R}^n) \hookrightarrow S[a,b]$$

is also obtained. The reverse side containing is obvious.

From equality (1) we obtain that

$$G(\mathbb{R}^n) = \liminf_{b \succ a : |b|, |a| \to \infty} S[a,b],$$

As it is known [9] in the inductive limit it is possible to take a countable subsequence of spaces $S[a,b]$. According to [8, Theorem 5.3.4] such an inductive limit of nuclear spaces is nuclear. Hence, the space $G(\mathbb{R}^n)$ is nuclear. By nuclearity of the space $G(\mathbb{R}^n)$, the strong dual space $G'(\mathbb{R}^n)$ is also nuclear [5, Theorem 21.5.3].

As it was proved in [10, Chap.7, Proposition 1.1] all embeddings

$$\big\{G_{\nu,[a,b]}(\mathbb{R}^n) \hookrightarrow G_{\mu,[a,b]}(\mathbb{R}^n) : \ \mu \succ \nu\big\},$$
$$\big\{G_{\nu,[a,b]}(\mathbb{R}^n) \hookrightarrow G_{\nu,[c,d]}(\mathbb{R}^n) : \ [a,b] \subset [c,d]\big\}$$

are compact. Hence, the inductive limit $G(\mathbb{R}^n) = \liminf_{\nu,[a,b]} G_{\nu,[a,b]}(\mathbb{R}^n)$ belongs to the class of (LN^*)-spaces in the sense of Silva [11]. Therefore, strong dual space $G'(\mathbb{R}^n)$ belongs to the class of (M^*)-spaces in the sense of Silva [11]. As a consequence, the spaces $G(\mathbb{R}^n)$ and $G'(\mathbb{R}^n)$ are reflexive. □

3. Spaces of polynomial ultradistributions. Let us introduce the basic definitions. By $\mathcal{L}^n(G, \mathbb{K})$ we shall denote the vector space of n-linear continuous forms on G, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Obviously, $\mathcal{L}^1(G, \mathbb{K}) := G'$ coincides with the topological dual vector space. On the space $\mathcal{L}^n(G, \mathbb{K})$ we have the topology of uniform convergence on bounded absolutely convex sets in the locally convex direct product $\prod_{j=1}^n G$ of n copies of G. In the case of the space G' this topology coincides with the strong topology $\beta(G', G)$ with respect to the duality $\langle G' \mid G \rangle$.

To every n-linear form $f_n \in \mathcal{L}^n(G, \mathbb{K})$ there corresponds the composition

$$p_n = f_n \circ \Delta_n, \qquad \Delta_n : \ G \ni \varphi \longmapsto {}^n\varphi := (\varphi, \dots, \varphi) \in \prod_{j \in \mathbb{N}} G,$$

which is usually referred as an n-homogeneous polynomial on the space G [3]. The vector space of all such n-homogeneous polynomials is denoted by $\mathcal{P}_n(G)$. On the space $\mathcal{P}_n(G)$ we have the topology of uniform convergence on bounded absolutely convex sets in G. By

$$\mathcal{P}(G) := \sum_{n\in\mathbb{N}} \mathcal{P}_n(G) = \Big\{ p(\varphi) = \sum_{n=1}^{m} p_n(\varphi) : \ p_n \in \mathcal{P}_n(G); \ m \in \mathbb{N} \Big\},$$

we denote the locally convex direct sum, which we call the space of all polynomials on G. Further we shall study some continuous derivations on the space $\mathcal{P}(G)$.

Let $\otimes^n G'$ be the algebraic tensor product of n copies of the strongly dual space G'. Assuming presence of a natural embedding $\otimes^n G' \subset \mathcal{L}^n(G, \mathbb{K})$ we also identify

$$\prod_{j=1}^{n} G \ni (\varphi_1, \dots, \varphi_n) \longmapsto \varphi_1 \otimes \dots \otimes \varphi_n \in \otimes^n G$$

in the sense that

$$\langle u_1 \otimes \dots \otimes u_n \mid \varphi_1 \otimes \dots \otimes \varphi_n \rangle = u_1(\varphi_1) \dots u_n(\varphi_n), \tag{2}$$

where $u_1, \dots, u_n \in G'$. In space $\otimes^n G'$ the operator of symmetrization

$$\pi_n' : \otimes^n G' \ni u_1 \otimes \dots \otimes u_n \longmapsto u_1 \odot \dots \odot u_n := \frac{1}{n!} \sum_{\varsigma \in \mathfrak{S}_n} u_{\varsigma(1)} \otimes \dots \otimes u_{\varsigma(n)}$$

is determined, where $\mathfrak{S}_n$ is the group of permutations on the set $\{1, \dots, n\}$. The range $\odot^n G' := \operatorname{Ran} \pi_{\mathbf{n}}'$ of the projector π_n' refers to the algebraic symmetric tensor product. Let further $\otimes_{\mathfrak{p}}^n G'$ (resp. $\odot_{\mathfrak{p}}^n G'$) be the completion of $\otimes^n G'$ (resp. $\odot^n G'$) in the projective topology of the tensor product of the strong dual space G'. By $\sum_{n\in\mathbb{N}} \otimes_{\mathfrak{p}}^n G'$ $\big($resp. $\sum_{n\in\mathbb{N}} \odot_{\mathfrak{p}}^n G'\big)$ we denote the locally convex direct sum of the corresponding spaces.

The same notation $\otimes^n G$, $\odot^n G$, $\otimes_{\mathfrak{p}}^n G'$, $\odot_{\mathfrak{p}}^n G'$ is used for the tensor products of space G. Let the projectors π_n and π_n' be each other's adjoints with respect to the duality $\langle \otimes^n G' \mid \otimes^n G \rangle$ determined by bilinear form (2). For any elements $u_1, \dots, u_n \in G'$ and $\varphi_1, \dots, \varphi_n \in G$ the projector π_n satisfies the relation

$$\langle u_1 \odot \dots \odot u_n \mid \varphi_1 \otimes \dots \otimes \varphi_n \rangle = \langle u_1 \otimes \dots \otimes u_n \mid \varphi_1 \odot \dots \odot \varphi_n \rangle,$$

$$\pi_n : \varphi_1 \otimes \dots \otimes \varphi_n \longmapsto \varphi_1 \odot \dots \odot \varphi_n := \frac{1}{n!} \sum_{\varsigma \in \mathfrak{S}_n} \varphi_{\varsigma(1)} \otimes \dots \otimes \varphi_{\varsigma(n)} .$$

We denote $\odot^n G := \operatorname{Ran} \pi_{\mathbf{n}}$. Such a projector π_n obviously has a continuous extension $\pi_n : \otimes_{\mathfrak{p}}^n G \longmapsto \odot_{\mathfrak{p}}^n G$ on the appropriate completions.

Further we consider the subspace of $G(\mathbb{R}^n)$ of functions which are symmetric with respect to the permutations of variables. The range of the corresponding projection

$$(\sigma_n \circ \varphi)(\tau) = \frac{1}{n!} \sum_{\varsigma \in \mathfrak{S}_n} \varphi(t_{\varsigma(1)}, \dots, t_{\varsigma(n)}), \quad \varphi \in G(\mathbb{R}^n)$$

is denoted by $\mathcal{G}(\mathbb{R}^n) := \operatorname{Ran} \sigma_n$. Obviously $\sigma_1(G) = G$. Let $\mathcal{G}'(\mathbb{R}^n)$ be the strong dual space to $\mathcal{G}(\mathbb{R}^n)$.

Lemma 2. *The following linear topological isomorphisms*

$$\mathcal{G}'(\mathbb{R}^n) \simeq \odot_{\mathsf{p}}^n G' \simeq \mathcal{P}_n(G)$$

are realized. The second of them is given by the formula

$$\odot_{\mathsf{p}}^n G' \ni \widetilde{p}_n \longmapsto \langle \widetilde{p}_n \mid {}^n\varphi \rangle = (\widetilde{p}_n \circ \Delta_n)(\varphi) =: p_n(\varphi) \in \mathcal{P}_n(G). \tag{3}$$

Proof. In the inductive limit

$$G(\mathbb{R}^n) = \liminf_{b \succ a, |a|, |b|, |\nu| \to \infty} G_{\nu,[a,b]}(\mathbb{R}^n)$$

it is possible to leave a countable subsequence of spaces $G_{\nu,[a,b]}(\mathbb{R}^n)$. As $G(\mathbb{R}^n)$ is an (LN^*)-space, any bounded subset of $G(\mathbb{R}^n)$ is contained and is bounded in some subspace $G_{\nu,[a,b]}(\mathbb{R}^n)$ and $G(\mathbb{R}^n)$ belongs to the known class (DF) [9, §6.6]. As a consequence, the space $G(\mathbb{R}^n)$ has a countable base of bounded sets and the dual space $G'(\mathbb{R}^n)$ is an (F)-space.

For strong dual spaces to nuclear (DF)-spaces topological isomorphism

$$(\otimes_{\mathsf{p}}^n G)' \simeq \otimes_{\mathsf{p}}^n G' \tag{4}$$

is known [9, §9.9]. In view of nuclearity of the space $G(\mathbb{R}^n)$ we can apply known result from [7] by virtue of which we get a topological isomorphism

$$G(\mathbb{R}^n) \simeq \otimes_{\mathsf{p}}^n G.$$

On the dense subspace $\otimes^n G$ this isomorphism is given by the equality

$$\varphi(\tau) = \sum_j \varphi_{j_1}(t_1) \ldots \varphi_{j_n}(t_n) \in \otimes^n G,$$

where $\varphi_{j_1}, \ldots, \varphi_{j_n} \in G$. From this isomorphism and the relation

$$(\sigma_n \circ \varphi)(\tau) = \frac{1}{n!} \sum_j \sum_{\varsigma \in \mathfrak{S}_n} \varphi_{j_1}(t_{\varsigma(1)}) \ldots \varphi_{j_n}(t_{\varsigma(n)}) =$$
$$= \frac{1}{n!} \sum_j \sum_{\varsigma \in \mathfrak{S}_n} \varphi_{j_{\varsigma(1)}}(t_1) \ldots \varphi_{j_{\varsigma(n)}}(t_n) = (\pi_n \circ \varphi)(\tau)$$

we get topological isomorphisms

$$\mathcal{G}(\mathbb{R}^n) \simeq \odot_{\mathsf{p}}^n G, \qquad \mathcal{G}'(\mathbb{R}^n) \simeq \odot_{\mathsf{p}}^n G'.$$

Thus the second isomorphism is obtained. On the other hand, for such spaces a linear topological isomorphism

$$\mathcal{L}^n(G, \mathbb{K}) \simeq (\otimes_{\mathsf{p}}^n G)' \tag{5}$$

is valid [9]. Combining equality (4), (5) and applying to them the continuous projector π_n' in a standard way (see [3]) we come to a topological isomorphism

$$\mathcal{L}_\varsigma^n(G, \mathbb{K}) \simeq \odot_{\mathsf{p}}^n G',$$

where $\mathcal{L}_\varsigma^n(G, \mathbb{K})$ is the subspace of symmetric n-linear forms in $\mathcal{L}^n(G, \mathbb{K})$. The nuclear (DF) spaces, as it is proved in [4], have a $(BB)_\infty$-property, therefore a topological isomorphism

$$\mathcal{L}_\varsigma^n(G, \mathbb{K}) \simeq \mathcal{P}_n(G)$$

takes place. From this we conclude that the operator (3) realizes the first of required linear topological isomorphisms. □

Theorem 3. *The following isomorphisms of topological algebras*

$$\mathcal{G}'_\infty \simeq \sum_{n\in\mathbb{N}} \odot_{\mathfrak{p}}^n G' \simeq \mathcal{P}(G)$$

are realized. The convolution in $\sum_{n\in\mathbb{N}} \odot_{\mathfrak{p}}^n G'$

$$\widetilde{p} * \widetilde{q} := \sum_n \Big(\sum_{k=1}^n \widetilde{p}_k \odot \widetilde{q}_{n-k} \Big), \quad \widetilde{q} := \sum \widetilde{q}_n \in \sum \odot_{\mathfrak{p}}^n G', \quad \widetilde{q}_n \in \odot_{\mathfrak{p}}^n G'$$

is transformed into the product of polynomials p, q *in the algebras* $\mathcal{P}(G)$.

Proof. As the isomorphisms directly follow from Lemma 2, let us concentrate on the operation of multiplication of polynomials. For any $\widetilde{p}_n \in \odot_{\mathfrak{p}}^n G'$ and $\widetilde{q}_k \in \odot_{\mathfrak{p}}^k G'$ we have

$$\widetilde{p}_n \odot \widetilde{q}_k \in (\odot_{\mathfrak{p}}^n G') \odot (\odot_{\mathfrak{p}}^k G') \subset \odot_{\mathfrak{p}}^{n+k} G'.$$

Therefore

$$p_n(\varphi)\, q_k(\varphi) = \langle \widetilde{p}_n \mid {}^n\varphi \rangle \big\langle \widetilde{q}_k \mid {}^k\varphi \big\rangle = \big\langle \widetilde{p}_n \otimes \widetilde{q}_k \mid {}^{n+k}\varphi \big\rangle =$$
$$= \big\langle \widetilde{p}_n \odot \widetilde{q}_k \mid {}^{n+k}\varphi \big\rangle = (\widetilde{p}_n \odot \widetilde{q}_k) \circ \Delta_{n+k}(\varphi),$$

so that $p_n q_k \in \mathcal{P}_{n+k}(G)$ and for any polynomials

$$p(\varphi) = \sum_n p_n(\varphi), \quad q(\varphi) = \sum_k q_k(\varphi)$$

from the space $\mathcal{P}(G)$ we get

$$p(\varphi)\, q(\varphi) = \sum_n \sum_{k=1}^n p_k(\varphi)\, q_{n-k}(\varphi) =$$
$$= \sum_n \sum_{k=1}^n (\widetilde{p}_n \odot \widetilde{q}_{n-k}) \circ \Delta_n(\varphi) = (\widetilde{p} * \widetilde{q}) \circ \Delta_n(\varphi).$$

Thus the operator

$$\sum_n \odot_{\mathfrak{p}}^n G' \ni \widetilde{p} = \sum_n \widetilde{p}_n \longmapsto p(\varphi) = \sum_n \widetilde{p}_n \circ \Delta_n(\varphi) \in \mathcal{P}(G)$$

realizes an isomorphism of algebras. □

4. Generalized derivations. Let $\mathcal{L}(G)$ be the algebra of linear continuous operators on G and $1\colon G \longmapsto G$ be the operator of multiplication by identity. For arbitrary $A, B \in \mathcal{L}(G)$ we compare two operators $\widetilde{\mathcal{A}}_\Sigma$ and $\widetilde{\mathcal{B}}_\Pi$ defined in $\sum_{n\in\mathbb{N}} \odot_{\mathfrak{p}}^n G'$. The operator $\widetilde{\mathcal{A}}_\Sigma$ is defined as the direct product

$$\widetilde{\mathcal{A}}_\Sigma := \prod_{n\in\mathbb{N}} \widetilde{\mathcal{A}}_n\,, \qquad \widetilde{\mathcal{A}}_n := \sum_{k=1} \underbrace{1' \otimes \ldots \otimes 1' \otimes A'}_{k} \otimes \underbrace{1' \otimes \ldots \otimes 1'}_{n-k},$$

where A' is the adjoint operator to A with respect to the duality $\langle G' \mid G\rangle$ and $1'$ is the adjoint to 1. The operator $\widetilde{\mathcal{B}}_\Pi$ is defined as the direct product

$$\widetilde{\mathcal{B}}_\Pi := \prod_{n\in\mathbb{N}} \widetilde{\mathcal{B}}^n, \qquad \widetilde{\mathcal{B}}^n := \underbrace{B' \otimes \ldots \otimes B'}_{n}.$$

The following commutative diagram

$$\begin{array}{ccccc} \mathcal{P}(G) & \longleftrightarrow & \sum \odot_{\mathsf{p}}^n G' & \longleftrightarrow & \mathcal{G}'_\infty \\ \widehat{\mathcal{A}}_\Sigma, \widehat{\mathcal{B}}_\Pi \downarrow & & \downarrow \widetilde{\mathcal{A}}_\Sigma, \widetilde{\mathcal{B}}_\Pi & & \downarrow \mathcal{A}'_\Sigma, \mathcal{B}'_\Pi \\ \mathcal{P}(G) & \longleftrightarrow & \sum \odot_{\mathsf{p}}^n G' & \longleftrightarrow & \mathcal{G}'_\infty \end{array}$$

uniquely determines the corresponding pair of linear operators $\widehat{\mathcal{A}}_\Sigma$, $\widehat{\mathcal{B}}_\Pi$ on $\mathcal{P}(G)$ and $\mathcal{A}'_\Sigma$, $\mathcal{B}'_\Pi$ on $\mathcal{G}'_\infty$.

Theorem 4. *If $A, B \in \mathcal{L}(G)$, then $\widehat{\mathcal{A}}_\Sigma$ is a continuous derivation and $\widehat{\mathcal{B}}_\Pi$ is a continuous homomorphism on the algebra $\mathcal{P}(G)$. For all operators $A, B \in \mathcal{L}(G)$ such that $B = I + A$ the equality*

$$\widehat{\mathcal{B}}_\Pi\, p = \sum_{j=0}^{m} \frac{\widehat{\mathcal{A}}_\Sigma^j}{j!}\, p, \qquad \forall\, p \in \mathcal{P}(G), \quad m = \deg p$$

holds. In this sense the homomorphism $\widehat{\mathcal{B}}_\Pi$ is the exponent of the derivation $\widehat{\mathcal{A}}_\Sigma$, i.e. $\widehat{\mathcal{B}}_\Pi = \exp(\widehat{\mathcal{A}}_\Sigma)$.

Proof. Note that the operators $\widetilde{\mathcal{A}}_\Sigma$ and $\widetilde{\mathcal{B}}_\Pi$ are adjoint to the direct product of operators

$$\mathcal{A}_\Sigma := \prod_{n\in\mathbb{N}} \mathcal{A}_n, \qquad \mathcal{A}_n := \sum_{k=1}^{n} \underbrace{1 \otimes \ldots \otimes 1 \otimes A}_{k} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-k},$$
$$\mathcal{B}_\Pi := \prod_{n\in\mathbb{N}} \mathcal{B}^n, \qquad \mathcal{B}^n := \underbrace{B \otimes \ldots \otimes B}_{n}$$

with respect to the duality $\Big\langle \sum \odot_{\mathsf{p}}^n G' \;\Big|\; \prod \odot_{\mathsf{p}}^n G \Big\rangle$. It is directly checked that on a dense in $\otimes_{\mathsf{p}}^n G$ subspace of functions

$$\phi = \sum_{l} \varphi_{l_1} \otimes \ldots \otimes \varphi_{l_n} \in \otimes^n G$$

the relations $\pi_n \circ \mathcal{B}^n = \mathcal{B}^n \circ \pi_n$ and $\pi_n \circ \mathcal{A}_n = \mathcal{A}_n \circ \pi_n$ take place for all n, i.e. the symmetric subspace $\odot^n G$ is invariant under the operators $\mathcal{B}^n$ and $\mathcal{A}_n$. From the continuity of the operators $\mathcal{A}_n$ and $\mathcal{B}^n$ it immediately follows that the operators $\widetilde{\mathcal{A}}_\Sigma$ and $\widetilde{\mathcal{B}}_\Pi$ ($\mathcal{A}'_\Sigma$, $\mathcal{B}'_\Pi$ and $\widehat{\mathcal{A}}_\Sigma$, $\widehat{\mathcal{B}}_\Pi$ also) are continuous.

It is easy to verify (see [2]) that the operator $\widehat{\mathcal{A}}_\Sigma$ satisfies the equality

$$(\widehat{\mathcal{A}}_\Sigma p)(\varphi) = (dp)(A\varphi)(\varphi), \qquad p \in \mathcal{P}(G),$$

where $(dp)(A\varphi)(\varphi)$ is the derivative of the polynomial p at the point $A\varphi \in G$. Therefore $\widehat{\mathcal{A}}_\Sigma$ and $\mathcal{A}'_\Sigma$ are derivations on $\mathcal{P}(G)$ and $\mathcal{G}'_\infty$ respectively. If $B = I + A$ and the polynomial $p \in \mathcal{P}(G)$ is such that $m = \deg p$, then by Taylor formula [3] we get that

$$(\widehat{\mathcal{B}}_\Pi p)(\varphi) = p(A\varphi + \varphi) = \sum_{j=0}^{m} \frac{(d^j p)(A\varphi)(\varphi)}{j!} = \sum_{j=0}^{m} \frac{\widehat{\mathcal{A}}_\Sigma^j}{j!} p(\varphi).$$

Since $(\widehat{\mathcal{B}}_\Pi p)(\varphi) = (p \circ B)(\varphi)$, the operator $\widehat{\mathcal{B}}_\Pi$ is a continuous homomorphism on the algebra $\mathcal{P}(G)$. □

In the dual space G' we shall define the operator of generalized differentiation by the standard relation

$$D : G' \ni u \longmapsto Du \in G', \qquad \langle Du \mid \varphi \rangle = -\langle u \mid D\varphi \rangle, \quad \varphi \in G.$$

Let

$$T : G' \ni u \longmapsto Tu \in G', \qquad \langle Tu \mid \varphi(t) \rangle = \langle u \mid t\varphi(t) \rangle, \quad \varphi \in G.$$

be the operator of multiplication by independent variable.

Corollary 5. *Let $A = D$ or $A = T$ and $B = I + A$. For each $p \in \mathcal{P}(G)$ the following Taylor expansions $\widehat{\mathcal{B}}_\Pi\, p = \exp(\widehat{\mathcal{A}}_\Sigma)p$ hold, in which the operators $\widehat{\mathcal{A}}_\Sigma$ are continuous derivations and the operators $\widehat{\mathcal{B}}_\Pi$ are continuous homomorphisms of the algebra $\mathcal{P}(S)$.*

5. Paley-Wiener type theorem. Further we define the normed space of entire analytic functions

$$E_{\nu,[a,b]}(\mathbb{C}^n) = \left\{\Phi : \mathbb{C}^n \ni z \longmapsto \Phi(z) \in \mathbb{C}, \quad \|\Phi\|_{E_{\nu,[a,b]}} < \infty\right\},$$

where

$$\|\Phi\|_{E_{\nu,[a,b]}} = \sup_{k \in \mathbb{Z}_+^n} \sup_{z \in \mathbb{C}^n} \frac{\left|z^k \Phi(z) e^{-H_{[a,b]}(\eta)}\right|}{\nu^k k^{k\aleph}}, \qquad \eta = \operatorname{Im} z$$

and $H_{[a,b]}(\eta) = \sup\limits_{\tau \in [a,b]} (\tau, \eta)$ is the supporting function of the cube $[a,b] \subset \mathbb{R}^n$. Let us consider the inductive limit of the spaces $E_{\nu,[a,b]}(\mathbb{C}^n)$; we will denote it by

$$E(\mathbb{C}^n) = \bigcup_{\nu \succ 0} \bigcup_{b \succ a} E_{\nu,[a,b]}(\mathbb{C}^n) = \liminf_{b \succ a : |b|,|a|,|\nu| \to \infty} E_{\nu,[a,b]}(\mathbb{C}^n),$$

where all injections $E_{\nu,[a,b]}(\mathbb{C}^n) \looparrowright E_{\nu',[a',b']}(\mathbb{C}^n)$, $(\nu' \succ \nu,\ [a,b] \subset [a',b'])$ are continuous.

The following statement is true [12].

Lemma 6. *The Fourier transformation*

$$G(\mathbb{R}^n) \ni \varphi \longmapsto \widehat{\varphi} \in E(\mathbb{C}^n), \qquad \widehat{\varphi}(z) = \int_{\mathbb{R}^n} \varphi(\tau) e^{-\mathrm{i}(\tau,z)} d\tau$$

realizes a linear topological isomorphism $G(\mathbb{R}^n) \simeq E(\mathbb{C}^n)$.

Let $G = G(\mathbb{R}^1)$, $E = E(\mathbb{C}^1)$. We will consider the spaces of polynomials $\mathcal{P}(G)$ and $\mathcal{P}(E)$. In the same way as before we consider the subspace of $E(\mathbb{C}^n)$ of functions symmetric with respect to permutation of variables. The range of the projection

$$(\sigma_n \circ \Phi)(z) = \frac{1}{n!} \sum_{\varsigma \in \mathfrak{S}_n} \Phi(z_{\varsigma(1)}, \dots , z_{\varsigma(n)}), \qquad \Phi \in E(\mathbb{C}^n)$$

is denoted by $\mathcal{E}(\mathbb{C}^n) = \mathrm{Ran}\ \sigma_n$. Obviously $\sigma_1(E) = E$. Let $\mathcal{E}'(\mathbb{C}^n)$ be the strong dual space to $\mathcal{E}(\mathbb{C}^n)$.

From Lemmas 6 and 2 linear topological isomorphisms

$$\mathcal{E}'(\mathbb{C}^n) \simeq \odot_{\mathfrak{p}}^n E' \simeq \mathcal{P}_n(E),$$

follow. The symmetric projective tensor product $\odot_{\mathfrak{p}}^n E'$ and the space of polynomials $\mathcal{P}_n(E)$ are determined similarly to $\odot_{\mathfrak{p}}^n G'$ and $\mathcal{P}_n(G)$ respectively. The second of these isomorphisms is determined by the formula

$$\odot_{\mathfrak{p}}^n E' \ni \widetilde{P}_n \longmapsto \left\langle \widetilde{P}_n \mid {}^n\Phi \right\rangle := P_n(\Phi) \in \mathcal{P}_n(E), \qquad \Phi \in E.$$

Thus, similarly to Theorem 3 we get

Lemma 7. *The following isomorphisms of topological algebras*

$$\mathcal{E}'_\infty := \sum_{n \in \mathbb{N}} \mathcal{E}'(\mathbb{C}^n) \simeq \sum_{n \in \mathbb{N}} \odot_{\mathfrak{p}}^n E' \simeq \mathcal{P}(E)$$

are realized. The convolution in $\sum_{n \in \mathbb{N}} \odot_{\mathfrak{p}}^n E'$

$$\widetilde{P} * \widetilde{Q} := \sum_n \Big(\sum_{k=1}^n \widetilde{P}_k \odot \widetilde{Q}_{n-k} \Big), \quad \widetilde{Q} := \sum \widetilde{Q}_n \in \sum \odot_{\mathfrak{p}}^n E', \quad Q_n \in \odot_{\mathfrak{p}}^n E'$$

is transformed into the product of polynomials P, Q in the algebra $\mathcal{P}(E)$.

Let F be the inverse Fourier transformation $F\colon E \longmapsto G$ and F' its adjoint. We shall define the following operators

$$\mathcal{F}^n : \otimes_{\mathfrak{p}}^n G' \longmapsto \otimes_{\mathfrak{p}}^n E', \quad \mathcal{F}^n := \underbrace{F' \otimes \cdots \otimes F'}_{n}, \quad \widetilde{\mathcal{F}}_\Pi = \prod_{n \in \mathbb{N}} \mathcal{F}^n.$$

The following commutative diagram

$$\begin{array}{ccc} \mathcal{P}(G) & \longleftrightarrow & \sum \odot_{\mathfrak{p}}^n G' \\ \widehat{\mathcal{F}}_\Pi \downarrow & & \downarrow \widetilde{\mathcal{F}}_\Pi \\ \mathcal{P}(E) & \longleftrightarrow & \sum \odot_{\mathfrak{p}}^n E' \end{array}$$

uniquely determines an operator $\widehat{\mathcal{F}}_\Pi$. We prove the following statement.

Theorem 8. *The Fourier transformation $\widehat{\mathcal{F}}_\Pi$ is a topological isomorphism from the algebra $\mathcal{P}(G)$ onto the algebra $\mathcal{P}(E)$.*

Proof. The relations $\operatorname{Ker} F' = \{0\}$ and $\overline{\operatorname{Ran} F'} = E'$ follow from Lemma 6. We prove that the equivalent conditions

$$\begin{array}{rcl} \operatorname{Ker} F' = \{0\} & \Longleftrightarrow & \overline{\operatorname{Ran} \widehat{\mathcal{F}}_\Pi} = \mathcal{P}(E) \\ \overline{\operatorname{Ran} F'} = E' & \Longleftrightarrow & \operatorname{Ker} \widehat{\mathcal{F}}_\Pi = \{0\} \end{array}$$

hold. In view of Lemma 7 it is enough to carry out the calculation for the algebras $\sum_n \odot_{\mathfrak{p}}^n G'$, $\sum_n \odot_{\mathfrak{p}}^n E'$ and the operator $\widetilde{\mathcal{F}}_\Pi$. From reflexivity of all spaces, we have

$$\operatorname{Ker} \widetilde{\mathcal{F}}^n \perp \operatorname{Ran} \mathcal{F}^n\,, \qquad \operatorname{Ran} \widetilde{\mathcal{F}}^n \perp \operatorname{Ker} \mathcal{F}^n\,,$$

where $\widetilde{\mathcal{F}}^n$ denotes the adjoint to $\mathcal{F}^n$. For the projective tensor products of nuclear reflexive spaces the equalities $\otimes_{\mathfrak{p}}^n \operatorname{Ker} F' = \operatorname{Ker}(F' \otimes \ldots \otimes F')$ and $\otimes_{\mathfrak{p}}^n \operatorname{Ker} F = \operatorname{Ker}(F \otimes \ldots \otimes F)$ are known [7]. From them it follows that

$$\operatorname{Ker} \widetilde{\mathcal{F}}^n = \odot_{\mathfrak{p}}^n \operatorname{Ker} F', \qquad \operatorname{Ker} \mathcal{F}^n = \odot_{\mathfrak{p}}^n \operatorname{Ker} F.$$

As a result, we get

$$\odot_{\mathfrak{p}}^n \operatorname{Ker} F' \perp \operatorname{Ran} \mathcal{F}^n\,, \qquad \operatorname{Ran} \widetilde{\mathcal{F}}^n \perp \odot_{\mathfrak{p}}^n \operatorname{Ker} F\,.$$

From here and also from the relations $\operatorname{Ker} F' \perp \operatorname{Ran} F$ and $\operatorname{Ran} F' \perp \operatorname{Ker} F$ we come to the necessary conclusion.

The continuity of the operator $\widehat{\mathcal{F}}_\Pi$ is proved in the same way as for the operator $\widehat{\mathcal{B}}_\Pi$ from Theorem 4. The continuity of the inverse operator is a consequence of the open mapping theorem. Finally, the fact that the operator $\widehat{\mathcal{F}}_\Pi$ is anisomorphism of algebras directly follows from its tensor form. □

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