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ORTHOGONAL RETRACTIONS AND M-EQUIVALENCE

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Two Tychonoff spaces are M-equivalent if their free topological groups are topologically isomorphic. We introduce the notion of orthogonal retracts in a Tychonoff space and as a development of Okunev's construction show that every pair of orthogonal retracts leads to a pair of M-equivalent spaces.

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Два тихоновских пространства M-эквивалентны, если их свободные топологические группы топологически изоморфны. Мы вводим понятие ортогональных ретрактов в тихоновском пространстве и с помощью развития конструкции Окунева доказываем, что каждой паре ортогональных ретрактов отвечает пара M-эквивалентных пространств.

1. Introduction

We refer to [7] for examples of categories and functors in topological algebra. In [9] one can find a collection of all known results on isomorphical classification generated by these functors. The papers [5] and [6] contain fundamental results on M-equivalence of Tychonoff spaces (see the definition below). In particular, a new method of the construction of M-equivalent spaces based on the notion of parallel retracts is developed in [5]. Its modification is considered in [8].

The aim of this note is to introduce the notion of orthogonal retracts and apply it to the construction of M-equivalent spaces.

2. Orthogonal retractions and the relation of strong M-equivalence

All spaces in this note (in particular, all quotient spaces used in the formulation of the claims) are supposed to be Tychonoff and all maps to be continuous if the opposite is not stated. The free [Markov] topological group of a Tychonoff space X is denoted by $F(X)$ $[F_M(X)]$. Let $F_n(X) \subset F(X)$ be the set of words with irreducible form of length $\leq n$. We shall write $X \simeq Y$ if the spaces X and Y are homeomorphic and $G_1 \simeq G_2$ if the topological groups G_1 and G_2 are topologically isomorphic.

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Definition 2.1. Spaces X and Y are called *M-equivalent* (denoted $X \overset{M}{\sim} Y$) if $F_M(X) \simeq F_M(Y)$.

Definition 2.2. Spaces X and Y are said to be *strongly M-equivalent* (denoted $X \overset{M}{\approx} Y$) if there exists a topological isomorphism $f\colon F_M(X) \to F_M(Y)$ such that $f(X) \subseteq F_n(Y)$ and $f^{-1}(Y) \subseteq F_m(X)$ for some n and m.

In [5] Okunev introduced the definition of parallel retractions: retractions r_1 and r_2 of a space X are *parallel* (written $r_1 \parallel r_2$) if $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$ and proved [5, Theorem 2.2] that the R-quotient spaces $X/r_1(X)$ and $X/r_2(X)$ are strongly M-equivalent.

Following Okunev ideas [5, Proposition 2.7] we introduce the following definition:

Definition 2.3. Retractions r_1 and r_2 of a space X are called *orthogonal* ($r_1 \perp r_2$) if the mappings $r_1 \circ r_2$ and $r_2 \circ r_1$ are constant.

As in [5] the term comes from the observation that the projections of the plane onto two orthogonal axes are orthogonal retractions. The images of the space X under orthogonal retractions K_1 and K_2 are called *orthogonal retracts* ($K_1 \perp K_2$) of the space X. The quotient mappings $X \to X/K_i$ are denoted by p_i.

Proposition 2.4. *If K_1 and K_2 are homeomorphic orthogonal retracts of a space X then the quotient spaces $Y_1{=}X/K_1$ and $Y_2{=}X/K_2$ are strongly M-equivalent.*

Proof. Let $f\colon K_2 \to K_1$ be a homeomorphism, $t_1 = f \circ r_2$, $t_2 = f^{-1} \circ r_1$. Denote by $\widehat{p_i}$ the extensions of p_i to the homomorphisms of the free topological groups. Define mappings $g, h\colon X \to F(X)$ putting $g(x) = xr_1^{-1}(x)t_1(x)$ and $h(x) = xr_2^{-1}(x)t_2(x)$.

Let us show that there exists a continuous mapping $g_1\colon Y_1 \to F(Y_2)$ that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & F(X) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \widehat{p}_2} \\ Y_1 & \xrightarrow[g_1]{} & F(Y_2) \end{array}$$

commutative.

Let $y \in Y_1$. There is a point $x \in X$ such that $p_1(x) = y$. Put $g_1(y) = \widehat{p}_2 g(x)$. The point $g_1(y)$ does not depend on the choice of x. Indeed, suppose that $p_1(z) = y$ for $z \in X$. Then $x, z \in K_1$ and $r_1(z) = z$, $r_1(x) = x$ so $g(x) = xr_1^{-1}(x)t_1(x) = t_1(x) = t_1 \circ r_1(x) = f \circ r_2 \circ r_1(x)$ and $g(z) = f \circ r_2 \circ r_1(z)$. Since the mapping $r_2 \circ r_1$ is constant, $g(x) = g(z)$ and thus $\widehat{p}_2 g(z) = \widehat{p}_2 g(x)$. The map g_1 is continuous, because so is the map $\widehat{p}_2 \circ g$ and Y_2 is a quotient space of the space X. In the same way we can show that there exists a continuous mapping $h_1\colon Y_2 \to F(Y_1)$ that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & F(X) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \widehat{p}_1} \\ Y_2 & \xrightarrow[h_1]{} & F(Y_1) \end{array}$$

commutative.

Extend the mappings g_1 and h_1 to the continuous homomorphisms $\overline{g}_1 \colon F(Y_1) \to F(Y_2)$, $\overline{h}_1 \colon F(Y_2) \to F(Y_1)$ and show that they are inverse to each other.

Denote $\{e_1\}=r_1 \circ r_2(X)$, $\{e_2\}=r_2 \circ r_1(X)$. Obviously, $t_2 \circ t_1(x) = r_2(x)$, $t_1 \circ t_2(x) = r_1(x)$, $t_2 \circ r_1(x) = t_2(x)$, $t_1 \circ r_2(x) = t_1(x)$ for each $x \in X$.

Let $y \in Y$ and $x \in p_1^{-1}(y)$. Then

$$\overline{h}_1\overline{g}_1(y) = \overline{h}_1(\widehat{p}_2 g(x)) = \overline{h}_1(\widehat{p}_2(x r_1^{-1}(x) t_1(x)) = (\overline{h}_1\widehat{p}_2(x))(\overline{h}_1\widehat{p}_2 r_1(x))^{-1}(\overline{h}_1\widehat{p}_2 t_1(x)) =$$

$$= \widehat{p}_1 h(x)(\widehat{p}_1 h r_1(x))^{-1}(\widehat{p}_1 h t_1(x)) = \widehat{p}_1(x \times r_2(x)^{-1} \times t_2(x) \times t_2 \circ r_1(x)^{-1} \times$$

$$\times r_2 \circ r_1(x) \times r_1(x)^{-1} \times t_1(x) \times r_2 \circ t_1(x)^{-1} \times t_2 \circ t_1(x)) =$$

$$= \widehat{p}_1(x \times r_2(x)^{-1} \times t_2(x) \times t_2(x)^{-1} \times e_2 \times r_1(x)^{-1} \times t_1(x) \times e_2^{-1} \times r_2(x)) =$$

$$= p_1(x) \times p_1(r_2(x))^{-1} \times p_1(e_2) \times p_1(r_1(x))^{-1} \times p_1(t_1(x)) \times p_1(e_2)^{-1} \times p_1(r_2(x)) = y,$$

because $p_1 r_1(x) = p_1 t_1(x)$ for all $x \in X$ and hence $\overline{h}_1\overline{g}_1 = 1_{F(Y_1)}$.

Similarly, $\overline{g}_1\overline{h}_1 = 1_{F(Y_2)}$. By the construction, Y_1 lies in $F_3(Y_2)$ and Y_2 lies in $F_3(Y_1)$. Hence $Y_1 \overset{M}{\approx} Y_2$. □

Proposition 2.5. *Subsets K_1 and K_2 of a space X are orthogonal retracts of X if and only if the following statements hold:*

a) $|K_1 \cap K_2| \leq 1$;

b) $p_2(K_1)$ is a retract in the quotient space X/K_2 and $p_1(K_2)$ is a retract in the quotient space X/K_1.

Proof. ($\Rightarrow$) Let $K_1 = r_1(X)$ and $K_2 = r_2(X)$ where r_1 and r_2 are orthogonal retractions of X. If $a, b \in K_1 \cap K_2$ then $a = r_1 \circ r_2(a) = r_1 \circ r_2(b) = b$ so $|K_1 \cap K_2| \leq 1$.

Let us prove that $p_2(K_1)$ is a retract in X/K_2. For this purpose we show that there exists a map $\overline{r}_1$ that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{r_1} & K_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2|K_1} \\ X/K_2 & \xrightarrow[\overline{r}_1]{} & p_2(K_1) \end{array}$$

commutative.

Let $y \in X/K_2$. There exists a point $x \in X$ such that $p_2(x) = y$. Put $\overline{r}_1(y) = p_2 r_1(x)$. Let us show that the mapping $\overline{r}_1$ is well-defined. Suppose that $p_2(x) = p_2(z) = y$ for $z \in X$. Then $x, z \in K_2$ and by the orthogonality of r_1 and r_2 we obtain $r_1(z)= r_1(x)$. Hence $p_2 r_1(x) = p_2 r_1(z)$. The map $\overline{r}_1$ is continuous, because so is the map $p_2 \circ r_1$ and X/K_2 is a quotient space of the space X.

Let $x \in p_2(K_1) \subset p_2(X)$. Then $\{\overline{r}_1(x)\} = p_2(r_1(p_2^{-1}(x))) = p_2(p_2^{-1}(x)) = \{x\}$ thus $\overline{r}_1$ is a retraction.

($\Leftarrow$) Let a) and b) hold. Denote by $\overline{r}_1 \colon X/K_2 \longrightarrow p_2(K_1)$ and $\overline{r}_2 \colon X/K_1 \longrightarrow p_1(K_2)$ the retractions. Let us show that there exists a map r_1 that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{r_1} & K_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2|K_1} \\ X/K_2 & \xrightarrow[\overline{r}_1]{} & p_2(K_1) \end{array}$$

commutative.

Since $p_2(K_1)$ is closed in X/K_2 the restriction $p_2|_{p_2^{-1}(p_2(K_1))}$ is a quotient map. If $K_1 \cap K_2 = \varnothing$ then $p_2|_{p_2^{-1}(p_2(K_1))} = p_2|_{K_1}$ so in this case $p_2|_{K_1}$ is a quotient bijection, thus a homeomorphism. If $K_1 \cap K_2$ is a singleton then $p_2|_{p_2^{-1}(p_2(K_1))} = p_2|_{K_1 \cup K_2}$. In this case $p_2\ |_{K_1 \cup K_2}$ is quotient. Since $K_1 \cup K_2$ is a bouquet, the restriction of $p_2|_{K_1\cup K_2}$ to K_1 is a homeomorphism [3, Ex. 2.4.12]. So $p_1|_{K_2}$ is a homeomorphism.

Hence for each point $x \in X$ there is a unique point $y \in K_1$ with $p_2(y) = \overline{r}_1 \circ p_2(x)$. Put $r_1(x) = y$. If $x \in K_1$ then $p_2(x) = \overline{r}_1 \circ p_2(x)$ and therefore r_1 is a retraction of X onto K_1. Since $p_2|_{K_1}$ is a homeomorphism and the diagram is commutative, the map r_1 is continuous. The set $p_2(K_2)$ is a singleton, therefore the set $p_2^{-1}(\overline{r}_1 p_2(K_2))$ is a singleton as well, thus $r_1(K_2)$ is singleton. Hence $r_1 \circ r_2$ is a constant mapping for any retraction $r_2\colon X \to K_2$. In the same way we can prove that there exists a continuous mapping r_2 that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ r_2\ } & K_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ X/K_1 & \xrightarrow[\ \overline{r}_2\ ]{} & p_1(K_2) \end{array}$$

commutative and $r_2 \circ r_1$ is a constant mapping for any retraction $r_1\colon X \to K_1$. □

Corollary 2.6. *If subsets K_1 and K_2 of a space X are orthogonal retracts of X then:*

a) $|K_1 \cap K_2| \le 1$;

b) $p_2(K_1)$ is a retract in the R-quotient space X/K_2 and $p_1(K_2)$ is a retract in the R-quotient space X/K_1.

Proof. The proof of (a) is contained in the proof of Proposition 2.5 and the proof of (b) is contained in [5, Pr. 2.7]. □

Recall that if K is closed in X then the R-quotient and quotient topologies on X/K coincide if and only if the latter is Tychonoff (in particular this holds in the case when X is normal or K is compact) [5]. The space X is called an *absolute retract* if for every space Y containing X as a closed subspace there exists a retraction $Y \to X$.

Corollary 2.7. *Let K_1 and K_2 be absolute retracts contained in a space X. Then K_1 and K_2 are orthogonal retracts if and only if $|K_1 \cap K_2| \le 1$.*

Proposition 2.8. *Let X be a space, $r_i : X \to A_i$, $\rho_j : X \to B_j$ (i,j=1,2) be retractions such that $r_1 \| r_2$, $\rho_1 \| \rho_2$ and $r_i \perp \rho_j$ for each i, j. Let $p_i\colon X \to X/A_i$ be the quotient maps. Then $(X/A_1)/p_1(B_1) \overset{M}{\approx} (X/A_2)/p_2(B_2)$.*

Proof. Since $r_1 \perp \rho_j$, we can define the retractions $\overline{\rho}_i\colon X/A_1 \to p_1(B_i)$ constructed in the proof of Proposition 2.5 which are parallel by constructions. Thus the sets $p_1(B_1)$ and $p_1(B_2)$ are parallel retracts of the space X/A_1 hence $(X/A_1)/p_1(B_1) \overset{M}{\approx} (X/A_1)/p_1(B_2)$. Let $q_2\colon X \to X/B_2$ be the quotient-mappings. Similarly as above, we can show that $(X/B_2)/q_2(A_1) \overset{M}{\approx} (X/B_2)/q_2(A_2)$ therefore

$$(X/A_1)/p_1(B_1) \overset{M}{\approx} (X/A_1)/p_1(B_2) \overset{M}{\approx} (X/B_2)/q_2(A_1) \overset{M}{\approx}$$
$$\overset{M}{\approx} (X/B_2)/q_2(A_2) \overset{M}{\approx} (X/A_2)/p_2(B_2).$$

□

Definition 2.9. [1, Definition 2.3] The *smash product* of spaces X and Y with base points x_0 and y_0 is the quotient space $(X, x_0) \wedge (Y, y_0) = X \times Y/(\{x_0\} \times Y \cup X \times \{y_0\})$.

Corollary 2.10. *For any* (x_1, y_1) *and* (x_2, y_2) *we have*

$$(X, x_1) \wedge (Y, y_1) \overset{M}{\approx} (X, x_2) \wedge (Y, y_2).$$

3. Orthogonality and the relations similar to M-equivalence

If X is a space, by X^+ we shall denote the sum of the space X and the one-point space.

Definition 3.1. Let $\mathfrak{R}$ be some equivalence relation defined on the class of spaces. Consider the following properties :

i) $X\mathfrak{R}Y$ implies $(X \oplus Z)\mathfrak{R}(Y \oplus Z)$ for any spaces X, Y and Z (*additivity*);

ii) $(X/K \oplus K)\mathfrak{R}X^+$ for any retract K of a Tychonoff space X(*Okunev's condition*).

Proposition 3.2. *Let* $\mathfrak{R}$ *be an equivalence relation defined on the class of spaces that satisfies additivity and Okunev's condition . Let* K_1 *and* K_2 *be orthogonal retracts of a space* X *such that* $K_1\mathfrak{R}K_2$*. Then the quotient spaces* $Y_1^+ = (X/K_1)^+$ *and* $Y_2^+ = (X/K_2)^+$ *are* $\mathfrak{R}$*-equivalent.*

Proof. Let $p_i : X \to X/K_i$ be the quotient maps. By [3, Pr. 2.4.14] the spaces $(X/K_1)/p_1(K_2)$ and $(X/K_2)/p_2(K_1)$ are homeomorphic. Let A be a space homeomorphic to $(X/K_1)/p_1(K_2)$. By Proposition 2.5, $p_1(K_2)$ is a retract of the space Y_1. The proof of Proposition 2.5 implies that $p_1(K_2)$ is homeomorphic to K_2. Hence by Okunev's condition,

$$Y_1^+\mathfrak{R}(Y_1/p_1(K_2) \oplus K_2)\mathfrak{R}((X/K_1)/p_1(K_2) \oplus K_2)\mathfrak{R}(A \oplus K_2).$$

Similarly,

$$Y_2^+\mathfrak{R}(Y_2/K_1 \oplus K_1)\mathfrak{R}((X/K_2)/p_2(K_1) \oplus K_1)\mathfrak{R}(A \oplus K_1).$$

By the additivity, $(A \oplus K_2)\mathfrak{R}\ (A \oplus K_1)$. Therefore $Y_1^+\mathfrak{R}Y_2^+$.

□

Corollary 3.3. *Let* X *be a space and* $\mathfrak{R}$ *be an equivalence relation satisfying the additivity and Okunev's conditions,* K_1,$K_2, \ldots, K_n$ *be retracts of* X *such that* $K_i \perp K_{i+1}$ *and* $K_i\mathfrak{R}K_{i+1}$ *for* $i = 1, \ldots, n-1$*. Then* $(X/K_1)^+\mathfrak{R}(X/K_n)^+$.

Corollary 3.4. *Let* $\mathfrak{R}$ *be an equivalence relation on the class of spaces satisfying the additivity and Okunev's conditions. Let* A *and* B *be spaces such that* $A\mathfrak{R}B$*. Let* $a^* \in A$, $b^* \in B$ *be arbitrary points. Then the sets* $\{a^*\} \times B$ *and* $A \times \{b^*\}$ *are orthogonal retracts of the space* $A \times B$ *and hence* $(A \times B/\{a^*\} \times B)^+\ \mathfrak{R}\ (A \times B/A \times \{b^*\})^+$.

We recall the following definitions.

Definition 3.5. The spaces X and Y are said to be

1. *M-equivalent* if for the free Markov topological groups we have $F_M(X) \simeq F_M(Y)$.
2. *Strongly M-equivalent* if there exists a topological isomorphism $f\colon F_M(X) \to F_M(Y)$ such that $f(X) \subseteq F_n(Y)$ and $f^{-1}(Y) \subseteq F_m(X)$ for some n and m.

3. *A-equivalent* [7] if for the free Markov abelian topological groups we have $A_M(X) \simeq A_M(Y)$.
4. *Strongly A-equivalent* [10] if there exists a topological isomorphism $f\colon A_M(X) \to A_M(Y)$ such that $f(X) \subseteq A_n(Y)$ and $f^{-1}(Y) \subseteq A_m(X)$ for some n and m.
5. *Uniformly F_A-equivalent* [11] if the spaces of topological groups $A_M(X)$ and $A_M(Y)$ endowed with the natural uniformities are uniformly homeomorphic.
6. *F_A-equivalent* if the spaces of the groups $A_M(X)$ and $A_M(Y)$ are homeomorphic.
7. *a-equivalent* [10] if for any abelian topological group G we have $C_p(X;G) \simeq C_p(Y;G)$.
8. *L-equivalent* [7] if for the free locally convex spaces we have $L(X) \simeq L(Y)$.
9. *Strongly L-equivalent* [10] if there exists a linear homeomorphism $f\colon L(X) \to L(Y)$ such that $f(X) \subseteq L_n(Y)$ and $f^{-1}(Y) \subseteq L_m(X)$ for some n and m.
10. *l-equivalent* [7] if for the spaces of continuous real-valued functions in the pointwise topology we have $C_p(X) \simeq C_p(Y)$.
11. *Strongly l-equivalent* [10] if there exists a linear homeomorphism $f : L_p(X) \to L_p(Y)$ of free locally convex spaces endowed with weak topologies such that $f(X) \subseteq L_p^n(Y)$ and $f^{-1}(Y) \subseteq L_p^m(X)$ for some n and m.
12. *u-equivalent* [7] if the spaces $C_p(X)$ and $C_p(Y)$ endowed with the natural uniformities are uniformly homeomorphic.
13. *t-equivalent* [7] if the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic.
14. *l^*-equivalent* [3] if for the spaces of continuous real-valued bounded functions in the pointwise convergence topology we have $C_p^*(X) \simeq C_p^*(Y)$.
15. *t^*-equivalent* if the spaces $C_p^*(X)$ and $C_p^*(Y)$ are homeomorphic.

Corollary 3.6. *The following equivalence relations satisfy the additivity and Okunev's condition.*

a) Relations 1-15.

b) The isomorphism of free topological algebras with signature including group operation (and hence an isomorphism of free topological groups in some β-variety) [7].

c) The isomorphism of free (abelian) compact groups [7].

Proof. Since the relation of strong M-equivalence satisfies Okunev's condition, all the above relations satisfy it too. The additivity for relations 1-4, 8-9, 11, b) and c) can be proved similarly to [4, St. 8.8], for relations 5-6 it holds since the groups $A_M(X \oplus Y)$ and $A_M(X) \times A_M(Y)$ are topologically isomorphic [8], for relations 7, 10, 12-13 it holds since the linear spaces $C_p(X \oplus Y; G)$ and $C_p(X;G) \times C_p(Y;G)$ are topologically isomorphic and for relations 14-15 it holds since the linear spaces $C_p^*(X \oplus Y)$ and $C_p^*(X) \times C_p^*(Y)$ are topologically isomorphic. □

Proposition 3.2 leads to the following question: find equivalence relations $\mathfrak{R}$ such that $X^+\mathfrak{R}Y^+$ implies $X\mathfrak{R}Y$. The following proposition gives a partial answer to this question.

Proposition 3.7. *Let X and Y be spaces such that $X^+ \overset{\mathfrak{R}}{\sim} Y^+$. Then $X \overset{\mathfrak{R}}{\sim} Y$ if $\mathfrak{R}$ is one of the following relations:*

a) F_A-equivalence.

b) Uniform F_A-equivalence.
c) A-equivalence.
d) L-equivalence.
e) l-equivalence.
f) strong A-equivalence.
g) strong L-equivalence.
h) strong l-equivalence.

Proof. Parts a) and b) hold, because the spaces $A_M(X)$ and $A_M(X \oplus N)$ are uniformly homeomorphic for every nonempty space X [11].

Part c): Let

$$A_0(X) = \left\{ x \in A_M(X) : x = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n, \sum_{i=1}^{n} \varepsilon_i = 0 \right\}.$$

If $A_M(X) \simeq A_M(Y)$ then $A_0(X) \simeq A_0(Y)$ [6, Pr. 3.6]. We are going to show that $A_0(X) \simeq A_G(X)$ where $A_G(X)$ is the Graev free topological group over the space X with the distinguished point $a \in X$. Define a mapping $i\colon X \to A_0(X)$ putting $i(x) = x - a$. Since $i(a) = 0$, there exists a continuous homomorphism $I\colon A_G(X) \to A_0(X)$ such that $I|X = i$. The identity mapping $id_X\colon X \to X$ can be extended to a continuous homomorphism $h\colon A_M(X) \to A_G(X)$. Since the restrictions of h and I^{-1} onto X coincide, $I^{-1} = h$ is a topological isomorphism.

Thus if $A_M(X^+) \simeq A_M(Y^+)$ then $A_G(X^+) \simeq A_G(Y^+)$ and since $A_M(X) \simeq A_G(X^+)$, we have $A_M(X) \simeq A_M(Y)$.

Part d): Put

$$L_0(X) = \left\{ x \in L(X) : x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n, \sum_{i=1}^{n} \lambda_i = 0 \right\}.$$

Similarly to the proof of part c) we can show that $L_0(X^+) = L(X)$ and then proceed similarly to the above.

Part e): Let $L_p(X)$ be the free locally convex space over X in the variety of locally convex spaces with the weak topology. Since the spaces $C_p(X)$ and $L_p(X)$ are dual, $C_p(X) \simeq C_p(Y)$ if and only if $L_p(X) \simeq L_p(Y)$. For the spaces $L_p(X)$ and $L_p(Y)$ the proof is similar to the above. □

Parts f), g), h) are obtained from parts c), d) and e) respectively by noting that if a topological isomorphism between $A_M(X)$ and $A_M(Y)$ (linear homeomorphism of $L(X)$ and $L(Y)$) is “strong” then the special topological isomorphism between $A_M(X)$ and $A_M(Y)$ (linear homeomorphism of $L(X)$ and $L(Y)$) constructed in [6, Pr 3.6, 3.7] is also “strong”.

For a cardinal number τ, we denote by D_τ the discrete space of cardinality τ.

Proposition 3.8. *Let $r_i : X \to K_i$ for $i = 1, 2$ be retractions of a space X such that $K_1 \overset{M}{\sim} K_2$, $r_1 \circ r_2(X) \simeq D_{\tau_1}$ and $r_2 \circ r_1(X) \simeq D_{\tau_2}$. Then $X/K_1 \oplus D_\tau \overset{M}{\sim} X/K_2 \oplus D_\tau$, for some $\tau \leq \tau_1 \times \tau_2$.*

Proof. The idea of the proof is to make a decomposition of X into its disjoint clopen subspaces $C_{s,t}$ such that the restrictions of $r_1 \circ r_2$ and $r_2 \circ r_1$ are constant maps and then proceed similarly to the case of orthogonal retracts. Put

$$A = r_1 \circ r_2(X) = \{a_s \mid s \in \tau_1\}, B = r_2 \circ r_1(X) = \{b_t \mid t \in \tau_2\},$$
$$A_s = (r_1 \circ r_2)^{-1}(a_s),\ B_t = (r_2 \circ r_1)^{-1}(b_t),\ C_{s,t} = A_s \cap B_t.$$

Let $P = \{(s,t) \in (\tau_1 \times \tau_2) \mid C_{s,t} \cap K_1 \cap K_2 \neq \oslash\}$, $\tau = |P|$ and $K^i_{s,t} = K_i \cap C_{s,t}$ for each s, t and i. For $i = 1,2$ we put $x \sim_i y$ if and only if either $x = y$ or there exist $(s_1,t_1),(s_2,t_2) \in (\tau_1 \times \tau_2)\backslash P$ such that $x \in K^i_{s_1,t_1}$, $y \in K^i_{s_2,t_2}$ or there exists $(s,t) \in P$ such that $x, y \in K^i_{s,t}$. Let $\sim_3$ be the smallest equivalence relation on X containing the relations $\sim_1$ and $\sim_2$. For $i = 1,2$ let $x \sim_{Ki} y$ if and only if $x = y$ or $x, y \in K_i$. Let $\approx_i$ be the smallest equivalence relation on X containing the relations $\sim_3$ and $\sim_{Ki}$.

For $i = 1,2,3$ let $q_i \colon X \to X/\sim_i$ be the quotient map. For $i = 1,2$ let $V_i = q_3(K_i)$. It is easy to check that the space V_i is discrete. Since $\sim_{Ki} \supset \sim_i$ then [3, Pr. 2.4.14] implies $(X/\approx_1) \simeq (X/\sim_3)/V_1 \simeq (X/\sim_2)/q_2(K_1)$ and $(X/\approx_2) \simeq (X/\sim_3)/V_2 \simeq (X/\sim_1)/q_1(K_2)$. The sets V_i are parallel retracts of the space $X/\sim_3$ hence $(X/\sim_3)/V_1 \overset{M}{\sim} (X/\sim_3)/V_2$ thus $(X/\approx_1) \overset{M}{\sim} (X/\approx_2)$. For $i = 1,2$ put $U_i = \{q_i(K^i_{s,t}) : (s,t) \in P\}$. The space U_i is homeomorphic to D_τ an U_i is a retract of the space $X/\sim_i$. For $i = 1,2$ we have $X/K_i \oplus D_\tau = ((X/\sim_i)/U_i) \oplus U_i \overset{M}{\sim} (X/\sim_i)^+$.

The space $X/\sim_1$ contains the retract $q_1(K_2)$ homeomorphic to K_2 hence

$$(X/\sim_1)^+ \overset{M}{\sim} (X/\sim_1)/q_1(K_2) \oplus q_1(K_2) \overset{M}{\sim} (X/\approx_2) \oplus K_2.$$

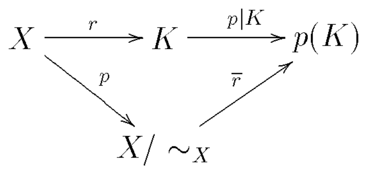
Therefore $X/K_1 \oplus D_\tau \overset{M}{\sim} (X/\approx_2) \oplus K_2 \overset{M}{\sim} (X/\approx_1) \oplus K_1 \overset{M}{\sim} X/K_2 \oplus D_\tau$. □

4. Some consequences from Okunev's construction

Let Y be a subspace of a space X and $\sim_Y$ be an equivalence relation defined on Y. We can extend this relation to the relation $\sim_X$ on X putting $x \sim_X y$ if and only if $x = y$ or $x, y \in Y$ and $x \sim y$. Sometimes we shall write $\sim_Y$ instead of $\sim_X$.

Proposition 4.1. *Let K be a retract of a space X. Let $\sim_K$ be an equivalence relation on K and $\sim_X$ be its extension onto X. Let $p : X \to X/\sim_X$ be the quotient mapping. Then the restriction $p|K$ is a quotient map and $p(K)$ is a retract of the quotient space $X/\sim_X$.*

Proof. Being a retract of X the set K is closed in X. Hence $p(K)$ is closed in $X/\sim_X$ and the map $p|_K = p|p^{-1}(p(K))$ is a quotient map and hence a homeomorphism. Let $r : X \to K$ be a retraction. Since $p \circ r(x) = p \circ r(y)$ provided $x \sim_X y$, there is a map $\overline{r} : X/\sim_X \to p(K)$ such that the diagram



is commutative. It is clear that $\overline{r}$ is a retraction onto $p(K)$ and since the map $r \circ (p|K)$ is continuous, the map $\overline{r}$ is continuous too. □

A map $p : X \to Y$ is called *R-quotient* provided $p(X) = Y$ and every function $f : X \to \mathbb{R}$ is continuous if and only if the composition $p \circ f$ is continuous.

Proposition 4.2. *Let K be a retract of a space X. Let $\sim_K$ be an of equivalence on K and $\sim_X$ be its extension onto X. Let $p : X \to X/\sim_X$ be an R-quotient mapping. Then the restriction $p|K$ is an R-quotient map and $p(K)$ is a retract of the R-quotient space $X/\sim_X$.*

Proof. Similarly to Proposition 4.1 we can define a map $\overline{r} : X/\sim_X \to p(K)$ and using [5, Pr. 1.1] we can verify that the map $\overline{r}$ is a retraction. Let us show that the map p_K is R-quotient. Let $f : p(K) \to \mathbb{R}$ be a function such that the composition $f_1 = f \circ (p|K)$ is continuous. Then the function $g_1 = f_1 \circ r$ is a continuous extension of f_1 onto X. Put $g = f \circ \overline{r} = p \circ g_1$. Since the map p is R-quotient, the map g is continuous thus the map $f = g|p(K)$ is also continuous. □

Proposition 4.3. *Let X be a space, K_1 and K_2 its retracts such that $X/K_1 \overset{M}{\sim} X/K_2$. Let $\sim_1$ and $\sim_2$ be equivalence relations on K_1 and K_2 respectively such that $(K_1/\sim_1) \overset{M}{\sim} (K_2/\sim_2)$. Then $(X/\sim_1)^+ \overset{M}{\sim} (X/\sim_2)^+$.*

Proof. Let $p : X \to X/\sim_1$ be the quotient map. By Proposition 4.1 $p(K_1)$ is a retract of the space $X/\sim_1$ and $p(K_1)$ is homeomorphic to $K_1/\sim_1$. Put $Z = X/\sim_1 \oplus K_1/\sim_1$. Since $p(K_1)$ and $K_1/\sim_1$ are parallel retracts of the space Z, by [5, Th. 2.2], $(X/\sim_1)^+ \overset{M}{\sim} (X/K_1) \oplus (K_1/\sim_1)$. Similarly, $(X/\sim_2)^+ \overset{M}{\sim} (X/K_2) \oplus (K_2/\sim_2)$. Since the relation of M-equivalence is additive,

$$(X/K_1) \oplus (K_1/\sim_1) \overset{M}{\sim} (X/K_2) \oplus (K_2/\sim_2),$$

therefore

$$(X/\sim_1)^+ \overset{M}{\sim} (X/K_1) \oplus (K_1/\sim_1) \overset{M}{\sim} (X/K_2) \oplus (K_2/\sim_2) \overset{M}{\sim} (X/\sim_2)^+.$$

□

Remark that the above theorem may be applied in the situations when K_1 and K_2 are either parallel or orthogonal retracts.

Let X and Y be spaces. Let $x \in X$ and $y \in Y$. By $(X, x) \vee (Y, y)$ we denote the bouquet of the spaces X and Y with distinguished points x and y. It was proved in [5, Pr. 2.6] that, up to M-equivalence, $(X, x) \vee (Y, y)$ does not depend on the distinguished points so we shall write $X \vee Y$.

Lemma 4.4. *If a space X contains a copy of $I = [0, 1]$ then $X \vee I \overset{M}{\sim} X$.*

Proof. Consider the space Z=$X \vee I$. Let I_1 be a copy of I contained in X. Then I and I_1 are parallel retracts of X. By [5, Th. 2.2] $(X/I) \vee I \overset{M}{\sim} X$. Let $I_2 = [0, 1/2] \subset I$. The spaces I_1 and I_2 are also parallel retracts of the space Z. So $(X/I) \vee I \overset{M}{\sim} X \vee I$ by [5, Th. 2.2]. Hence $X \vee I \overset{M}{\sim} X$. □

Proposition 4.5. *Let X and Y be linearly connected spaces such that $X \overset{A}{\sim} Y$. Let $A \subseteq X$ and $B \subseteq Y$ be finite subspaces with $|A| = |B|$. Then $X/A \overset{A}{\sim} Y/B$.*

Proof. Put $n = |A|$. At first we suppose that $n = 2$. Let $A = \{a, b\}$ $(a \neq b)$. By [3, Ex. 6.3.12] there exists an embedding $i : I \hookrightarrow X$ such that $i(0) = a$ and $i(1) = b$. Consider the space $Z{=}X \vee I$. Since the space I is an absolute retract, the spaces $i(I)$ and I are parallel retracts of the space Z. Define an equivalence relations on X putting $x \sim_1 y$ if and only if $x = y$ or $\{x, y\} = \{0, 1\}$ and $x \sim_2 y$ if and only if $x = y$ or $\{x, y\} = \{i(0), i(1)\}$. Let S be a circle.

Applying Proposition 4.3 for $X = Z$, $K_1 = i(I)$ and $K_2 = I$ we obtain $(X/Y) \vee I^+ \overset{M}{\sim} X \vee S^+ \overset{M}{\sim} X \oplus S$. Applying Lemma 4.4 we obtain $(X/A)^+ \overset{A}{\sim} (X/A) \vee I^+ \overset{A}{\sim} X \oplus S \overset{A}{\sim} Y \oplus S \overset{A}{\sim} (Y/B)^+$. By Proposition 3.7 we have $X/A \overset{A}{\sim} Y/B$.

Suppose that the statement is already proved for $|A| \leq n$. Let now $|A| = n$. Choose points $a \in A$ and $b \in B$ and put $A' = A \backslash \{a\}$, $B' = B \backslash \{b\}$. Then $X/A' \overset{A}{\sim} Y/B'$ and the spaces X/A' and Y/B' are linearly connected. Let $p : X \to X/A'$ and $q : Y \to Y/B'$ be the quotient maps. Since $p(A)$ and $q(B)$ are discrete sets of cardinality 2, we see that $X/A \overset{A}{\sim} (X/A')/p(A) \overset{A}{\sim} (Y/B')/p(B) \overset{A}{\sim} Y/B$. □

Proposition 4.6. *Let $r_i : X \to K_i$ be retractions such that $r_1 \circ r_2(X) = r_2 \circ r_1(X)$ and $K_1 \overset{M}{\sim} K_2$. Then $X/K_1 \oplus (K_1 \cap K_2) \overset{M}{\sim} X/K_2 \oplus (K_1 \cap K_2)$.*

Proof. Put $A = K_1 \cap K_2$. It is easy to check that $r_1 \circ r_2(X) = r_2 \circ r_1(X) = A$. Put $Y = X/A$ and $\overline{K_i} = K_i/A$. Let $p : X \to Y$, $p_i : Y \to Y/\overline{K_i}$ be the quotient maps. By Proposition 4.1 the sets $\overline{K_i}$ are retracts of the space Y. Let $\overline{r_i} : Y \to \overline{K_i}$ be retractions such that $\overline{r_i} \circ p = p \circ r$. Then $\overline{r_1} \circ \overline{r_2}(X) = p(r_1 \circ r_2(X)) = p(A)$ is a singleton. Similarly $\overline{r_2} \circ \overline{r_1}(X) = p(A)$ is a singleton. Thus $\overline{r_1}$ and $\overline{r_2}$ are orthogonal retractions. From the proof of Proposition 3.2 using [5, Th. 2.5] we obtain $Y/\overline{K_1}/p_1(\overline{K_2}) \vee \overline{K_2} \overset{M}{\sim} Y/\overline{K_1}$ and $Y/\overline{K_2}/p_2(\overline{K_1}) \vee \overline{K_1} \overset{M}{\sim} Y/\overline{K_2}$. Note that $Y/\overline{K_i} \overset{M}{\sim} X/K_i$ and $(Y/\overline{K_1})/p_1(\overline{K_2}) \overset{M}{\sim} (Y/\overline{K_2})/p_2(\overline{K_1})$ by [3, Pr. 2.4.14].

For the restrictions $r_1 |_{K_2}: K_2 \to A$ and $r_2 |_{K_1}: K_1 \to A$ we have $r_1 \circ r_2(X) \subseteq r_2(X)$ and $r_2 \circ r_1(X) \subseteq r_1(X)$ thus they are retractions. Using [5, Th. 2.5] we have $\overline{K_i} \vee A \overset{M}{\sim} K_i$. Now we obtain

$X/K_1 \oplus A \overset{M}{\sim} Y/\overline{K_1} \oplus A \overset{M}{\sim} (Y/\overline{K_1}/p_1(\overline{K_2}) \vee \overline{K_2}) \oplus A \overset{M}{\sim} (Y/\overline{K_1})/p_1(\overline{K_2}) \oplus (A \vee \overline{K_2}) \overset{M}{\sim} (Y/\overline{K_1})/p_1(\overline{K_2}) \oplus K_2$.

Similarly $X/K_1 \oplus A \overset{M}{\sim} (Y/\overline{K_2})/p_2(\overline{K_1}) \oplus K_1$.

By the additivity $(Y/\overline{K_1})/p_1(\overline{K_2}) \oplus K_2 \overset{M}{\sim} (Y/\overline{K_2})/p_2(\overline{K_1}) \oplus K_1$. Hence $X/K_1 \oplus A \overset{M}{\sim} X/K_2 \oplus A$. □

Note 4.7. *Taking $X = [0, 1]$, $K_1 = [0, 1]$ and $K_2 = [0, 1/2]$ we can show that the component $K_1 \cap K_2$ is essential in the last proposition.*

Question 4.8. Let X be a space and $r_i \colon X \to K_i$, $i = 1, \ldots, n$ its retractions. We say that $\{r_1, r_2, \ldots. r_n\}$ are *independent* if $r_{\sigma(1)} \circ r_{\sigma(2)} \circ \cdots \circ r_{\sigma(n)} = const_\sigma$ for any permutation σ on $\{1, 2, \ldots, n\}$. If $n = 2$ then the independence coincides with the orthogonality. Are there generalizations of Proposition 3.2 in the case $n \geq 3$?

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