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ON THE BOUNDEDNESS OF L-M- AND L-μ-INDEX OF THE DIRICHLET SERIES

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We find conditions on the exponents of the Dirichlet series with an arbitrary abscissa of absolute convergence in order that the relations $\ln M(\sigma, F) = O(\Phi(\sigma))$, $\sigma \uparrow A$ and $\ln \mu(\sigma, F) = O(\Phi(\sigma))$, $\sigma \uparrow A$ be equivalent, where Φ is a certain convex function defined on $(-\infty, A)$, $A \in (-\infty, +\infty]$. The obtained result was applied to proving equivalence of the boundedness of l-M and l-μ-index of the Dirichlet series.

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Найдены условия на показатели ряда Дирихле $F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}$ с абсциссой $A \in (-\infty, +\infty]$ абсолютной сходимости, достаточные для того, чтобы соотношения $\ln M(\sigma, F) = O(\Phi(\sigma))$ и $\ln \mu(\sigma, F) = O(\Phi(\sigma))$ при $\sigma \uparrow A$ были равносильными, где $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$, $\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$ и Φ — некоторая выпуклая на $(-\infty, A)$ функция. Полученный результат применен к доказательству эквивалентности ограниченности l-M- и l-μ-индексов ряда Дирихле.

1. Introduction. Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an increasing to $+\infty$ sequence of nonnegative numbers ($\lambda_0 = 0$) and $S(\Lambda; A)$ be a class of Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

with the abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$. For $\sigma < A$ we put $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ and let $\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$ be the maximal term of series (1).

We suppose that a positive continuous function l on $(-\infty, A)$ satisfies the condition $\int_{\sigma}^{A} l(t)dt = +\infty$ for each $\sigma < A$ and, as in [1], we say that a function $F \in S(\Lambda; A)$ is of bounded l-M-index if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and all $\sigma \in (-\infty, A)$

$$\frac{M(\sigma, F^{(n)})}{n! l^n(\sigma)} \leq \max\left\{\frac{M(\sigma, F^{(k)})}{k! l^k(\sigma)} : 0 \leq k \leq N\right\}, \tag{2}$$

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and is of bounded l-μ-index [2] if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and all $\sigma \in (-\infty, A)$ inequality (2) holds with $\mu(\sigma, F^{(j)})$ instead of $M(\sigma, F^{(j)})$.

In this paper we will find a connection between the boundedness of l-M-index and the boundedness of l-μ-index for some class of functions l.

By $\Omega(A)$ we denote the class of functions Φ defined on $(-\infty, A)$ which are positive and unbounded, and moreover their derivatives Φ' are positive continuously differentiable functions which increase to $+\infty$ on $(-\infty, A)$. For $\Phi \in \Omega(A)$ let φ be the inverse function to Φ' and $\Psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(\sigma)$ be a function associated with Φ in the sense of Newton. Then [3] the function Ψ increases to A on $(-\infty, A)$ and the function φ increases to A on $(0, +\infty)$.

We consider only the cases $A = +\infty$ and $A = 0$ because the case $-\infty < A < +\infty$ reduces to the case $A = 0$ using the replacement of s by $s - A$. Our aim is to prove the following theorem.

Theorem 1. *Let either $A = 0$ or $A = +\infty$ and a function $\Phi \in \Omega(A)$ satisfy the conditions:*

1) $\Phi(\sigma)/\Phi'(\sigma) \searrow 0$, $\sigma \uparrow A$,
2) $\Phi'(\sigma + O(\Phi(\sigma)/\Phi'(\sigma))) = O(\Phi'(\sigma))$, $\sigma \uparrow A$,
3) $\ln \Phi'(\sigma) = o(\Phi(\sigma))$, $\sigma \uparrow A$,

and in the case $A = 0$ also the condition

4) $|\sigma|\Phi'(\sigma)/\Phi(\sigma) \to +\infty$, $\sigma \uparrow 0$.

Assume that

$$\ln n(t) = O(\Phi(\Psi(\varphi(t)))), \quad t \to +\infty, \tag{3}$$

where $n(t) = \sum_{\lambda_n \le t} 1$ is the counting function of the sequence Λ.

Then $F \in S(\Lambda, A)$ is of bounded l-M-index if and only if F is of bounded l-μ-index.

The proof of Theorem 1 is based on the following three statements.

Lemma 1 [1]. *Let $A \in (-\infty, +\infty]$, $F \in S(\Lambda, A)$ and $\ln n(t) \le t/\alpha(t)$, $t \ge t_0$, where α is a positive continuous increasing to $+\infty$ on $[0, +\infty)$ function such that $\alpha(t) = o(t)$, $t \to +\infty$. Suppose that $\Phi \in \Omega(A)$ satisfies the conditions:*

1) $\Phi'(\sigma) = O(\Phi'(\Psi(\sigma)))$, $\sigma \uparrow A$;
2) $\Phi'(\sigma + 2/\alpha(a\Phi'(\sigma))) = O(\Phi'(\sigma))$, $\sigma \uparrow A$, *for each* $a \in (0, +\infty)$;
3) $2\Phi'(\sigma)/\alpha(\Phi'(\sigma)) < \Phi(\sigma) + (A - \sigma)\Phi'(\sigma)$ *for all* $\sigma \in [\sigma_0, A)$.

In order that F be of bounded Φ'-M-index (i.e. l-M-index with $l(\sigma) = \max\{\Phi'(\sigma), 1\}$) it is necessary and sufficient that $\ln M(\sigma, F) = O(\Phi(\sigma))$, $\sigma \uparrow A$.

Lemma 2 [2]. *Let $A \in (-\infty, +\infty]$ and $F \in S(\Lambda, A)$. Suppose that $\Phi \in \Omega(A)$ satisfies the conditions:*

1) $\Phi'(\sigma) = O(\Phi'(\Psi(\sigma)))$, $\sigma \uparrow A$;
2) $\Phi'(\sigma + 1/\Phi'(\sigma)) = O(\Phi'(\sigma))$, $\sigma \uparrow A$;
3) $1/\Phi'(\sigma) < (A - \sigma)$, $\sigma_0 \le \sigma < A$.

In order that F be of bounded Φ'-μ-index it is necessary and sufficient that $\ln\,\mu(\sigma, F) = O(\Phi(\sigma))$, $\sigma \uparrow A$.

Theorem 2. *Let either $A = 0$ or $A = +\infty$ and a function $\Phi \in \Omega(A)$ satisfy the conditions:*

1) $\Phi(\sigma)/\Phi'(\sigma) \searrow 0$, $\sigma \uparrow A$,

2) $\Phi(\sigma + O(\Phi(\sigma)/\Phi'(\sigma))) = O(\Phi(\sigma))$, $\sigma \uparrow A$,

3) $\ln\,\Phi'(\sigma) = o(\Phi(\sigma))$, $\sigma \uparrow A$,

and in the case $A = 0$ also the condition

4) $|\sigma|\Phi'(\sigma)/\Phi(\sigma) \to +\infty$, $\sigma \uparrow 0$.

In order that for each function $F \in S(\Lambda, A)$ the correlations

$$\ln\,M(\sigma, F) = O(\Phi(\sigma)),\ \sigma \uparrow A, \tag{4}$$

and

$$\ln\,\mu(\sigma, F) = O(\Phi(\sigma)),\ \sigma \uparrow A. \tag{5}$$

be equivalent it is necessary and sufficient that condition (3) holds.

We note that condition 3) of Theorem 2 in Lemmas 1 and 2 is unnecessary.

2. Proof of Theorem 2. At first we remark that from the Cauchy inequality $\mu(\sigma, F) \le M(\sigma, F)$ it follows that (4) implies (5). In order to prove that (5) implies (4) we need the following lemma.

Lemma 3 [3]. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega(A)$ and $F \in S(\Lambda, A)$. In order that $\ln\mu(\sigma, F) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_0,\, A)$, it is necessary and sufficient that $\ln|a_n| \le -\lambda_n\Psi(\varphi(\lambda_n))$ for all $n \ge n_0$.*

Using Lemma 3 we will prove the following lemma.

Lemma 4. *Let either $A = 0$ or $A = +\infty$, $F \in S(\Lambda, A)$ and a function $\Phi \in \Omega(A)$ satisfy the conditions:*

1) $\Phi(\sigma)/\Phi'(\sigma) \searrow 0$, $\sigma \uparrow A$,

2) $\Phi(\sigma + O(\Phi(\sigma)/\Phi'(\sigma))) = O(\Phi(\sigma))$, $\sigma \uparrow A$,

and in the case $A = 0$ also the condition

3) $|\sigma|\Phi'(\sigma)/\Phi(\sigma) \to +\infty$, $\sigma \uparrow 0$.

If condition (3) holds then (5) implies (4).

Proof. From (5) we have $\ln\mu(\sigma, F) \le K_1\Phi(\sigma)$ for some constant $K_1 \ge 1$ and all $\sigma \in [\sigma_0,\, A)$ and from Lemma 3 it follows that for all $n \ge 0$

$$\ln|a_n| \le -\lambda_n\Psi(\varphi(\lambda_n/K_1)). \tag{6}$$

From (3) we have $\ln\,n(t) \le K_2\Phi(\Psi(\varphi(t)))$ for some constant $K_2 > 0$ and for all $t \ge t_0$. Let us put $\beta(\sigma) = (K_2+1)\Phi(\sigma)/\Phi'(\Psi^{-1}(\sigma))$. Then $\sigma + \beta(\sigma) < A$ for all $\sigma \in [\sigma_0,\, A)$. Indeed, if $A = +\infty$ then the inequality is obvious, and if $A < +\infty$ then it holds due to condition 3).

From the inequality $\sigma + \beta(\sigma) < A$ it follows that the function $\Psi^{-1}(\sigma + \beta(\sigma))$ is defined on $[\sigma_0, A)$.

Since $\Psi'(\sigma) = \Phi(\sigma)\Phi''(\sigma)/(\Phi'(\sigma))^2$, $\Psi(\sigma) < \sigma$ and $\Phi'(\sigma)/\Phi(\sigma)$ is nondecreasing function, we have

$$\left(\frac{\Phi(\Psi(\sigma))}{\Phi'(\sigma)}\right)' = \frac{\Phi''(\sigma)\Phi(\sigma)\Phi(\Psi(\sigma))}{(\Phi'(\sigma))^3}\left(\frac{\Phi'(\Psi(\sigma))}{\Phi(\Psi(\sigma))} - \frac{\Phi'(\sigma)}{\Phi(\sigma)}\right) \le 0,$$

that is the functions $\Phi(\Psi(\sigma))/\Phi'(\sigma)$ and $\beta(\sigma)$ are nonincreasing.

Hence it follows that $\beta(\sigma) \ge (K_2+1)\Phi(\sigma+\beta(\sigma))/\Phi'(\Psi^{-1}(\sigma+\beta(\sigma)))$, and also in view of (3) and condition 1) we have $\ln n(t) = o(t)$, $t \to +\infty$.

Let us put $\gamma(\sigma) = K_1\Phi'(\Psi^{-1}(\sigma+\beta(\sigma)))$. Then in view of (6), for $\sigma \ge \sigma_0$ we have

$$\sum_{\lambda_n > \gamma(\sigma)} \exp\{-\lambda_n(\Psi(\varphi(\lambda_n/K_1)) - \sigma)\} \le \sum_{\lambda_n > \gamma(\sigma)} \exp\{-\lambda_n(\Psi(\varphi(\gamma(\sigma)/K_1)) - \sigma)\} =$$

$$= \sum_{\lambda_n > \gamma(\sigma)} \exp\{-\lambda_n\beta(\sigma)\} \int\limits_{\gamma(\sigma)}^{\infty} \exp\{-t\beta(\sigma)\}dn(t) \le$$

$$\le \beta(\sigma)\int\limits_{\gamma(\sigma)}^{\infty} n(t)\exp\{-t\beta(\sigma)\}dt \le \beta(\sigma)\int\limits_{\gamma(\sigma)}^{\infty} \exp\{-t\beta(\sigma) + K_2\Phi(\Psi(\varphi(t)))\}dt \le$$

$$\le \beta(\sigma)\int\limits_{\gamma(\sigma)/K_1}^{\infty} \exp\{-t(\beta(\sigma) - K_2\Phi(\Psi(\varphi(t)))/t\}dt \le$$

$$\le \beta(\sigma)\int\limits_{\gamma(\sigma)/K_1}^{\infty} \exp\left\{-t\left(\beta(\sigma) - \frac{K_2\Phi(\sigma+\beta(\sigma))}{\Phi'(\Psi^{-1}(\sigma+\beta(\sigma)))}\right)\right\}dt \le$$

$$\le \beta(\sigma)\int\limits_{\gamma(\sigma)/K_1}^{\infty} \exp\left\{-t\left(\beta(\sigma) - \frac{K_2\Phi(\sigma)}{\Phi'(\Psi^{-1}(\sigma))}\right)\right\}dt =$$

$$= \beta(\sigma)\int\limits_{\Phi'(\Psi^{-1}(\sigma+\beta(\sigma)))}^{\infty} \exp\{-t\beta(\sigma)/(K_2+1)\}dt \le K_2+1,$$

that is $M(\sigma, F) \le n(\gamma(\sigma))\mu(\sigma, F) + K_2 + 1$, thus,

$$\ln M(\sigma, F) \le \ln \mu(\sigma, F) + \ln n(\gamma(\sigma)) + o(1), \quad \sigma \to A,$$

and in order to obtain (4) it is enough to prove that $\ln n(\gamma(\sigma)) = O(\Phi(\sigma))$, $\sigma \to A$.

From nonincreasing of $\Phi(\Psi(\varphi(t)))/t$ we have $\Phi(\Psi(\varphi(t/K_1))) \ge \Phi(\Psi(\varphi(t)))/K_1$ and from condition 2) it follows that $\Phi(\sigma+\beta(\sigma)) \le \Phi(\sigma + (K_2+1)\Phi(\sigma)/\Phi'(\sigma)) \le K_3\Phi(\sigma)$ for all $\sigma \in [\sigma^*, A)$. Therefore, from (3) we obtain

$$\varlimsup_{\sigma\uparrow A}\frac{\ln n(\gamma(\sigma))}{\Phi(\sigma)} = \varlimsup_{\sigma\uparrow A}\frac{\ln n(K_1\Phi'(\Psi^{-1}(\sigma+\beta(\sigma))))}{\Phi(\sigma)} \le$$
$$\le K_3\varlimsup_{\sigma\uparrow A}\frac{\ln n(K_1\Phi'(\Psi^{-1}(\sigma+\beta(\sigma))))}{\Phi(\sigma+\beta(\sigma))} = K_3\varlimsup_{t\to+\infty}\frac{\ln n(t)}{\Phi(\Psi(\varphi(t/K_1))} \le K_3K_1K_2,$$

and, thus, Lemma 4 is proved. □

In order to ascertain the necessity of condition (3) in Lemma 4 we need the following lemma.

Lemma 5 [4, 5]. *Let γ be a positive continuous increasing to $+\infty$ on $[0,+\infty)$ function and $\overline{\lim\limits_{n\to\infty}}\dfrac{\ln n}{\gamma(\lambda_n)}>1$. Then there exists a subsequence $(\lambda_k^\star)$ of the sequence (λ_n) such that $k\le\exp\{\gamma(\lambda_k^\star)\}+1\,(k\in\mathbb{N})$ and $k_j\ge\exp\{\gamma(\lambda_{k_j}^\star)\}$ for some increasing sequence (k_j) of positive integers.*

Lemma 6. *Let either $A=0$ or $A=+\infty$ and a function $\Phi\in\Omega(A)$ satisfy the conditions:*

1) $\Phi(\sigma)/\Phi'(\sigma)\searrow 0$, $\sigma\uparrow A$,

2) $\Phi(\sigma+O(\Phi(\sigma)/\Phi'(\sigma)))=O(\Phi(\sigma))$, $\sigma\uparrow A$,

3) $\ln\,\Phi'(\sigma)=o(\Phi(\sigma))$, $\sigma\uparrow A$,

and in the case $A=0$ also the condition

4) $|\sigma|\Phi'(\sigma)/\Phi(\sigma)\to+\infty$, $\sigma\uparrow 0$.

If condition (3) does not hold then there exists a function $F\in S(\Lambda,A)$ for which correlation (5) does not imply (4).

Proof. If condition (3) does not hold then there exists a positive continuous increasing to $+\infty$ on $[0,+\infty)$ function l such that

$$\overline{\lim_{n\to\infty}}\frac{\ln n}{\Phi(\Psi(\varphi(\lambda_n)))l(\lambda_n)}>1.$$

Since $\Phi(\Psi(\varphi(t)))/t\to 0, t\to+\infty$, we may suppose the that function l is slowly increasing so that $\Phi(\Psi(\varphi(t)))l(t)/t\to 0, t\to+\infty$.

By Lemma 5 then there exists a subsequence $(\lambda_k^\star)$ of sequence (λ_n) such that

$$k\le\exp\{\Phi(\Psi(\varphi(\lambda_k^\star)))l(\lambda_k^\star)\}+1\quad(k\in\mathbb{N})$$

and

$$k_j\ge\exp\{\Phi(\Psi(\varphi(\lambda_{k_j}^\star)))l(\lambda_{k_j}^\star)\}$$

for some increasing sequence (k_j) of positive integers.

We choose $a_n=0$ if $\lambda_n\ne\lambda_k^\star$, and $a_n=a_k^\star$ if $\lambda_n=\lambda_k^\star$, where

$$a_k^\star=\exp\{-\lambda_k^\star\Psi(\varphi(\lambda_k^\star))\}.$$

Since $\Phi(\Psi(\varphi(t)))l(t)/t\to 0, t\to+\infty$, we have $(\ln\,k)/\lambda_k^\star\to 0\,(k\to\infty)$. On the other hand, $\dfrac{1}{\lambda_k^\star}\ln\,\dfrac{1}{a_k^\star}\to A\,(k\to\infty)$. Therefore, Dirichlet series (1) with such coefficients has the abscissa of absolute convergence A, and by Lemma 3 $\ln\,\mu(\sigma,F)\le\Phi(\sigma)$, that is (5) holds.

Now we put $m_j = \left[k_j - \sqrt{k_j}\,\right]$. Then for all j large enough

$$\lambda^\star_{m_j} \geq \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln(m_j-1)}{l(\lambda^\star_{m_j})}\right)\right)\right) \geq \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln\left(k_j-\sqrt{k_j}-2\right)}{l(\lambda^\star_{m_j})}\right)\right)\right) \geq$$
$$\geq \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{m_j})} - \frac{2}{l(\lambda^\star_{m_j})\sqrt{k_j}}\right)\right)\right) = \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{m_j})}\right)\right)\right) -$$
$$-\left\{\Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{m_j})}\right)\right)\right) - \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{m_j})} - \frac{2}{l(\lambda^\star_{m_j})\sqrt{k_j}}\right)\right)\right)\right\} \geq$$
$$\geq \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{k_j})}\right)\right)\right) - \delta_j \geq \lambda^\star_{k_j} - \delta_j,$$

where

$$\delta_j = \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{m_j})}\right)\right)\right) - \Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{m_j})} - \frac{2}{l(\lambda^\star_{m_j})\sqrt{k_j}}\right)\right)\right) =$$
$$= \frac{2\Phi''(\Psi^{-1}(\Phi^{-1}(\xi_j)))}{l(\lambda^\star_{m_j})\sqrt{k_j}\,\Psi'(\Psi^{-1}(\Phi^{-1}(\xi_j)))\Phi'(\Phi^{-1}(\xi_j))} =$$
$$= \frac{2\{\Phi'(\Psi^{-1}(\Phi^{-1}(\xi_j)))\}^2}{l(\lambda^\star_{m_j})\sqrt{k_j}\,\Phi(\Psi^{-1}(\Phi^{-1}(\xi_j)))\Phi'(\Phi^{-1}(\xi_j))}, \quad \frac{\ln k_j}{l(\lambda^\star_{m_j})} - \frac{2}{l(\lambda^\star_{m_j})\sqrt{k_j}} \leq \xi_j \leq \frac{\ln k_j}{l(\lambda^\star_{m_j})},$$

whence for $j \to \infty$

$$\delta_j = o\left(k_j^{-1/2}\{\Phi'(\Psi^{-1}(\Phi^{-1}(\xi_j)))\}^2\right) = o\left(k_j^{-1/2}\left\{\Phi'\left(\Psi^{-1}\left(\Phi^{-1}\left(\frac{\ln k_j}{l(\lambda^\star_{m_j})}\right)\right)\right)\right\}^2\right).$$

From condition 2), using 1) and 4), it follows that $\Phi(\sigma) \leq K_4\Phi(\Psi(\sigma))$ for all $\sigma \geq \sigma_0$ where $K_4 = \mathrm{const} > 0$. Therefore, putting $x_j = \Psi^{-1}(\Phi^{-1}(\ln k_j/l(\lambda^\star_{m_j})))$ and using condition 3), for $j \to \infty$ we obtain

$$\delta_j = o(\{\Phi'(x_j)\}^2 \exp\{-(1/2)l(\lambda^\star_{m_j})\Phi(\Psi(x_j))\}) = o(\exp\{2\ln\Phi'(x_j) - \Phi(x_j)\}) = o(1).$$

In the case $A = +\infty$ we put $\sigma_j = \varphi(\lambda^\star_{k_j})$. Then for all j large enough

$$M(\sigma_j, F) \geq \sum_{k=m_j}^{k_j} \exp\{-\lambda^\star_k\Psi(\varphi(\lambda^\star_k)) + \sigma_j\lambda^\star_k\} \geq$$
$$\geq (k_j - m_j + 1)\exp\{-\lambda^\star_{k_j}\Psi(\varphi(\lambda^\star_{k_j})) + \sigma_j\lambda^\star_{m_j}\} \geq$$
$$\geq \exp\left\{\Phi(\Psi(\varphi(\lambda^\star_{k_j})))l(\lambda^\star_{k_j})/2 - \lambda^\star_{k_j}\Psi(\varphi(\lambda^\star_{k_j})) + \varphi(\lambda^\star_{k_j})\lambda^\star_{k_j} - \varphi(\lambda^\star_{k_j})\delta_j\right\} =$$
$$= \exp\left\{\Phi(\varphi(\lambda^\star_{k_j})) + \Phi(\Psi(\varphi(\lambda^\star_{k_j})))l(\lambda^\star_{k_j})/2 - \varphi(\lambda^\star_{k_j})\delta_j\right\} \geq$$
$$\geq \exp\left\{\Phi(\varphi(\lambda^\star_{k_j}))\left(1 + l(\lambda^\star_{k_j})/(2K_4)\right) - \varphi(\lambda^\star_{k_j})\delta_j\right\},$$

that is

$$\ln M(\sigma_j, F) \geq \Phi(\sigma_j)\left(1 + l(\lambda^\star_{k_j})/(2K_4)\right) - \sigma_j\delta_j.$$

Since $\delta_j \to 0\,(j \to \infty)$ and $\sigma/\Phi(\sigma) \to 0\,(\sigma \to +\infty)$, we have $\delta_j\sigma_j/\Phi(\sigma_j) \to 0(j \to +\infty)$ and from the latter inequality we obtain

$$\ln M(\sigma_j, F) \geq l(\lambda^\star_{k_j})\Phi(\sigma_j)/(3K_4), \tag{7}$$

that is in this case Lemma 6 is proved.

In the case $A = 0$ we put $\sigma_j = \varphi(\lambda^\star_{m_j})$. Then for all j large enough

$$\begin{aligned} &M(\sigma_j, F) \geq (k_j - m_j + 1)\exp\{-\lambda^\star_{m_j}\Psi(\varphi(\lambda^\star_{m_j})) + \sigma_j\lambda^\star_{k_j}\} \geq \\ &\geq \exp\left\{\Phi(\Psi(\varphi(\lambda^\star_{k_j})))l(\lambda^\star_{k_j})/2 - \lambda^\star_{m_j}\Psi(\varphi(\lambda^\star_{m_j})) + \varphi(\lambda^\star_{m_j})\lambda^\star_{m_j} + \varphi(\lambda^\star_{m_j})(\lambda^\star_{k_j} - \lambda^\star_{m_j})\right\} \geq \\ &\geq \exp\left\{\Phi(\Psi(\varphi(\lambda^\star_{m_j})))l(\lambda^\star_{m_j})/2 + \Phi(\varphi(\lambda^\star_{m_j})) + \varphi(\lambda^\star_{m_j})\delta_j\right\} \geq \\ &\geq \exp\left\{\Phi(\varphi(\lambda^\star_{m_j}))\left(1 + l(\lambda^\star_{m_j})/(2K_4)\right) + \varphi(\lambda^\star_{m_j})\delta_j\right\} \end{aligned}$$

Since $\delta_j \to 0\,(j \to \infty)$ and $\varphi(\lambda^\star_{m_j}) \to 0\,(j \to \infty)$ from the latter inequality we obtain again (7) with m_j instead of k_j. □

Theorem 2 is an immediate consequence of Lemma 4 and 6.

3. Proof of Theorem 1. Using condition 1) of Theorem 1 it is easy to show that $\sigma \leq \Psi(\sigma + \Phi(\sigma)/\Phi'(\sigma))$, i.e. $\Psi^{-1}(\sigma) \leq \sigma + \Phi(\sigma)/\Phi'(\sigma)$ and, thus, by condition 2) of Theorem 1 $\Phi'(\Psi^{-1}(\sigma)) = O(\Phi'(\sigma))$ and $\Phi'(\sigma) = O(\Phi'(\Psi(\sigma)))$ as $\sigma \uparrow A$. Therefore, conditions 1) and 2) of Theorem 1 imply condition 1) of Lemmas 1 and 2 and condition 2) of Lemma 2.

In the case $A = 0$ condition 3) of Lemma 2 has the form $\Phi'(\sigma)|\sigma| > 1$ and follows from condition 4) of Theorem 1. Thus, all conditions of Lemma 2 are valid.

Further, if we put $1/\alpha(t) = K_2\Phi(\Psi(\varphi(t)))/t$, where $K_2 = \text{const} > 0$, then $\alpha(t) = o(t)$, $t \to +\infty$ and by condition 1) of Theorem 1 $\alpha(t) \nearrow +\infty$, $t \to +\infty$. By such a choice of α condition (3) is equivalent to the condition $\ln n(t) \leq t/\alpha(t)$, $t \geq t_0$, that is the exponents of the Dirichlet series (1) satisfy the condition of Lemma 1. Condition 2) of Lemma 1 has the form

$$\Phi'\left(\sigma + \frac{2K_2\Phi(\Psi(\varphi(a\Phi'(\sigma))))}{a\Phi'(\sigma)}\right) = O(\Phi'(\sigma)), \quad \sigma \uparrow A, \tag{8}$$

for every $a \in (0, +\infty)$. If $a \leq 1$ then $\Phi(\Psi(\varphi(a\Phi'(\sigma)))) \leq \Phi(\Psi(\sigma)) \leq \Phi(\sigma)$ and by condition 2) of Theorem 1 relation (8) holds. If $a > 1$ then using condition 1) of Theorem 1, by analogy as it was done in the proof of Lemma 4,

$$\Phi(\Psi(\varphi(a\Phi'(\sigma))))/(a\Phi'(\sigma)) \leq \Phi(\Psi(\sigma))/\Phi'(\sigma) \leq \Phi(\sigma)/\Phi'(\sigma)$$

and again by condition 2) of Theorem 1 we obtain (8), that is condition 2) of Lemma 1 holds for each $a \in (0, +\infty)$. Condition 3) of Lemma 1 for such a choice of α has the form $2K_2\Phi(\Psi(\sigma)) < \Phi(\sigma) + |\sigma|\Phi'(\sigma)$ in the case $A = 0$ and follows from condition 4) of Theorem 1. Thus, all conditions of Lemma 1 are valid too.

Concerning the conditions of Theorem 2 we have only to prove that condition 2) of Theorem 1 implies condition 2) of Theorem 2. One can easy obtain the latter statement from the following inequality

$$\ln\Phi\left(\sigma + \frac{K\Phi(\sigma)}{\Phi'(\sigma)}\right) - \ln\Phi(\sigma) = \int\limits_{\sigma}^{\sigma + K\Phi(\sigma)/\Phi'(\sigma)} \frac{\Phi'(t)}{\Phi(t)}dt \leq \frac{\Phi'(\sigma + K\Phi(\sigma)/\Phi'(\sigma))}{\Phi(\sigma)}\frac{K\Phi(\sigma)}{\Phi'(\sigma)}.$$

In view of Lemma 1, 2 and Theorem 2 the proof of Theorem 1 is now complete.

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