УДК 517.5

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ON THE WIMAN THEOREM FOR ABSOLUTELY CONVERGENT DIRICHLET SERIES

O. B. Skaskiv, Ya. Z. Stasyuk. *On the Wiman theorem for absolutely convergent Dirichlet series*, Matematychni Studii, **20** (2003) 133–142.

For functions f analytic in the unit disc and such that $(1-r)K_f(r)\to+\infty\ \ (r\to 1-0)$, where $K_f(r)=(\ln M_f(r))'$, $M_f(r)=\max\{|f(z)|:|z|=r\}$ it is proved that

$$M_f(r)=(1+o(1))\max\{\operatorname{Re} f(z):|z|=r\}=-(1+o(1))\min\{\operatorname{Re} f(z):|z|=r\}$$

as $r\to 1-0$ $(r\notin E)$, $\operatorname{meas}(E\cap[r,1))=o(1-r)$ $(r\to 1-0)$.

О. Б. Скаскив, Я. З. Стасюк. *О теореме Вимана для абсолютно сходящихся рядов Дирихле* // Математичні Студії. – 2003. – Т.20, №2. – С.133–142.

Для аналитических в единичном круге функций f таких, что $(1-r)K_f(r)\to+\infty$ $(r\to 1-0)$, где $K_f(r)=(\ln M_f(r))'$, $M_f(r)=\max\{|f(z)|:|z|=r\}$, доказано, что

$$M_f(r)=(1+o(1))\max\{\operatorname{Re} f(z):|z|=r\}=-(1+o(1))\min\{\operatorname{Re} f(z):|z|=r\}$$

при $r\to 1-0$ $(r\notin E)$, $\operatorname{meas}(E\cap[r,1))=o(1-r)$ $(r\to 1-0)$.

1. Introduction. For an entire function $f(z)$, let us define $M_f(r)=\max\{|f(z)|:|z|=r\}$, $B_f(r)=\max\{\operatorname{Re} f(z):|z|=r\}$, $A_f(r)=\min\{\operatorname{Re} f(z):|z|=r\}$, $r>0$. By the Wiman theorem [1],

$$B_f(r)=-(1+o(1))A_f(r)=(1+o(1))M_f(r) \tag{1}$$

as $r\to+\infty$ outside a certain set E of finite logarithmic measure, i.e. $\int_{E\cap[1,+\infty)} d\ln r<+\infty$.

For functions f analytic in the unit disc $\mathrm{U}=\{z:|z|<1\}$, reasoning in a similar way as in [2, p.266–270] and using the scheme of the proof of Wiman's theorem (see [1, 3]) one can obtain that relation (1) holds as $r\to 1-0$ outside a certain set E of finite logarithmic measure on $(0,1)$, i.e. $\int_{E\cap(0,1)}\frac{dr}{1-r}<+\infty$, when the condition

$$\varliminf_{r\to1-0}\frac{\ln K_f(r)}{-\ln(1-r)}>1$$

2000 *Mathematics Subject Classification:* 30B10, 30B50, 30D40.

holds where $K_f(r) = (\ln M_f(r))'$ is the derivative from the right. In fact, the analogue of the last statement is true in a general class $S(0;1)$ of functions F analytic in the strip $\{z : 0 < \operatorname{Re} z < 1\}$, bounded in the strip $\{z : 0 < \operatorname{Re} z < x\}$ for all $x \in (0,1)$ and such that

$$\varliminf_{x \to 1-0} \frac{\ln L(x,F)}{-\ln(1-x)} > 1, \tag{2}$$

where $L(x,F) = (\ln M(x,F))'$ is the derivative from the right of a convex function $\ln M(x,F)$, $M(x,F) = \sup\{|F(x+iy)| : y \in \mathbb{R}\}$. For such functions the relation

$$B(x,F) = -(1+o(1))A(x,F) = (1+o(1))M(x,F), \tag{3}$$

where $B(x,F) = \sup\{\operatorname{Re} F(x+iy) : y \in \mathbb{R}\}$, $A(x,F) = \inf\{\operatorname{Re} F(x+iy) : y \in \mathbb{R}\}$ holds as $x \to 1-0$ outside a certain set E of finite logarithmic measure on the interval $(0,1)$.

We will replace condition (2) by the condition

$$(1-x)L(x,F) \to +\infty \quad (x \to 1-0), \tag{4}$$

and obtain the following estimate for the exceptional set E in relation (3):

$$D^1 E = \varlimsup_{x \to 1-0} \frac{1}{1-x} \operatorname{meas}\,(E \cap [x,1)) = 0,$$

where $\operatorname{meas} E$ is the Lebesgue measure of a measurable set E on the line.

Condition (4) transforms into the condition

$$(1-r)K_f(r) \to +\infty \quad (r \to 1-0) \tag{5}$$

for analytic in U functions f and the exceptional set E in relation (1) obtains a description $D^1 E = 0$.

Remark, that for the function $f(z) = \frac{1}{1-z}$ we have $M_f(r) = \frac{1}{1-r}$, $K_f(r) = \frac{1}{1-r}$, $A_f(r) = \frac{1}{1+r}$, $B_f(r) = \frac{1}{1-r}$. Therefore, $(1-r)K_f(r) \equiv 1$, $A_f(r) = O(1)$, $A_f(r) = o(B_f(r))$ $(r \to 1-0)$, i.e. condition (5) and relation (1) do not hold. Obviously, the same is true for any single-value branch in U of the function $f(z) = \frac{1}{(1-z)^\alpha}$, $\alpha > 0$. The above reasonings yield that condition (5) cannot be essentially weakened.

2. Main result. For the sake of convenience, we will consider the class $S(-1;0)$ of analytic in the strip $\{z : -1 < \operatorname{Re} z < 0\}$ functions, bounded in $\{z : -1 < \operatorname{Re} z \le x\}$ for all $x \in (-1;0)$ and such that

$$|x|L(x,F) \to +\infty \quad (x \to -0) \tag{6}$$

instead of the class $S(0;1)$.

Theorem 1. *Let Φ be a positive increasing to $+\infty$ on $(-1,0)$ function such that*

$$|x|\Phi(x) \to +\infty \quad (x \to -0) \tag{7}$$

and let $F \in S(-1;0)$ be a function satisfying

$$L(x,F) \ge \Phi(x) \quad (x_0 \le x < 0). \tag{8}$$

Then relation (3) holds as $x \to -0$ ($x \in (-1,0) \setminus E$) and for every positive nondecreasing on $(-1,0)$ function h such that

$$h(x) = o(\Phi(x)),\ h\left(x + o\left(\frac{1}{h(x)}\right)\right) = O(h(x)) \quad (x \to -0), \tag{9}$$

the set E has a zero h-density at the point $x = 0$

$$D_h E = \overline{\lim_{x\to -0}}\, h(x) \operatorname{meas}(E \cap [x,0]) = 0.$$

Remark. The theorem formulated above makes sense when $\overline{\lim\limits_{x\to -0}}\, h(x)|x| > 0$ (in the opposite case, $D_h E = 0$ for any set $E \subset [-1,0)$). That is why we assume that $h(x) \geq 1/|x|$ $(-1 < x < 0)$ throughout this paper.

Taking $h(x) = \frac{1}{|x|}$, $\Phi(x) = L(x,F)$ one can obtain the following corollary from Theorem 1.

Corollary 1. *For every function $F \in S(-1;0)$ relation (3) is true as $x \to -0$ ($x \in (-1,0) \setminus E$) and* $\operatorname{meas}(E \cap [x,0]) = o(|x|)$ $(x \to -0)$.

3. Auxiliary results. We need the following version of the Borel-Nevanlinna lemma.

Lemma 1. *Let $u(x)$ be a positive nondecreasing on $[-1,0)$ function and let Φ and h be the functions from Theorem 1. If*

$$u(x) \geq \Phi(x) \quad (x_0 \leq x < 0)$$

then there exists a function $\delta(u) \nearrow +\infty$ ($u \to -0$) such that the inequality

$$|u(x+\tau) - u(x)| < u(x)/\delta(x)$$

holds for all $x \in (-1,0) \setminus E$ and all $\tau \in \mathbb{R}$, $|\tau| \leq \psi(x) \equiv \delta(x)/u(x)$ and the set E has a zero h-density at the point $x = 0$.

Proof. Let $\delta > 1$. There is no loss of generality in assuming that $\min\{u(x)|x|, (1-|x|)u(x)\} > \delta$ $(x_0 \leq x < 0)$. Then $x + \frac{\delta}{u(x)} < 0$, $x - \frac{\delta}{u(x)} > -1$. Denote

$$E^1(\delta) = \left\{x \in (x_0,0) : u\left(x + \frac{\delta}{u(x)}\right) \geq \left(1 + \frac{1}{\delta}\right) u(x)\right\},$$

$$E^2(\delta) = \left\{x \in (x_0,0) : u(x) \geq \left(1 + \frac{1}{\delta}\right) u\left(x - \frac{\delta}{u(x)}\right)\right\}.$$

Let us prove that $D_h(E^1(\delta) \cup E^2(\delta)) = 0$. Suppose that $E^1(\delta)$ and $E^2(\delta)$ contain points arbitrarily close to the point $x = 0$. Otherwise, for $E^1(\delta)$ we have $u\left(x + \frac{\delta}{u(x)}\right) < \left(1 + \frac{1}{\delta}\right) u(x)$ $(x_1 < x < 0)$, $u\left(x + \frac{\delta}{u(x)}\right) - u(x) < u(x)/\delta$ and $\operatorname{meas}(E^1(\delta) \cap [x_1,0)) = 0$. Similarly, for $E^2(\delta)$ we have $u(x) < \left(1 + \frac{1}{\delta}\right) u\left(x - \frac{\delta}{u(x)}\right)$ $(x_2 < x < 0)$ and $\operatorname{meas}(E^2(\delta) \cap [x_2,0)) = 0$. Denote $E^j(\delta,x) = E^j(\delta) \cap [x,0]$. Let us define the following sequences for $j = 1$ and $j = 2$:

$$r_1^{(j)} = \operatorname{essinf}\{x : x \in E^j(\delta,x_0)\}, \quad R_1^{(j)} = r_1^{(j)} + \frac{\delta}{u(r_1^{(j)} + 0)}.$$

Assume that $r_k^{(j)}, R_k^{(j)}$ for $k \le n$ are already defined. Now define

$$r_{n+1}^{(j)} = \text{essinf}\{x : x \in E^j(\delta, R_n^{(j)})\}, \quad R_{n+1}^{(j)} = r_{n+1}^{(j)} + \frac{\delta}{u(r_{n+1}^{(j)} + 0)}.$$

It is clear that (see also [4, c. 67])

$$E^j(\delta) \setminus E^* \subset \bigcup_{n=1}^{+\infty} [r_n^{(j)}, R_n^{(j)}],$$

where E^* is at most a countable set.

Let us prove that if $x^{(j)} = \text{essinf}\{x : x \in E^j(\delta, r)\}$, $a^{(j)} = u(x^{(j)} + 0) = \lim\limits_{x \to x^{(j)}+0} u(x)$ then $u\left(x^{(1)} + \frac{\delta}{a^{(1)}} + 0\right) \ge \left(1 + \frac{1}{\delta}\right) a^{(1)}$ and $a^{(2)} \ge \left(1 + \frac{1}{\delta}\right) u\left(x^{(2)} - \frac{\delta}{a^{(2)}} + 0\right)$. First suppose that for a certain sequence $x_k \downarrow x^{(1)}$ from the set $E^1(\delta, r)$ we have $v_k = x_k + \frac{\delta}{u(x_k)} \to v_0 + 0$, $v_0 = x^{(1)} + \frac{\delta}{a^{(1)}}$. Then passing to the limit as $k \to +\infty$ in the inequality $u(v_k) \ge \left(1 + \frac{1}{\delta}\right) u(x_k)$ we obtain $u(v_0 + 0) \ge \left(1 + \frac{1}{\delta}\right) u(x^{(1)} + 0)$, i.e. the required inequality. If for $x_k \downarrow x^{(1)}$ from the set $E^1(\delta, r)$ we have $v_k \to v_0 - 0$ then passing to the limit as $k \to +\infty$ we see that $u(v_0 - 0) \ge \left(1 + \frac{1}{\delta}\right) u(x^{(1)} + 0)$. Since $u(v_0 + 0) \ge u(v_0 - 0)$, we obtain the required relation. Therefore, the statement concerning the first inequality is proved because $\lim\limits_{x \to x^{(1)}+0} \left(x + \frac{\delta}{u(x)}\right)$ exists. Similarly, if $x_k \downarrow x^{(2)}$ for a certain sequence from $E^2(\delta, r)$ then $w_k = x_k - \frac{\delta}{u(x_k)} \ge x_{k+1} - \frac{\delta}{u(x_{k+1})} = w_{k+1}$. Thus, $w_k \downarrow w_0 \overset{\text{def}}{=} x^{(2)} - \frac{\delta}{a^{(2)}}$, and by the inequality $u(x_k) \ge (1 + \frac{1}{\delta}) u(w_k)$ we obtain $u(x^{(2)} + 0) \ge (1 + \frac{1}{\delta}) u(w_0 + 0)$, i.e. the desired conclusion.

By what was proved above,

$$u(r_{n+1}^{(1)} + 0) \ge u(R_n^{(1)} + 0) = u\left(r_n^{(1)} + \frac{\delta}{u(r_n^{(1)} + 0)} + 0\right) \ge \left(1 + \frac{1}{\delta}\right) u(r_n^{(1)} + 0), \tag{10}$$

and since $r_{n+1}^{(2)} \ge R_n^{(2)} = r_n^{(2)} + \frac{\delta}{u(r_n^{(2)} + 0)} \ge r_n^{(2)} + \frac{\delta}{u(r_{n+1}^{(2)})}$, we have

$$u(r_{n+1}^{(2)} + 0) \ge \left(1 + \frac{1}{\delta}\right) u\left(r_{n+1}^{(2)} - \frac{\delta}{u(r_{n+1}^{(2)} + 0)} + 0\right) \ge \left(1 + \frac{1}{\delta}\right) u(r_n^{(2)} + 0). \tag{11}$$

It follows from (10) and (11) that $r_n^{(j)} \uparrow 0$ $(n \to +\infty)$ and

$$\delta \frac{u(r_{n+1}^{(j)} + 0) - u(r_n^{(j)} + 0)}{u(r_n^{(j)} + 0)} \ge 1.$$

Arguing in a similar way as in [5], for $r \in [R_n^{(j)}, r_{n+1}^{(j)}]$ we have

$$\text{meas}\, E^j(\delta, r) \le \sum_{k=n+1}^{+\infty} (R_k^{(j)} - r_k^{(j)}) \le \frac{\delta}{u(r_{n+1}^{(j)} + 0)} +$$

$$+\delta^2 \sum_{k=n+2}^{+\infty} \frac{u(r_k^{(j)} + 0) - u(r_{k-1}^{(j)} + 0)}{u(r_k^{(j)} + 0) u(r_{k-1}^{(j)} + 0)} = \frac{\delta(1 + \delta)}{u(r_{n+1}^{(j)} + 0)} \le \frac{\delta(1 + \delta)}{\Phi(r_{n+1}^{(j)})}. \tag{12}$$

In the same manner we can see that for $r \in [r_n^{(j)}, R_n^{(j)}]$

$$\text{meas}\, E^j(\delta, r) \le \sum_{k=n}^{+\infty}(R_k^{(j)} - r_k^{(j)}) \le \frac{\delta}{u(r_n^{(j)}+0)} +$$

$$+\delta^2 \sum_{k=n+1}^{+\infty} \frac{u(r_k^{(j)}+0) - u(r_{k-1}^{(j)}+0)}{u(r_k^{(j)}+0)u(r_{k-1}^{(j)}+0)} = \frac{\delta(1+\delta)}{u(r_n^{(j)}+0)} \le \frac{\delta(1+\delta)}{\Phi(r_n^{(j)})}. \qquad (13)$$

It is clear that $h(r) \le h(r_{n+1}^{(j)})$ for $r \le r_{n+1}^{(j)}$ and

$$h(r) \le h(r_n^{(j)} + R_n^{(j)} - r_n^{(j)}) \le h\left(r_n^{(j)} + \frac{\delta}{u(r_n^{(j)})}\right) =$$

$$= h\left(r_n^{(j)} + o\left(\frac{1}{h(r_n^{(j)})}\right)\right) = O(h(r_n^{(j)})) \quad (n \to +\infty)$$

for $r \le R_n^{(j)}$. Therefore, from (12) and (13) for $r \in [r_n^{(j)}, r_{n+1}^{(j)}]$ we obtain

$$h(r)\, \text{meas}\, E^j(\delta, r) = O\left(\max\left\{\frac{h(r_n^{(j)})}{\Phi(r_n^{(j)})}, \frac{h(r_{n+1}^{(j)})}{\Phi(r_{n+1}^{(j)})}\right\}\right) = o(1) \quad (n \to +\infty).$$

Thus, $D_h E^j(\delta) = 0$, $j \in \{1, 2\}$ and this yields $D_h(E^1(\delta) \cup E^2(\delta)) = 0$. Now let $t_n \uparrow 0$ $(n \to +\infty)$ be a sequence such that for all $r \in [t_n, 0)$

$$h(r)\, \text{meas}\, E(n+1, r) \le \frac{1}{n^2},$$

where $E(\delta) = E^1(\delta) \cup E^2(\delta)$.

Define $\delta(x) = n+1$ for $x \in [t_n, t_{n+1})$ and $E_1 = \bigcup_{n=1}^{+\infty}(E(n+1, t_n) \cap [t_n, t_{n+1}))$. For $r \in [t_n, t_{n+1})$ we have

$$h(r)\, \text{meas}\, (E_1 \cap [r, 0)) \le h(r)\, \text{meas}\, E(n+1, r) +$$

$$+ \sum_{k=n+1}^{+\infty} \frac{h(r)}{h(t_k)} h(t_k)\, \text{meas}\, E(k+1, t_k) \le \sum_{k=n}^{+\infty} \frac{1}{k^2} = o(1) \quad (n \to +\infty),$$

that is $D_h E_1 = 0$. For $x \in [t_n, t_{n+1}) \setminus E_1$

$$u\left(x + \frac{\delta(x)}{u(x)}\right) = u\left(x + \frac{n+1}{u(x)}\right) < \left(1 + \frac{1}{n+1}\right) u(x) = \left(1 + \frac{1}{\delta(x)}\right) u(x). \qquad (14)$$

Since the inequality $u(x) < \left(1 + \frac{1}{\delta}\right) u\left(x - \frac{\delta}{u(x)}\right)$, $\delta > 1$ yields $u\left(x - \frac{\delta}{u(x)}\right) > \left(1 - \frac{1}{\delta}\right) u(x)$, for $x \in [t_n, t_{n+1}) \setminus E_1$ we obtain

$$u\left(x - \frac{\delta(x)}{u(x)}\right) = u\left(x - \frac{n+1}{u(x)}\right) > \left(1 - \frac{1}{n+1}\right) u(x) = \left(1 - \frac{1}{\delta(x)}\right) u(x). \qquad (15)$$

From (14) and (15) for all $x \in (x_0, 0) \setminus E_1$ $(D_h E_1 = 0)$ and $\tau \in \mathbb{R}$, $|\tau| \le \delta(x)/u(x)$ it follows that

$$|u(x+\tau) - u(x)| \le \max\left\{u\left(x + \frac{\delta(x)}{u(x)}\right) - u(x), u(x) - u\left(x - \frac{\delta(x)}{u(x)}\right)\right\} < \frac{u(x)}{\delta(x)}.$$

This is precisely the assertion of the lemma. $\square$

Theorem 2. *Let the functions Φ and h be the same as in Theorem 1. Let the function $F \in S(-1;0)$ satisfy condition (8) and let $\varepsilon(x) \downarrow 0$ $(x \to -0)$ be an arbitrary function. Then there exist a function $\delta(x) \nearrow +\infty$ $(x \to -0)$ and a set $E \subset (-1,0)$, $D_h E = 0$ such that the relation*

$$F(z) = F(z_0)\exp\left\{(L(x,F)+\Delta_1)s + \sum_{n=2}^{+\infty}\Delta_n s^n\right\},$$

where

$$|\Delta_n| \le 2r_0^{-n}\ln\left(1+\frac{r_0 c(x)L(x,F)}{\delta(x)}\right),$$

holds as $x \to -0$ $(x \notin E)$ for all z_0, $\operatorname{Re} z_0 = x$ satisfying

$$|F(z_0)| \ge M(x,F)/(1+\varepsilon(x))$$

and for all $z = z_0 + s$, $s \in \mathbb{C}$, $|s| \le r_0 < \frac{\delta(x)}{L(x,F)c(x)}$, $c(x) = 1 + e(1+\varepsilon(x))$.

Proof. Let us apply Lemma 1 to the function $u(x) = L(x,F)$. For all $x \in (-1,0)\setminus E$, $D_h E = 0$ and $\tau \in \mathbb{R}$, $|\tau| \le \psi(x) \equiv \delta(x)/L(x,F)$ we have

$$|L(x+\tau,F) - L(x,F)| \le \frac{1}{\psi(x)}. \tag{16}$$

Let $\varepsilon(x) \downarrow 0$ $(x \to -0)$ and let a point $z_0 = x + iy$ be such that $|F(z_0)| \ge M(x,F)/(1+\varepsilon(x))$. From the monotonicity of $L(x,F)$ it follows ([2, c.147]) that for all $x \in (-1,0)$, $h \in \mathbb{R}$, $|x| - 1 < h < |x|$

$$\ln M(x+h,F) - \ln M(x,F) \le hL(x+h,F).$$

Therefore, for $x \in (-1,0)$, $h \in \mathbb{R}$, $|h| \le \psi(x)$ using (16) we obtain

$$\ln M(x+h,F) - \ln M(x,F) - hL(x,F) \le h(L(x+h,F) - L(x,F)) =$$
$$= |h||L(x+h,F) - L(x,F)| \le 1.$$

Hence, for $x \notin E$, $\eta \in \mathbb{C}$, $|\operatorname{Re}\eta| \le \psi(x)$

$$\left|\frac{F(z_0+\eta)}{F(z_0)}e^{-\eta L(x,F)}\right| \le$$
$$\le (1+\varepsilon(x))\exp\{\ln M(x+\operatorname{Re}\eta,F) - \ln M(x,F) - \operatorname{Re}\eta L(x,F)\} \le e(1+\varepsilon(x)).$$

According to the Schwartz lemma [6, c. 223], applied in the disc $\{\eta : |\eta| \le \psi(x)\}$ to the function $g(\eta) = \frac{F(z_0+\eta)}{F(z_0)}e^{-\eta L(x,F)} - 1$ for all $\eta \in \mathbb{C}$, $|\eta| < \psi(x)$ we have

$$|g(\eta)| \le (1+e(1+\varepsilon(x)))\frac{|\eta|}{\psi(x)} = c(x)\frac{|\eta|}{\psi(x)}. \tag{17}$$

Note that for $|\eta| < \frac{\psi(x)}{c(x)}$ and $x \in (-1,0)\setminus E$ (17) yields

$$\left|\frac{F(z_0+\eta)}{F(z_0)}e^{-\eta L(x,F)}\right| \ge 1 - |g(\eta)| > 0.$$

Thus, $F(z_0+\eta)\neq 0$ for $|\eta|<\frac{\psi(x)}{c(x)}$ and that is why the function

$$G(\eta)=\int\limits_0^{\eta}\frac{F'(z_0+\tau)}{F(z_0+\tau)}d\tau-\eta L(x,F)$$

is analytic in the disc $\{\eta:|\eta|<\psi(x)/c(x)\}$, $G(0)=0$. Let

$$G(\eta)=\sum_{j=1}^{+\infty}\Delta_j\eta^j.$$

For $|\eta|\le q<\psi(x)/c(x)$ inequality (17) implies

$$\operatorname{Re}G(\eta)=\ln|1+g(\eta)|\le\ln(1+|g(\eta)|)\le\ln\left(1+\frac{qc(x)}{\psi(x)}\right). \tag{18}$$

Therefore, by the modified Cauchy inequality ([7, c. 30]) in the disc $\{\eta:|\eta|\le q\}$ we have

$$|\Delta_j|\le 2q^{-j}\max\{\operatorname{Re}G(\eta):|\eta|=q\}\le 2q^{-j}\ln\left(1+\frac{qc(x)}{\psi(x)}\right).$$

It suffices to note that for $|\eta|<\psi(x)/c(x)$

$$F(z_0+\eta)=F(z_0)\exp\left\{\eta L(x,F)+\sum_{j=1}^{+\infty}\Delta_j\eta^j\right\}.$$

□

Corollary 2. *Let Φ, h and $F\in S(-1;0)$ be the same as in Theorem 1 and let a function $b(x)>0$ be such that $L(x,F)b(x)=O(1)$ $(x\to-0)$. Then there exists a set $E\subset[-1,0)$, $D_hE=0$ such that for z_0, $\operatorname{Re}z_0=x$ satisfying $|F(z_0)|=(1+o(1))M(x,F)$ and for all $|\eta|\le b(x)$ the relation*

$$F(z_0+\eta)=F(z_0)(1+\omega(\eta))e^{\eta L(x,F)}$$

holds as $x\to-0$ $(x\notin E)$ with $\omega(\eta)=o(1)$ $(\eta\to 0)$.

Proof. Let $b(x)=b_0(x)/L(x,F)$ with $b_0(x)=O(1)$ $(x\to-0)$. Taking $|\eta|\le b(x)$ in Theorem 2 we obtain

$$\left|\sum_{j=1}^{+\infty}\Delta_j\eta^j\right|\le 2\ln\left(1+\frac{r_0c(x)L(x,F)}{\delta(x)}\right)\sum_{j=1}^{+\infty}\left(\frac{b(x)}{r_0}\right)^j$$

where $c(x)=1+e(1+\varepsilon(x))$ and $\varepsilon(x)\downarrow 0$ $(x\to-0)$. Choose $r_0=(1+o(1))\frac{\delta(x)}{L(x,F)c(x)}$. Then

$$\left|\sum_{j=1}^{+\infty}\Delta_j\eta^j\right|\le 2(1+o(1))\frac{b(x)/r_0}{1-b(x)/r_0}=2(1+o(1))\frac{b(x)}{r_0-b(x)}=$$
$$=2(1+o(1))\frac{c(x)b_0(x)}{\delta(x)(1+o(1))-c(x)b_0(x)}=o(1)$$

as $x\to-0$. Applying Theorem 2 we complete the proof of Corollary 1. □

4. Proof of Theorem 1. We will prove the theorem using the scheme from [3] (see also [1, 8]) and applying Corollary 2. First take $\eta = i(\pi - \arg F(z_0))/L(x, F)$, $x \to -0$ $(x \notin E)$.

$$F(z_0 + \eta) = (1 + o(1))|F(z_0)|e^{i\pi} = -(1 + o(1))|F(z_0)|$$

and, therefore, $A(x, F) \le \operatorname{Re} F(z_0 + \eta) \le -(1 + o(1))M(x, F)$. The obvious inequality $|A(x, F)| \le M(x, F)$ yields $A(x, F) = -(1 + o(1))M(x, F)$ $(x \to -0, x \notin E)$. Now take $\eta = -i \arg F(z_0)/L(x, F)$. We obtain

$$F(z_0 + \eta) = (1 + o(1))|F(z_0)| \quad (x \to -0, x \notin E)$$

and, thus, $B(x, F) \ge \operatorname{Re} F(z_0 + \eta) = (1 + o(1))M(x, F)$. Applying the inequality $B(x, F) \le M(x, F)$ and noting that $|A(x, F)| \le M(x, F) \le (1 + o(1))B(x, F)$ $(x \to -0, x \notin E)$ we complete the proof of Theorem 1.

5. Corollaries for Dirichlet series and functions analytic in a disc. It is easily seen that absolutely convergent in the half-plane $\Pi_0 = \{z : \operatorname{Re} z < 0\}$ Dirichlet series

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}, \quad \lambda_n \in \mathbb{R}_+ \ (n \ge 0), \tag{19}$$

belongs to the class $S(-1; 0)$ and ([7, p. 82]) in the case when $\lambda_n \uparrow +\infty$ $(n \to +\infty)$ the sum F of convergent series (19) belongs to $S(-1; 0)$. Therefore, Theorem 1 implies the following corollary.

Corollary 3. *Let either series (19) be absolutely convergent in* Π_0, *or the condition* $\lambda_n \uparrow +\infty$ $(n \to +\infty)$ *hold and series (19) be convergent in* Π_0. *Suppose that condition (6) is satisfied. Then there exists a set* $E \subset (-1, 0)$ *such that relation (3) is true as* $x \to -0$ $(x \notin E)$ *and* $\operatorname{meas}(E \cap [x, 0]) = o(|x|)$ $(x \to -0)$.

Proof. It suffices to apply Theorem 1 with the functions $\Phi(x) = L(x, F)$ and $h(x) = \frac{1}{|x|}$. □

Corollary 4. *Let an analytic in the unit disc* U *function* $f(z)$ *satisfy condition (5). Then relation (1) holds as* $r \to 1 - 0$ *outside a certain set* $E_1 \subset [0, 1)$ *such that* $D^1 E_1 = 0$.

Proof. Let $F(z) = f(e^z)$, $z = x + iy$. Then $B(x, F) = B_f(e^x)$, $A(x, F) = A_f(e^x)$, $M(x, F) = M_f(e^x)$ and, obviously, $F \in S(-1; 0)$. If E_1 is the image of the set E from Theorem 1 under the mapping $r = e^x : (-1, 0) \to (0, 1)$ then

$$\operatorname{meas}(E_1 \cap [R, 1)) = \int_{E_1 \cap [R,1)} dr = \int_{E \cap [\ln R, 0)} e^x dx \le \int_{E \cap [\ln R, 0)} dx =$$
$$= \operatorname{meas}(E \cap [\ln R, 0)) = o(|\ln R|) \quad (R \to 1 - 0)$$

and $|\ln R| \sim 1 - R$ $(R \to 1 - 0)$. This implies $\operatorname{meas}(E_1 \cap [R, 1)) = o(1 - R)$ $(R \to 1 - 0)$, i.e. $D^1 E_1 = 0$. On the other hand,

$$|x|L(x, F) = |x|e^x K_f(e^x) \sim (1 - e^x)K_f(e^x) \to +\infty \quad (x \to -0).$$

Therefore, applying Theorem 1 with $\Phi(x) = L(x, F)$ and $h(x) = \frac{1}{|x|}$ we obtain

$$B_f(r) = -(1 + o(1))A_f(r) = (1 + o(1))M_f(r)$$

as $r \to 1 - 0$ $(r \notin E_1)$. □

6. Application of Lemma 1 to meromorphic functions in a disc. The following statement provides new information on the exceptional set in lemma on the logarithmic derivative for meromorphic functions in the unit disc U which grow rather quickly.

Corollary 5. *Let f be a meromorphic in the unit disc* U *function such that a certain positive nondecreasing on the interval* $(0,1)$ *function* Φ_0 *satisfies the following conditions*

$$(1-r)\Phi_0(r) \to +\infty \quad (r \to 1-0),$$
$$T(r,f) \geq \Phi_0(r) \quad (r_0 \leq r < 1).$$

If h_0 is a nondecreasing positive on (0,1) function satisfying

$$h_0(r) \geq \frac{1}{1-r}, \quad h_0(r) = o(\Phi_0(r)), \quad h_0\left(r + o\left(\frac{1}{h_0(r)}\right)\right) = O(h_0(r)) \quad (r \to 1-0),$$

then

$$m\left(r, \frac{f'}{f}\right) = O(\ln T(r,f))$$

as $r \to 1-0$ *outside a certain set* E_2 *such that*

$$D^1_{h_0} E_2 = \varlimsup_{r \to 1-0} h_0(r) \,\mathrm{meas}\,(E_2 \cap [r,1)) = 0$$

where $T(r,f)$ *and* $m\left(r, \frac{f'}{f}\right)$ *are the Nevanlinna characteristics ([4, c. 21–22]).*

Proof. By means of Lemma 2.3 [4, c. 63] in the case when $f(0) \neq 0, \infty$ for $0 < r < R < 1$ we have

$$m\left(r, \frac{f'}{f}\right) < 4\ln^+ T(R,F) + 4\ln^+\ln^+ \frac{1}{|f(0)|} +$$
$$+5\ln^+ R + 6\ln^+ \frac{1}{R-r} + \ln^+ \frac{1}{r}. \tag{20}$$

Let us apply Lemma 1 with the functions $u(x) = T(1+x,f)$, $h(x) = h_0(1+x)$, $\Phi(x) = \Phi_0(1+x)$, $x < 0$. Taking $r = 1+x$, $R = 1 + x + \frac{\delta(x)}{u(x)}$ as $x \to -0$ ($x \notin E, D_h E = 0$) in (20) we obtain

$$m\left(r, \frac{f'}{f}\right) < 4\ln^+\left(1 + \frac{1}{\delta(x)}\right) u(x) + 6\ln^+ \frac{u(x)}{\delta(x)} = O(\ln T(r,f)).$$

It remains to note that for the set $E_2 \subset (0,1)$ which is the image of the set E under the mapping $r = 1+x$ the equality $D_h E = 0$ yields

$$\varlimsup_{r \to 1-0} h_0(r) \,\mathrm{meas}\,(E_2 \cap [r,1)) = \varlimsup_{x \to -0} h(x) \,\mathrm{meas}\,(E \cap [x,0)) = 0.$$

□

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Received 19.08.2003