УДК 517.5

B. V. Vynnytskyi, I. B. Sheparovych

ON BASIS $\{\psi(\lambda_n z)\}$

B. V. Vynnytskyi, I. B. Sheparovych. *On basis* $\{\psi(\lambda_n z)\}$, Matematychni Studii, **20** (2003) 97–100.

We obtaine conditions for existence of a sequence (λ_n) such that the system $\{\psi(\lambda_n z)\}$ is a basis in the space of holomorphic functions in $\{z : |z| < R\}$ simultaneously for each $R \in (0; +\infty]$.

Б. В. Винницкий, И. Б. Шепарович. *О базисности* $\{\psi(\lambda_n z)\}$ // Математичні Студії. – 2003. – Т.20, №1. – С.97–100.

Получены условия существования последовательности (λ_n) такой, что система $\{\psi(\lambda_n z)\}$ является базисом в пространстве голоморфных в круге $\{z : |z| < R\}$ функций одновременно для всех $R \in (0; +\infty]$.

Let (λ_n) be a sequence of different complex numbers satisfying $0 \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \leqslant |\lambda_n| \leqslant \ldots$ and $\lim_{n\to\infty} \lambda_n = \infty$; ψ an entire function with Taylor's coefficients ψ_n; $\varkappa_n(\psi) = |\psi_{n-1}/\psi_n|$; $\widehat{\psi}$ Newton's majorant of ψ [1, 2]; A_R the space of holomorphic function in the disk $\{z : |z| < R\}$ with the topology of uniform convergence on every compact from this disk. In [3] there were found conditions, under which there exists a sequence (λ_n) such that the system

$$\{\psi(\lambda_n z)\}_{n=1}^{\infty} \tag{1}$$

forms a basis in the space A_R for given $R \in (0; +\infty]$. The purpose of this paper consists in finding conditions on ψ, under which there exists sequence (λ_n) such that the system (1) forms a basis in the space A_R simultaneously for each $R \in (0; +\infty]$. We shall prove the following proposition.

Theorem 1. *In order that there exists the sequence (λ_n) such that system (1) forms a basis in the space A_R simultaneously for each $R \in (0; +\infty]$, it is necessary that*

a) $(\forall n \geqslant 0) : \psi_n \neq 0$;

b) for some sequence (δ_n) of complex numbers $\delta_n \to 0$ an entire function ψ^* with Taylor's coefficients $\psi_n^* = \psi_n e^{n\delta_n}$ is Newton's majorant of ψ;

c)

$$(\forall \sigma > 1)\ (\exists \sigma_1 > 1)\ (\forall k \in \mathbb{N})\ (\forall n \geqslant k)\ : \frac{\varkappa_k(\widehat{\psi})}{\varkappa_n(\widehat{\psi})} \leqslant \frac{\sigma_1^k}{\sigma^n}.$$

2000 *Mathematics Subject Classification:* 30B60.

© B. V. Vynnytskyi, I. B. Sheparovych, 2003

and sufficient that the following conditions be satisfied: a), b) and

c′)

$$(\forall\alpha>0)(\forall\varepsilon>0)(\exists c_1)(\forall k\geqslant 1)(\forall n\geqslant k):\quad \frac{\varkappa_k(\widehat{\psi})}{\varkappa_n(\widehat{\psi})}\leqslant c_1\alpha^{n-k}(1+\varepsilon)^k$$

To prove this theorem we shall use the following statements in the form from [3–5].

Theorem A. *In order that there exists a sequence (λ_n) such that system (1) forms a basis in the space A_R, $R\in(0;+\infty)$, it is necessary and sufficient that the following conditions be satisfied : a), b) and*

$$c'')\quad (\exists\sigma>1)\ (\exists\sigma_1>1)\ (\forall k\in\mathbb{N})\ (\forall n\geqslant k):\quad \frac{\varkappa_k(\widehat{\psi})}{\varkappa_n(\widehat{\psi})}\leqslant\frac{\sigma_1^k}{\sigma^n}.$$

Theorem B. *In order that there exists a sequence (λ_n) such that system (1) forms a basis in the space A_∞, it is necessary and sufficient that the following conditions be satisfied: a), c) and*

b′) for some bounded sequence (δ_n) an entire function ψ^* with Taylor's coefficients $\psi_n^*=\psi_n e^{n\delta_n}$ is Newton's majorant of ψ .

We note, that if condition $c'')$ holds, then [3] $\ln M_\psi(r)=O(\ln^2 r)$ $(r\to+\infty)$, and if $c)$ holds, then $\ln M_\psi(r)=o(\ln^2 r)$ $(r\to+\infty)$.

Theorem C. *Let $R\in(0;+\infty]$ be a fixed number and an entire function ψ satisfy the conditions a), b), c) if $0<R<+\infty$, and conditions a), b′), c) if $R=+\infty$. Then in order that system (1) forms a basis in the space A_R, it is necessary and sufficient that the following conditions be satisfied :*

d) $(\forall\rho_*\in(0;R))(\exists R_*\in(0;R))(\exists c_1)(\forall k\geqslant 1)(\forall n\geqslant k):\ \prod_{m=k}^{n}|\lambda_m/\varkappa_m(\psi^*)|\leqslant c_1R_*^k/\rho_*^n,$

e) $(\forall\overline{\rho}\in(0;R))(\exists\overline{R}\in(0;R))(\exists c_2)(\forall k>1)(\forall n\geqslant k):\ \prod_{m=k}^{n}|\varkappa_{m-1}(\psi^*)/\lambda_m|\leqslant c_2\overline{R}^n/\overline{\rho}^k,$

f) the product

$$L(z)=(\lambda_1-z)\prod_{n=2}^{\infty}(1-z/\lambda_n)\tag{2}$$

uniformly converges on every compact from $\mathbb{C}$ and the function L satisfies the conditions

$$|\lambda_nL'(\lambda_n)|\geqslant M_L((1+o(1))|\lambda_n|),\quad n\to\infty,\tag{3}$$

$$\ln M_L(r)=N((1+o(1))r),\quad n\to\infty,\tag{4}$$

if $R\neq+\infty$, and satisfies the condition

$$(\exists R_0>0)(\exists n_0)(\forall n\geqslant n_0):|\lambda_nL'(\lambda_n)|\geqslant M_L(R_0|\lambda_n|),\quad n\to\infty,$$

if $R=+\infty$.

Theorem D. *For each entire transcendental function F there exists an entire function L, which has infinite set of zeros $\{\lambda_n\}_{n=1}^{\infty}$, all its zeros are simple, and such that:*

g) $M_L(r) = M_F((1+o(1))r), \quad r \to +\infty,$

h) $|\lambda_n L'(\lambda_n)| = M_L((1+o(1))|\lambda_n|), \quad n \to \infty,$

k) $\ln M_L(r) = N((1+o(1))r), \quad r \to +\infty,$

l) $|\lambda_n| = (1+o(1))\varkappa_n(\widehat{F}), \quad n \to \infty.$

Proof. Necessity follows at once from Theorems A and B. We prove sufficiency. Without loss of generality we assume that $\delta_n \equiv 0$ (look [3]). Then $\widehat{\psi}_k = |\psi_k|$ and therefore $\varkappa_k(\psi) = \varkappa_k(\widehat{\psi}) =: \varkappa_k$. Let F be an entire function with Taylor's coefficients $F_n = 1/\prod_{k=1}^n \rho_k$ $(n \geqslant 1)$, $F_0 = 1$. where $\rho_n = \sqrt{\varkappa_{n-1}\varkappa_n}$ and $\varkappa_0 := \varkappa_1/2$. Then $\varkappa_n(F) = \sqrt{\varkappa_{n-1}\varkappa_n}$. By the function F construct the entire function L, whose existence is generated by Theorem D. We shall show that the sequence of zeros (λ_n) of L satisfies following conditions

$$(\forall R \in (0;+\infty])(\forall \rho_* \in (0;R))(\exists R_* \in (0;R))(\exists c_1 \in (0;+\infty))(\forall k \geqslant 1)(\forall n \geqslant k):$$

$$\prod_{m=k}^{n} |\lambda_m/\varkappa_m| \leqslant c_1 R_*^k/\rho_*^n, \tag{5}$$

$$(\forall R \in (0;+\infty])(\forall \overline{\rho} \in (0;R))(\exists \overline{R} \in (0;R))(\exists c_2 \in (0;+\infty))(\forall k > 1)(\forall n \geqslant k):$$

$$\prod_{m=k}^{n} |\varkappa_{m-1}/\lambda_m| \leqslant c_4 \overline{R}^n/\overline{\rho}^k. \tag{6}$$

c) follows from c'). Therefore [3] $\ln M_F(r) = o(\ln^2 r)$ $(r \to +\infty)$.

Thus in accordance with g), L has form (2), and conditions (3), (4) are fulfilled. Moreover, $|\lambda_n| = (1+o(1))\sqrt{\varkappa_{n-1}\varkappa_n}$, $n \to \infty$. Hence, proceeding from c'), for each $R \in (0;+\infty], \rho_* \in (0;+\infty]$ and $\varepsilon > 0$ we obtain

$$\prod_{m=k}^{n} |\lambda_m|/\varkappa_m = \prod_{m=k}^{n} (1+\varepsilon)\sqrt{\frac{\varkappa_{m-1}}{\varkappa_m}} \leqslant (1+\varepsilon)^{n-k+1}\sqrt{\frac{\varkappa_{k-1}}{\varkappa_n}} \leqslant$$

$$\leqslant c_1(1+\varepsilon)^{n-k+1}(1+\varepsilon)^{\frac{k-1}{2}}\alpha^{\frac{n-k+1}{2}} \leqslant c_1\frac{(\rho_*\sqrt{1+\varepsilon})^k}{\rho_*^n}, \quad n \geqslant k \geqslant k_0(\varepsilon).$$

Thus (5) holds. Similarly for each $R \in (0;+\infty], \overline{\rho} \in (0;+\infty]$ and $\varepsilon > 0$ from c') we have

$$\prod_{m=k}^{n} \frac{\varkappa_{m-1}}{|\lambda_m|} \leqslant \prod_{m=k}^{n} (1+\varepsilon)\sqrt{\frac{\varkappa_{m-1}}{\varkappa_m}} \leqslant (1+\varepsilon)^{n-k+1}\sqrt{\frac{\varkappa_{k-1}}{\varkappa_n}} \leqslant c_1\frac{(\bar{\rho}\sqrt{1+\varepsilon})^n}{\bar{\rho}^k}, \quad n \geq k \geq k_0(\varepsilon),$$

that is (6) holds too. Therefore, Theorem 1 follows from Theorems C and D. □

Remark. Condition c) follows from c'), but c) and c') are not equivalent.

In fact, define the sequence $(\varkappa_k)$ by the equalities $\varkappa_1 = \varkappa_{m_0} = 1$, $\varkappa_k = 3^{3^{m_i}}$, if $m_i < k \leq m_{i+1}$, where $m_i = 3^i$. Let $m_i < k \leq m_{i+1}$, $m_j < n \leq m_{j+1}$. Therefore, if $i = j$, then

$$\frac{\varkappa_k}{\varkappa_n} = 1 = \frac{\sigma^n}{\sigma^n} \leqslant \frac{(\sigma^3)^k}{\sigma^n}, \quad \forall \sigma > 1, \ \forall n \geqslant k.$$

If $j = i + 1$, then

$$\frac{\varkappa_k}{\varkappa_n} = 1 = \frac{\sigma^n}{\sigma^n} \leqslant \frac{(\sigma^9)^k}{\sigma^n}, \quad \forall \sigma > 1,\ \forall n \geqslant k.$$

Finally, if $i < j$, then $\forall \sigma > 1, \quad \forall k \geqslant k_0, \quad \forall n \geqslant k$

$$\frac{\varkappa_k}{\varkappa_n} \leqslant \frac{\varkappa_{m_{i+1}}}{\varkappa_{m_j}} = \frac{3^{3^{m_{i+1}}}}{3^{3^{m_j}}} = 3^{-3^{m_j}(1-3^{m_{i+1}-m_j})} \leqslant 3^{-3^{m_j}\cdot 2/3} = \frac{1}{3^{2/3\cdot\frac{1}{3}m_{j+1}}} \leqslant \frac{1}{\sigma^n} \leqslant \frac{\sigma^k}{\sigma^n}.$$

As we saw the sequence $(\varkappa_k)$ satisfies $c)$ but does not satisfy $c')$, because from the condition $c')$ it follows that for some $\alpha > 0$ and $\varepsilon > 0$ $\varkappa_k/\varkappa_{2k} \to 0$ when $k \to \infty$, whereas for the constructed sequence it does not hold.

REFERENCES

1. Валирон Ж. Аналитические функции. – М.: ГИТТЛ, 1957. – 235 с.
2. Шеремета М. М. Цілі ряди Діріхле. – К.: ІСДО, 1993. – 168 с.
3. Винницкий Б. В. *Об описании базисов из обобщенных систем экспонент* // Матем. сб. –1988. – Т. 135(177) , №1. – С.59–79.
4. Винницкий Б. В. *О построение целой функции произвольного порядка с заданными асимптотическими свойствами* // Укр. мат. журн. –1988. – Т. 38, №2. –С. 143–148.
5. Винницький Б. В. Системи експонент і їх узагальнення в просторах аналітичних функцій. Дис. докт. фіз.-мат. наук. – Дрогобич, 1996. –300 с.

Institute of Physics and Mathematics,
Drohobych State Pedagogical University

Received 10.08.2002