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APPROXIMATION PROPERTIES OF THE LIE-ALGEBRAIC SCHEME

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In this paper one of the approximate methods based on a Lie-algebraic approach to solving Cauchy problems for partial differential equations is investigated. Approximation properties for quasirepresentations of the main differential operators in case of more than one space variable are proved.

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В статье исследован один из приближенных методов решения задач Коши для уравнений в частных производных, основанный на Ли-алгебраической схеме дискретных аппроксимаций. Доказаны аппроксимационные свойства квазипредставлений основных дифференциальных операторов в случае нескольких пространственных переменных.

Introduction

A Lie-algebraic method of discrete approximations had been proposed by F. Calogero in 1983 for calculating the eigenvalues of linear differential operators [4]. There were also some other papers devoted to applications of the Lie-algebraic approach to analytical solving some class of differential equations [5, 10]. In 1988 a new paper introducing an analytical-numerical method for solving partial differential equations based on the Lie-algebraic approach appeared [7]. High velocity of convergence demonstrated by this method was shown in [8]. Numerical tests for the nonhomogeneous heat equation performed making use of the Lie-algebraic scheme are presented in [2]. Here we investigate the latter method and prove some special properties for quasirepresentations of the main differential operators in the multidimensional case.

The problem investigated in this paper has the following form:

$$\begin{cases} u_t = K(t;x,\partial)\,u + f(t;x),\ x \in \Omega \\ u|_{t=0} = \varphi \in B, \end{cases} \tag{1}$$

where $(\forall\, t \geq 0)\ u(t;\,\cdot\,), f(t;\,\cdot\,) \in B,\ x = (x_1, \dots\ x_q), q \in \mathbb{N}$, is a vector of space variables, $K\colon B \to B$ is a differential operator depending on the evolution parameter $t \geq 0$, $B = L_2(\Omega)$ (definition of the space B may include also some boundary conditions [6]), a closed connected region $\Omega \subseteq D = \prod_{i=1}^{q} [a_j, b_j\,]$ with D being a q-dimensional cube.

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1. The Lie-algebraic method of discrete approximations

Now we introduce some data from the theory of Lie algebras and the Lagrange interpolation necessary for the further exposition. We assume that the differential operator K appearing in problem (1) belongs to the universal enveloping algebra $U(\mathcal{G})$ of the Heisenberg-Weyl algebra of differential operations $\mathcal{G} = \bigoplus_{j=1}^{q} \mathcal{G}_j = \bigoplus_{j=1}^{q}\{x_j, \partial_{x_j}, 1\}$ [7], where x_j is the operator of multiplication by the variable x_j, ∂_{x_j} the operator of differentiation, $\mathbf{1}$ the identity operator, $\bigoplus$ denotes the direct sum. The Lie brackets of the Lie algebra $\mathcal{G}$ are defined as follows: $[L_1, L_2] = L_1 \circ L_2 - L_2 \circ L_1$ for $L_1, L_2 \in \mathcal{G}$, where "$\circ$" means the usual superposition of operators.

As it was mentioned above, within the Lie-algebraic method we try to find the corresponding representations of all elements involved in (1), both functions and operators [3]. With this aim we define a sequence of linear mappings $\Phi_{(n)} = \pi_{(n)} \circ P_{(n)}{}^{1}$, $(n) \in \mathbb{Z}_+^q$. Here $P_{(n)}$, $(n) = (n_1, n_2, \dots n_q)$, $n_i \in \mathbb{N}$, $i \in \{1, \dots, q\}$, are the projectors onto the space $\mathcal{P}_{(n)}$ of algebraic polynomials of degree not exceeding $n_i - 1$ with respect to the variable x_i, $i \in \{1, \dots, q\}$. In order to define these projectors more precisely let us consider a lattice Θ of the cube D with nodes $\left(x_k^{(1)}, x_k^{(2)}, \dots x_k^{(n_k)}\right)$, $k \in \{1, \dots, q\}$, being the mesh points with respect to each variable and hence

$$\Theta = \left\{x_{(i)} = \left(x_1^{(i_1)}, x_2^{(i_2)}, \dots, x_q^{(i_q)}\right) \in D, 1 \le i_k \le n_k, 1 \le k \le q\right\}.$$

Then the operators $P_{(n)}$ acting on functions $u \in B$ are defined as follows:

$$P_{(n)}(x; u) = \sum_{(i) \in \mathrm{Im}} u(x_{(i)}) L_{(i)}(x),$$

where $\mathrm{Im} = \{(i) = (i_1, i_2, \dots, i_q) : \ 1 \le i_k \le n_k, \ 1 \le k \le q\}$ is the set of multi-indices determining the nodes of the lattice Θ, $L_{(i)}(x) = \prod_{k=1}^{q} l_{k,i_k}(x_k)$ and $l_{k,i_k}(x_k)$, $1 \le i_k \le n_k$, are fundamental Lagrange polynomials with respect to the variable x_k, $k \in \{1, \dots, q\}$. The isomorphism $\pi_{(n)}$ between the spaces $\mathcal{P}_{(n)}$ and $B_{(n)} = \bigotimes_{j=1}^{q} \mathbb{R}^{n_j}$ acts as follows:

$$(\forall g \in \mathcal{P}_{(n)}) \pi_{(n)} g = g_{(n)} := \left\{\frac{g\left(x_{(i)}\right)}{q_{(i)}}\right\}_{(i) \in \mathrm{Im}},$$

where $q_{(i)}$ are some constants determined by the lattice Θ on D. Thereby we can represent $P_{(n)}(x; u)$ as $P_{(n)}(x; u) = \langle u_{(n)}, e_{(n)} \rangle$ with $e_{(n)}$ being an appropriate basis in $\mathcal{P}_{(n)}$.

Thus the problem of finding the representations of functions involved in (1) in the space $B_{(n)}$ is solved. Within the Lie-algebraic method the operator K is also replaced by some appropriate matrix operator acting in the space $B_{(n)}$, namely the operator $K_{(n)}(t) := K\left(t; \bar{X}^{(n)}, \bar{Z}^{(n)}\right)$ where

$$\bar{X}^{(n)} = \left(X_1^{(n)}, X_2^{(n)}, \dots, X_q^{(n)}\right),$$

$$X_j^{(n)} = I^{(n_1)} \otimes \dots \otimes I^{(n_{j-1})} \otimes X_j^{(n_j)} \otimes I^{(n_{j+1})} \otimes \dots \otimes I^{(n_q)},$$

$j \in \{1, \dots, q\}$, and $\bar{Z}^{(n)}$ is defined analogously. The matrices $X_j^{(n_j)}$, $Z_j^{(n_j)}, I^{(n_j)} \in \mathbb{R}^{n_j \times n_j}$ defined in [4] are called quasirepresentations of the operators $x_j, \partial_{x_j}, \mathbf{1}$, $j \in \{1, \dots, q\}$,

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[1] Here and in the sequel $(\forall x \in \mathrm{Dom}(\psi))\ \varphi \circ \psi(x) \stackrel{\mathrm{df}}{=} \varphi(\psi(x))$.

respectively in the space $B_{(n)}$. Similarly to $X_j^{(n)}$, $Z_j^{(n)}$, $j \in \{1, \ldots, q\}$, we define $I^{(n)} = I^{(n_1)} \otimes \ldots \otimes I^{(n_q)}$ where $I^{(n_k)} = \{\delta_{ij}\}_{1 \le i,j \le n_k}$.

Therefore, problem (1) reduces to a sequence of Cauchy problems for ordinary differential equations with respect to multi-index (n):

$$\begin{cases} \dfrac{du_{(n)}}{dt} = K_{(n)}(t)\, u_{(n)} + f_{(n)}(t), \\ u_{(n)}\big|_{t=0} = \varphi_{(n)} \in B_{(n)}, \end{cases} \tag{2}$$

which can be solved either analytically (when it is possible) or by means of some numerical method, for instance the Runge-Kutta method. An approximate solution of problem (1) is constructed by means of an approximate solution of problem (2) as follows:

$$\tilde{u}(t^{(k)};\, x) = \Big\langle \bigotimes_{j=1}^{q} e_j(x_j),\, u_{(n)}\big(t^{(k)}\big) \Big\rangle,$$

where $t^{(k)}$, $k \in \{0,\, n_t\}$, are nodes of the mesh $\bigcup_{j=1}^{n_t-1} \big[t^{(j)},\, t^{(j+1)}\big] = [0,\, T]$ with respect to time variable $t \in \mathbb{R}$.

2. The approximation error estimations

Now we proceed to formulating and proving some statements concerning the accuracy of approximation of the basic operators x_j, ∂_{x_j} and $\mathbf{1}$ by means of their quasirepresentations $X_j^{(n_j)}$, $Z_j^{(n_j)}$, $I^{(n_j)}$, $j \in \{1, \ldots, q\}$, respectively. Henceforth we use the notion of the space $W_\infty^{(r)}(\mathbf{M};\, D) \overset{\mathrm{df}}{=} \overline{Q_\infty^{(r)}(\mathbf{M};\, D)}$, $(r) = (r_1, \ldots,\, r_q)$, $\mathbf{M} = (M_1, \ldots,\, M_q)$, where

$$Q_\infty^{(r)}(\mathbf{M};\, D) = \Big\{ v \in C^{(r)}(D;\, \mathbb{R}) : \Big|\frac{\partial^{r_i} v}{\partial\, x_i^{r_i}}\Big|_\infty \le M_i,\ i \in \{1, \ldots, q\} \Big\}$$

with $|f|_\infty = \max_{x \in D} |f(x)|$ for any $f \in C(D)$.

The following proposition holds.

Proposition. *If $u \in W_\infty^{(n)}(\mathbf{M};\, D)$ then the following estimation*

$$\Big| P_{(n)}\,(x;\ \partial_{x_i} u) - \partial_{x_i} P_{(n)}\,(x;\ u) \Big|_\infty \le M_i (b_i - a_i)^{n_i - 1} \frac{\Lambda_{(n)}}{n_i!}, \tag{3}$$

holds, where $i \in \{1, \ldots,\, q\}$, $\Lambda_{(n)} = \max_{x \in D} \sum_{(i) \in \mathrm{Im}} \big| L_{(i)}\,(x) \big|$ is the usual Lebesgue constant [1].

Proof. Without loss of generality we prove this proposition for the case $q = 2$, $i = 1$. In the following we will use the formulae

$$\partial_{x_1} u(x_1,\, x_2) = \partial_{x_1} P_{n_1}(x_1;\ u) + \partial_{x_1} \Big(\omega_{n_1}(x_1) \Big) \cdot \frac{\dfrac{\partial^{n_1} u}{\partial x_1^{n_1}}\,(\xi,\, x_2)}{n_1!}, \tag{4}$$

where $(x_1,\, x_2) \in D = [a_1,\, b_1] \times [a_2,\, b_2]$, $\xi \in [a_1,\, b_1]$, $\omega_{n_1}(x_1) = \prod_{k=1}^{n_1} \big(x_1 - x_1^{(k)}\big)$, and $P_{n_1}(x_1;\ u)$ is the Lagrange polynomial constructed with respect to the variable x_1. Expression (4) is a particular case of the Lagrange formulae for the Lagrange interpolation [1] applied to a function of two variables when one variable is regarded as a parameter.

One can verify that

$$P_{(n_1,n_2)}\Big(x_1,\, x_2;\; \partial_{x_1} P_{n_1}(x_1;\; u)\Big) = \partial_{x_1} P_{(n_1,n_2)}(x_1,\, x_2;\; u) \tag{5}$$

and

$$P_{(n_1,n_2)}\Big(x_1,\, x_2;\; \partial_{x_1}\Big(\omega_{n_1}(x_1)\Big)\cdot \frac{\partial^{n_1} u}{\partial x_1^{n_1}}(\xi,\, x_2)\Big) =$$

$$= \partial_{x_1}\Big(\omega_{n_1}(x_1)\Big)\cdot P_{n_2}\Big(x_2;\; \frac{\partial^{n_1} u}{\partial x_1^{n_1}}(\xi,\, x_2)\Big). \tag{6}$$

Now let us act on the both sides of (4) by the operator $P_{(n_1,n_2)}(x_1,\, x_2;\, \cdot)$. Taking into account (5) and (6), one obtains

$$P_{(n_1,n_2)}\big(x_1,\, x_2;\; \partial_{x_1} u\big) = \partial_{x_1} P_{(n_1,n_2)}\big(x_1,\, x_2;\; u\big) +$$

$$+\frac{1}{n_1!}\partial_{x_1}\big(\omega_{n_1}(x_1)\big)\cdot P_{n_2}\Big(x_2;\; \frac{\partial^{n_1} u}{\partial x_1^{n_1}}(\xi,\, x_2)\Big). \tag{7}$$

It is easy to check that

$$\Big|\partial_{x_1}\big(\omega_{n_1}(x_1)\big)\Big|_{\infty} \le \big(b_1 - a_1\big)^{n_1-1}\Lambda_{n_1}\big(x_1^{(1)},\dots,x_1^{(n_1)}\big), \tag{8}$$

where $\Lambda_{n_1}\big(x_1^{(1)},\dots,x_1^{(n_1)}\big)$ is the Lebesgue constant for Lagrangian polynomials with respect to the variable x_1.

Since $u \in W_{\infty}^{(n)}(M;\, D)$, the inequality

$$\Big|\frac{\partial^{n_1} u}{\partial\, x_1^{n_1}}(x_1,\, x_2)\Big|_{\infty} \le M_1 \tag{9}$$

holds.

Estimation of (7) by means of (8) and (9) ends the proof:

$$\Big|P_{(n_1,n_2)}\big(x_1,\, x_2;\; \partial_{x_1} u\big) - \partial_{x_1} P_{(n_1,n_2)}\big(x_1,\, x_2;\; u\big)\Big|_{\infty} \le$$

$$\le M_1\frac{1}{n_1!}(b_1 - a_1)^{(n_1-1)}\Lambda_{n_1}\big(x_1^{(1)},\dots,x_1^{(n_1)}\big)\cdot\Lambda_{n_2}\big(x_2^{(1)},\dots,x_2^{(n_2)}\big) =$$

$$= M_1(b_1 - a_1)^{(n_1-1)}\frac{\Lambda_{(n_1,n_2)}}{n_1!}.$$

□

Theorem 1. *Under the assumption of the previous proposition*

$$\left|\left\langle Z_i^{(n)}\, u_{(n)} - \left(\frac{\partial\, u}{\partial\, x_i}\right)_{(n)},\ e_{(n)}\right\rangle\right|_\infty \le M_i (b_i - a_i)^{n_i - 1} \frac{\Lambda_{(n)}}{n_i!}, \tag{10}$$

$$\left|\left\langle X_i^{(n)}\, u_{(n)} - \left(x_i\, u\right)_{(n)},\ e_{(n)}\right\rangle\right|_\infty = 0, \tag{11}$$

$$\left|\left\langle \left(I^{(n)} - \left[Z_i^{(n)},\, X_i^{(n)}\right]\right) u_{(n)},\ e_{(n)}\right\rangle\right|_\infty \le \hat{M}_i (b_i - a_i)^{n_i - 1} \frac{\Lambda_{(n)}}{(n_i - 1)!} \tag{12}$$

holds, where $i \in \{1, \dots, q\}$, $\hat{M}_i = \mathrm{const} > 0$.

Proof. One can verify that $\left\langle Z_i^{(n)}\, u_{(n)},\ e_{(n)}\right\rangle = \dfrac{\partial}{\partial x_i} P_{(n)}(x;\ u)$. Taking into account (3) and the latter equality one obtains (10). Equality (11) can be easily verified as well. Thus let us proceed to a proof of (12).

By means of the following transformations one obtains

$$\begin{aligned}
&\left\langle \left(I^{(n)} - \left[Z_i^{(n)},\, X_i^{(n)}\right]\right) u_{(n)},\ e_{(n)}\right\rangle = \left\langle u_{(n)},\ e_{(n)}\right\rangle - \left\langle Z_i^{(n)}\, X_i^{(n)}\, u_{(n)},\ e_{(n)}\right\rangle + \\
&+\left\langle X_i^{(n)}\, Z_i^{(n)}\, u_{(n)},\ e_{(n)}\right\rangle = P_{(n)} u + P_{(n)}\big(x_i\, \partial_{x_i} P_{(n)} u\big) - P_{(n)}\big(\partial_{x_i} P_{(n)}\, (x_i u)\big) = \\
&= \Big[P_{(n)} u - P_{(n)}\big(\partial_{x_i}\ (x_i u)\big) + P_{(n)}\big(x_i\, \partial_{x_i}\, u\big)\Big] + \Big[P_{(n)}\big(x_i\, \partial_{x_i}\, P_{(n)} u\big) - P_{(n)}\big(x_i\, \partial_{x_i}\, u\big)\Big] + \\
&+\Big[P_{(n)}\big(\partial_{x_i}\ (x_i u)\big) - P_{(n)}\big(\partial_{x_i} P_{(n)}\, (x_i u)\big)\Big] = P_{(n)}\Big(\big(1 - [\partial_{x_i},\, x_i]\big)\, u\Big) + \\
&+\Big[P_{(n)}\Big(\partial_{x_i} v_i\Big) - P_{(n)}\Big(\partial_{x_i} P_{(n)} v_i\Big)\Big] = P_{(n)}\Big(\partial_{x_i} v_i\Big) - P_{(n)}\Big(\partial_{x_i} P_{(n)} v_i\Big),
\end{aligned}$$

where $v_i = x_i\, u - u$, $i \in \{1, \dots, q\}$. In the preceding expressions we have used the abbreviation $P_{(n)} u = P_{(n)}\, (x;\ u)$.

It is easy to observe that for all $i \in \{1, \dots, q\}$

$$\left|\frac{\partial^{n_i} v_i}{\partial x_i^{n_i}}\right|_\infty = \left| n_i \frac{\partial^{n_i - 1} u}{\partial x_i^{n_i - 1}} + (x_i - 1)\, \frac{\partial^{n_i} u}{\partial x_i^{n_i}}\right|_\infty \le n_i \tilde{M}_i + (|b_i| + 1)\, M_i = c_{1,i}\, n_i + c_{2,i}, \tag{13}$$

where $\tilde{M}_i$ is a constant bounding $\left|\dfrac{\partial^{n_i - 1} u}{\partial x_i^{n_i - 1}}\right|_\infty$, $i \in \{1, \dots, q\}$, (the latter expression is bounded due to properties of the space $W_\infty^{(n)}\, (\mathbf{M};\ D)$ [1]), $c_{1,i} = \tilde{M}_i$, $c_{2,i} = (|b_i| + 1)\, M_i$.

Estimation (3) in accordance with (13) and the inequality $n_i \ge 1$ implies

$$\left| P_{(n)}\Big(\partial_{x_i} v_i\Big) - P_{(n)}\Big(\partial_{x_i} P_{(n)} v_i\Big)\right|_\infty \le$$

$$\le \frac{c_{1,i}\, n_i + c_{2,i}}{n_i}\, (b_i - a_i)^{n_i - 1}\, \frac{\Lambda_{(n)}}{(n_i - 1)!} \le \hat{M}_i (b_i - a_i)^{n_i - 1} \frac{\Lambda_{(n)}}{(n_i - 1)!},$$

where $\hat{M}_i = c_{1,i} + c_{2,i}$. □

Remark 1. For an equispaced distribution of all intervals $[a_i, b_i]$, $i \in \{1, \dots, q\}$, in the case of the Lagrange interpolation over the cube D the Lebesgue constant can be evaluated as follows: $\Lambda_{(n)} = \prod_{j=1}^{q} \Lambda_{n_j}\big(x_j^{(1)}, \dots, x_j^{(n_j)}\big) \le 2^{N-q}$, where $N = \sum_{j=1}^{q} n_j$ [1].

From Theorem 1 and Remark 1 it follows that the operators $\tilde{Z}_i^{(n)} \colon \mathcal{P}_{(n)} \to \mathcal{P}_{(n)}$, $(\forall g \in \mathcal{P}_{(n)})\ \tilde{Z}_i^{(n)} g := \langle Z_i^{(n)} g_{(n)}, e_{(n)} \rangle$, approximate the operator ∂_{x_i} defined in the domain $C^1(D)$ [9], with velocity $\frac{\alpha^{n_i}}{n_i!}$, $\alpha = \text{const} > 0$ as $n_i \to \infty$ and $n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_q$ are fixed. Moreover, the considered approximation of the Heisenberg-Weyl algebra operators preserves the important basic identity $\big[\partial_{x_i}, x_i\big] = 1$, namely the expression $\Big[\tilde{Z}_i^{(n)}, \tilde{X}_i^{(n)}\Big] - \tilde{I}^{(n)}$ approximates the operator $\mathbf{0} \colon B \to B$, $(\forall g \in B)\ \mathbf{0} g := 0$, with $\tilde{X}_i^{(n)}$, $\tilde{I}^{(n)}$ being defined analogously to $\tilde{Z}_i^{(n)}$.

Conclusions

The Lie-algebraic method uses the so-called Lagrangian collocation when approximating both functions and differential operators. This peculiarity of the method under regard causes unusual type of approximation error convergence, namely $O\big(\frac{\alpha^n}{n!}\big)$ as $n \to \infty$, $\alpha > 0$ while convergence of the standard finite-difference method is in general proportional to n^{-p}, $p > 0$. This essential difference can be explained by means of the type of derivative approximation: within the finite difference method only local representation of derivative is used while within the Lie-algebraic approach all mesh points are involved in the expression of derivative approximation.

The parameter $\alpha \in \mathbb{R}_+$ depends on the length of an interpolation interval. Therefore the minimal number of mesh points providing sufficiently good approximation grows if the interpolation interval length increases.

All results presented in this paper concern the multi-dimensional Lie-algebraic approximation. The remarkable fact to be mentioned here is that application of this method does not need significant modification with respect to one-dimensional case.

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