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DAUGAVET TYPE INEQUALITIES FOR NARROW OPERATORS IN THE SPACE L_1

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First the classical Daugavet Equation $\|I+T\| = 1 + \|T\|$ (DE) for L_1 was established for compact operators T and then generalized for wide classes of operators in a number of papers. We prove some inequalities which generalize the DE for an arbitrary into isomorphism $J : L_1 \to L_1$ instead of the identity I.

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Классическое равенство Даугавета $\|I+T\| = 1 + \|T\|$ (РД) для пространства L_1, установленное изначально для компактных операторов T, неоднократно обобщалось в различных работах на более широкие классы операторов. Мы приводим неравенства, обобщающие РД на случай, когда рассматривается произвольное изоморфное вложение $J : L_1 \to L_1$ вместо тождественного оператора I.

0. Preliminaries

We use the standard terminology and usual notations as in [22, 23]. For a Banach space X we denote $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \le 1\}$, by $\mathcal{L}(X)$ and $\mathcal{K}(X)$ we denote the space of all bounded linear and respectively compact operators acting in X and by I we denote the identity operator in X. $\mathcal{L}(X, Y)$ stands for the set of all bounded linear operators acting from X to a Banach space Y. According to [27], an operator $T \in \mathcal{L}(L_1, Y)$ is called narrow if for every measurable set $A \subset [0, 1]$ and every $\varepsilon > 0$ there is $x \in L_1$ such that $x^2 = \chi(A)$, $\int x d\mu = 0$ and $\|Tx\| < \varepsilon$ (by $\chi(A)$ we denote the characteristic function of a set A). The set of all narrow operators $\mathrm{Narr}\,(L_1, Y)$ contains all compact operators; on the other hand, there exists a narrow projection P of L_1 onto a subspace isomorphic to L_1 (see [27]).

A classical result of I. K. Daugavet [4] which states that for every $T \in \mathcal{K}(C[0, 1])$ the following equation is satisfied:

$$\|I + T\| = 1 + \|T\| \qquad (DE)$$

has been generalized in several directions in the last twenty years by a number of authors.

First direction. Proving (DE) for $T \in \mathcal{K}(X)$ for new classes of Banach spaces X. The first result of the kind is due to G. Ya. Lozanovskii [24] for $X = L_1$. A Banach space X is

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said to have the Daugavet property (DP) if (DE) is satisfied for every $T \in \mathcal{K}(X)$. One can find other development in [1], [13], [34].

Second direction. Proving (DE) for new classes of operators $T \in \mathcal{M}(X) \subset \mathcal{L}(X)$ containing compact operators $\mathcal{K}(X) \subset \mathcal{M}(X)$ for a Banach space X with (DP): weakly compact operators, narrow operators, not strongly Enflo operators etc. (see [8], [9], [10], [13], [27], [33]). A special kind of this direction is proving (DE) for pairs of Banach spaces: a pair (X, Y) of Banach spaces with $X \subset Y$ is said to have (DP) with respect to a class of operators $\mathcal{M} \subset \mathcal{L}(X, Y)$ if

$$\|J + T\| = 1 + \|T\|$$

for each $T \in \mathcal{M}$ where $J \colon X \longrightarrow Y$ is the inclusion operator (see [15], [32], [34]).

Third direction. Comparison (DP) with other geometric properties of Banach spaces. The best results in this direction are obtained in [16]. So, a Banach space X has (DP) if and only if (DE) is satisfied for all rank one operators $T \in \mathcal{L}(X)$; if X has the (DP) then X cannot be embedded into a Banach space with an unconditional basis (this fact was originally proved in [11]), both X and X^* do not have the Radon-Nikodym property (the first part was earlier proved in [35]). There is a nice geometric characterization of (DP) in terms of the unit ball of X discovered by V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin and D. Werner: X has (DP) if and only if For every $y \in S_X$, every $x^* \in S_{X^*}$ and every $\varepsilon > 0$ there is $x \in S_X$ such that $x^*(x) \geq 1 - \varepsilon$ and $\|x + y\| \geq 2 - \varepsilon$.

One can find other development concerning this direction in [32] and [34].

Our paper is devoted to the last direction to which we shall pay the maximal attention.

Fourth direction. Proving weaker conditions than (DE) for wide classes of operators and spaces. First Y. Benyamini and P.-K. Lin [3] proved that for every $p, 1 < p < \infty, p \neq 2$ and every $\varepsilon > 0$ there is $\delta_p(\varepsilon) > 0$ such that for each $T \in \mathcal{K}(L_p(\mu))$ with atomless μ

$$\|I + T\| \geq 1 + \delta_p(\|T\|). \qquad (BLI)$$

(Note that the spaces L_p with $1 < p < \infty$ do not have (DP)). Then the following weaker inequality for narrow projections was proved in [28]: for every $p, 1 \leq p < \infty, p \neq 2$ there is $k_p > 1$ such that if $P \neq 0$ is a narrow projection in $L_p(\mu)$ with atomless μ then

$$\|I - P\| \geq k_p. \qquad (PI)$$

Later the above two results were generalized in [27] by proving (BLI) for narrow operators in $L_p(\mu)$ for $1 < p < \infty, p \neq 2$ (note that for $p = 1$ (DE) one can consider as (BLI) with $\delta_1(\varepsilon) = \varepsilon$).

The weakest condition of the kind

$$\|I - P\| > 1 \qquad (RI)$$

was proved by B. Randrianantoanina for finite-dimensional projections $P \neq 0$ in any real r.i. function space X on $[0, 1]$ which is not isometric to L_2 in [30] and by N. J. Kalton and B. Randrianantoanina for one-dimensional projections P in any separable real order-continuous Kothe function space on (Ω, μ) with finite atomless μ if X does not contain a band (see [19]) isomorphic to L_2 in [19].

C. Franchetti [6] proved that the best value of k_p in (PI) is attained for the simplest projection $k_p = \|I - A\|$ where

$$(Ax)(t) \equiv \int x d\mu. \qquad (FO)$$

Then C. Franchetti and E. M. Semenov [7] proved (RI) for (FO) in any real r.i. function space on an atomless probability measure space (Σ, μ) which is not isometric to L_2. After that E.M.Semenov asked several natural questions:

(i) Whether (PI) holds in every rearrangement invariant function space X on $[0,1]$ with absolutely continuous norm but L_2 for narrow projections $P \neq 0$ (of course, with some constant $k_X > 1$ instead of k_p)?

(ii) Does the validation of (PI) for one-dimensional projections in a function space X imply (PI) for nontrivial finite dimensional projections or even for nontrivial narrow projections (if they are well defined in X)?

(iii)–(iv) The same questions for (RI) instead of the (PI).

T. Oikhberg [25] introduced the notion of Pseudo-Daugavet property based on (BLI): A Banach space X is said to have (PDP) if there exists a function $\phi_X \colon (0,\infty) \longrightarrow (0,\infty)$ such that (BLI) holds for every nonzero compact operator T. He has proved (PDP) for non-commutative L_p-spaces in [25] and for subspaces of $\mathcal{L}(l_p)$ containing all compact operators in [26].

The following versions of the Semenov questions could be asked for (PDP) in (i) and (ii) instead of (PI).

These questions were partially affirmatively answered also in [29], [31].

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1. Norm estimations of the sum of an into isomorphism and a narrow operator

The following simple observation is, in some sense, the best possible.

Proposition 1. *Given a Banach space X having (DP) with respect to a set $\mathcal{M} \subset \mathcal{L}(X)$, let $J \in \mathcal{L}(X)$ be an into isomorphism and $T \in \mathcal{L}(X)$ be such that $T(X) \subset J(X)$ and $J^{-1}T \in \mathcal{M}$. Then*

$$\|J+T\| \, \|J\| \, \|J^{-1}\| \geq \|J\| + \|T\|. \tag{$*$}$$

Proof.

$$\|J+T\| = \|J(I+J^{-1}T)\| = \sup_{\|x\|=1} \|J(I+J^{-1}T)x\| \geq \sup_{\|x\|=1} \|J^{-1}\|^{-1}\|(I+J^{-1}T)x\| =$$

$$\|J^{-1}\|^{-1}\|I+J^{-1}T\| = \|J^{-1}\|(1+\|J^{-1}T\|).$$

Then

$$\|J^{-1}T\| = \sup_{\|x\|=1} \|J^{-1}Tx\| \geq \sup_{\|x\|=1} \|J\|^{-1}\|Tx\| = \|J\|^{-1}\|T\|.$$

Thus, $\|J+T\| \geq \|J^{-1}\|^{-1}(1+\|J\|^{-1}\|T\|)$. □

Later on we shall consider operators acting in the space L_1 only.

Corollary 2. *Let $J \in \mathcal{L}(L_1)$ be an into isomorphism, $T \in \mathrm{Narr}\,(L_1)$ and $T(L_1) \subset J(L_1)$. Then $(*)$ holds.*

For the proof it is enough to note that if $T \in \mathrm{Narr}\,(L_1)$ and $S \in \mathcal{L}(L_1)$ then trivially $ST \in \mathrm{Narr}(L_1)$.

As a consequence we obtain that if $J \in \mathcal{L}(L_1)$ is an into isometry, $T \in \mathrm{Narr}\,(L_1)$ and $T(L_1) \subset J(L_1)$ then

$$\|J + T\| = 1 + \|T\|. \tag{$**$}$$

We show that for an into isomorphism J the condition $T(L_1) \subset J(L_1)$ is essential while for an into isometry this is not the case.

Proposition 3. *For every $\varepsilon > 0$ there exist an into isomorphism $J \in \mathcal{L}(L_1)$ with $\|J\| = 1 + \varepsilon$ and $\|J^{-1}\| = 1$ and $T \in \mathrm{Narr}\,(L_1)$ with $\|T\| = \varepsilon$ such that $\|J + T\| = 1$ while $1 + \varepsilon = \|J + T\|\|J\|\|J^{-1}\| < \|J\| + \|T\| = 1 + 2\varepsilon$.*

Proof. Let $T \in \mathrm{Narr}\,(L_1)$ be any narrow operator satisfying $\|T\| = \varepsilon$ and $\mathrm{supp}\; Tx \subset [\frac{1}{2}, 1]$ for each $x \in L_1$. Put

$$Jx(t) = -Tx(t) + \begin{cases} 2x(2t), \text{ if } t \in [0, \frac{1}{2}) \\ 0, \text{ if } t \in [\frac{1}{2}, 1]. \end{cases}$$

for each $x \in L_1$. Then

$$\|x\| \le \|Jx\| = \|x\| + \|Tx\| \le (1 + \varepsilon)\|x\|$$

for every $x \in L_1$ and evidently $\sup\limits_{\|x\|=1} \|Jx\| = 1 + \varepsilon$. So, J is an into isomorphism with the desired properties and

$$(J + T)x(t) = \begin{cases} 2x(2t), \text{ if } t \in [0, \frac{1}{2}) \\ 0, \text{ if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Thus, $\|J + T\| = 1$ and the proposition is proved. □

Remark 4. Note that $\|J + T\| = \|J^{-1}\|^{-1}$ in the above example and in general, if X is a Banach space, $J \in \mathcal{L}(X)$ is an into isomorphism and $T \in \mathcal{L}(X)$ is not, then trivially

$$\|J + T\| \ge \|J^{-1}\|^{-1}. \tag{$* * *$}$$

Thus, $(* * *)$ is the best general estimate.

The next theorem implies, in particular, that if $J \in \mathcal{L}(L_1)$ is an into isometry then the condition $T(L_1) \subset J(L_1)$ could be omitted in Proposition 1. On the other hand, it implies that the lost of norm in the sum $\|J + T\|$ (as in Proposition 3) could be estimated by norms of J, J^{-1} and T.

Theorem 5. *Let $J \in \mathcal{L}(L_1)$ be an into isomorphism, $T \in \mathrm{Narr}\,(L_1)$. Then*

$$\|J + T\| \ge \|T\| + \max\Big\{\frac{2}{\|J\|\|J^{-1}\|^2} - \|J\|, \quad \frac{8}{3\,\|J^{-1}\|} - \frac{5}{3}\|J\|\Big\}.$$

Proof. Fix $\varepsilon > 0$. It is not difficult to see that there is an $A \subset [0, 1]$ of positive measure such that for $x = \frac{\chi(A)}{\mu(A)}$ we have $\|x\| = 1$ and $\|Tx\| \ge \|T\| - \varepsilon$ (indeed, choose an integer m so that for the finite σ-algebra Σ_m generated by the sets $I_k^m = [\frac{k-1}{m}, \frac{k}{m})$ for $k \in \{1, ..., m\}$ we have $\|T|\Sigma_m\| \ge \|T\| - \varepsilon$. Then a simple argument shows that $\|T|\Sigma_m\| = m \max\limits_k \|T\chi(I_k^m)\|$).

Then we need the following two known lemmas. The first one is a simple n-fold usage of the definition of a narrow operator and the second one of a theorem due to L. E. Dor.

Lemma 6 ([27, p.55]). (we formulate it only for the setting of L_1). *Let Y be a Banach space and $T \in \mathcal{L}(L_1, Y)$ be a narrow operator. Then for every measurable set $A \in [0,1]$, every $\varepsilon > 0$ and every integer n there exists a partition $A = A' \bigsqcup A''$ into measurable subsets with $\mu(A') = 2^{-n}\mu(A)$ and $\mu(A'') = (1 - 2^{-n})\mu(A)$ such that $\|Th\| < \varepsilon$ where $h = (2^n - 1)\chi(A') - \chi(A'')$.*

For $x \in L_1$ and a measurable $A \subset [0,1]$ denote

$$\|x|A\| \stackrel{\text{def}}{=} \int_A |x| d\mu.$$

Lemma 7 ([5]). *Let $\{x_n\} \in L_1$ and a number $\theta \in (0,1]$ be such that for each integer n and each scalars $\{a_k\}_{k=1}^n$*

$$\Big|\sum_{k=1}^n a_k x_k\Big| \geq \theta \sum_{k=1}^n |a_k|.$$

Then there is a disjoint sequence $\{A_n\}_{n=1}^\infty$ of measurable subsets of $[0,1]$ such that for any n

$$\|x_n|A_n\| \geq \theta^2$$

and also there is a disjoint sequence $\{B_n\}_{n=1}^\infty$ such that for any n

$$\|x_n|B_n\| \geq 1 - \frac{4}{3}(1-\theta).$$

Continue our proof. For each n by Lemma 6 choose an $h_n \in L_1$ and a decomposition $A = A'_n \bigsqcup A''_n$ so that $\mu(A'_n) = 2^{-n}\mu(A)$ and $\|Th_n\| < \varepsilon$ where

$$h_n = \frac{1}{\mu(A)}\Big((2^n - 1)\chi(A'_n) - \chi(A''_n)\Big).$$

Then

$$x + h_n = \frac{2^n}{\mu(A)}\chi(A'_n)$$

and $\|x + h_n\| = 1$. Denote by $\{e_n\}$ the unit vector basis of the space l_1. It is not difficult to see that the normalized sequence $\{x + h_n\}$ contains a subsequence $\{x + h_{i(n)}\}$ which is $(1+\varepsilon)$-equivalent to disjoint functions in L_1, i.e. to $\{e_n\}$.

Suppose $\|J\| = 1$. Then $\|J^{-1}\| \geq 1$ and $\theta = \|J^{-1}\|^{-1}(1+\varepsilon)^{-1} \leq 1$. Putting $x_n = J(x + h_{i(n)})$, we obtain for any n and any scalars $\{a_k\}_{k=1}^n$

$$\Big|\sum_{k=1}^n a_k x_k\Big| = \Big|J\sum_{k=1}^n a_k(x + h_{i(k)})\Big| \geq \|J^{-1}\|^{-1}\Big|\sum_{k=1}^n a_k(x + h_{i(k)})\Big| \geq \theta \sum_{k=1}^n |a_k|.$$

Using Lemma 7, choose a disjoint sequence $A_n \subset [0,1]$ with

$$\|x_n|A_n\| \geq \theta^2.$$

Then choose a number m so that $\|Tx|A_m\| < \varepsilon$. Estimate:

$$\begin{aligned} \|(J+T)(x+h_{i(m)})\| \geq \|x_m + Tx\| - \varepsilon = \\ = \|(x_m+Tx)|A_m\| + \|(x_m+Tx)|([0,1]\setminus A_m)\| - \varepsilon \geq \\ \geq \theta^2 - \varepsilon + \|Tx|([0,1]\setminus A_m)\| - \|x_m|([0,1]\setminus A_m)\| - \varepsilon = \\ = \theta^2 - 2\varepsilon + \|Tx - Tx|A_m\| - \|x_m|([0,1]\setminus A_m)\| \geq \\ \geq \theta^2 - 2\varepsilon + \|Tx\| - \varepsilon - \|x_m|([0,1]\setminus A_m)\| \geq \theta^2 + \|T\| - 4\varepsilon - \|x_m|([0,1]\setminus A_m)\| = \\ = \theta^2 + \|T\| - 4\varepsilon - \|x_m\| + \|x_m|A_m\| \geq 2\theta^2 + \|T\| - 4\varepsilon - 1. \end{aligned}$$

By arbitrariness of ε,

$$\|J+T\| \geq \frac{2}{\|J^{-1}\|^2} - 1 + \|T\|.$$

Using the second part of Door's theorem one can show that

$$\|J+T\| \geq 2\left(1 - \frac{4}{3}(1 - \|J^{-1}\|^{-1})\right) - 1 + \|T\| = \frac{8}{3\|J^{-1}\|} - \frac{5}{3} + \|T\|.$$

Suppose now no restriction on $\|J\|$. Since $\left\|\left(\frac{J}{\|J\|}\right)^{-1}\right\| = \left\|\|J\|J^{-1}\right\| = \|J\|\|J^{-1}\|$, we have

$$\|J+T\| = \|J\|\left\|\frac{J}{\|J\|} + \frac{T}{\|J\|}\right\| \geq \|J\|\left(\frac{2}{\|J\|^2\|J^{-1}\|^2} - 1 + \left\|\frac{T}{\|J\|}\right\|\right) = \frac{2}{\|J\|\|J^{-1}\|^2} - \|J\| + \|T\|$$

and respectively

$$\|J+T\| \geq \frac{8}{3\,\|J^{-1}\|} - \frac{5}{3}\|J\| + \|T\|.$$

The theorem is proved. □

Remark 8. For an into isometry $\|J\| = \|J^{-1}\| = 1$ we obtain that Theorem 5 implies a probably well known Daugavet property for a pair (X, Y) of isometric copies of the space L_1 with $X \subset Y$.

Remark 9. A theorem of D. E. Alspach [2] states that there exist $\varepsilon_0 > 0$ and a function $\tau(\varepsilon) > 0$ for $0 < \varepsilon < \varepsilon_0$ where $\tau(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $J \in \mathcal{L}(L_1)$ is an into isomorphism with $\|J\|\|J^{-1}\| < 1 + \varepsilon$ then there is an into isometry $S \in \mathcal{L}(L_1)$ with $\|J - S\| < \tau(\varepsilon)$. Whence we obtain the following:

Let $J \in \mathcal{L}(L_1)$ be an into isomorphism with $\|J\|\|J^{-1}\| < 1 + \varepsilon < 1 + \varepsilon_0$. Then

$$\|J+T\| \geq \|S+T\| - \tau(\varepsilon) = 1 + \|T\| - \tau(\varepsilon).$$

Theorem 10. *Suppose $J, T \in \mathcal{L}(L_1)$. Let T be narrow and J be an isomorphic embedding with:*

$$\text{if } \mu(A_n) \longrightarrow 0 \text{ then } J\left(\frac{\chi(A_n)}{\mu(A_n)}\right) \longrightarrow 0 \text{ in measure.} \qquad (MC)$$

Then

$$\|J+T\| \geq \|J^{-1}\|^{-1} + \|T\|.$$

Proof. Fix $\varepsilon > 0$. Like in the proof of Theorem 5, choose a measurable subset $A \subset [0,1]$ with

$$\|Tx\| \geq \|T\| - \varepsilon \quad \text{where} \quad x = \frac{\chi(A)}{\mu(A)}.$$

Then for each n choose by Lemma 6 a partition $A = A_n' \bigsqcup A_n''$ onto measurable subsets of measure $\mu(A_n') = 2^{-n}\mu(A)$ with $\|Th_n\| < \varepsilon\mu(A)$ where $h_n = (2^n - 1)\chi(A_n') - \chi(A_n'')$.

Note that

$$x + \frac{h_n}{\mu(A)} = \frac{\chi(A_n')}{\mu(A_n')}.$$

By conditions of the theorem, $J\left(\frac{\chi(A_n')}{\mu(A_n')}\right) \longrightarrow 0$ in measure.

It is not hard to see that if (x_n) is a measure null sequence then for each $y \in L_1$

$$\limsup_n \|x_n + y\| = \limsup_n \|x_n\| + \|y\|.$$

Therefore, there is an integer n such that

$$\left\|J\left(x + \frac{h_n}{\mu(A)}\right) + Tx\right\| \geq \limsup_n \left\|J\left(x + \frac{h_n}{\mu(A)}\right)\right\| + \|Tx\| \geq$$

$$\|J^{-1}\|^{-1} + \|Tx\| \geq \|J^{-1}\|^{-1} + \|T\| - \varepsilon.$$

Thus,

$$\left\|(J+T)\left(x + \frac{h_n}{\mu(A)}\right)\right\| \geq \left\|J\left(x + \frac{h_n}{\mu(A)}\right) + Tx\right\| - \left\|T\frac{h_n}{\mu(A)}\right\| \geq \|J^{-1}\|^{-1} + \|T\| - 2\varepsilon.$$

Since

$$\left\|x + \frac{h_n}{\mu(A)}\right\| = 1,$$

the theorem is proved. □

Remark. In particular, (MC) holds if T is continuous with respect to an L_p-norm for some $0 \leq p < 1$ (or equivalently, T can be extended to a continuous operator in L_p, $0 \leq p < 1$). Note also that continuous linear operators acting in the spaces L_p, $0 \leq p < 1$ are characterized by Kwapień's theorem [21] for $p = 0$ and Kalton's theorem [18] for $0 < p < 1$ (see also [20]).

2. Further remarks and open problems

By $L_1(A)$ for measurable $A \subset [0,1]$ we denote the subspace of L_1 consisting of all x with $x(t) = 0$ for $t \in [0,1] \setminus A$.

In [17] the authors introduced a new notion of narrow operators acting from a Banach space with (DP). Using Proposition 3.11 from [17] it can be formulated as follows:

Definition. Let X be a Banach space possessing (DP), Y be any Banach space. An operator $T \in \mathcal{L}(X,Y)$ is called *(KSW)-narrow* if for every $x, y \in S(X)$, $\varepsilon > 0$ and every slice $S(x^*, \varepsilon_1) = \{x \in B(X) : x^*(x) \geq 1 - \varepsilon_1\}$ containing y there is an $v \in S(x^*, \varepsilon_1)$ such that $\|x + v\| > 2 - \varepsilon$ and $\|T(y - v)\| < \varepsilon$.

In [17,Theorem 6.1] (see also [34]) it is proved that for operators T acting from L_1 to a Banach space X the new notion of narrow operators is equivalent to the following one:

Definition. An operator $T \in \mathcal{L}(L_1, X)$ is called *(KSW)-narrow* if for each $\varepsilon > 0$, $\varepsilon_1 > 0$ and a measurable $A \subset [0,1]$ there exists an $x \in L_1(A)$ such that:

(i) $x(t) \geq -1$ a.e. at A;

(ii) $\int x d\mu = 0$;

(iii) $\mu\{t \in A : x(t) > -1\} \leq \varepsilon_1$;

(iv) $\|Tx\| \leq \varepsilon$.

It can be easily verified (cf. Lemma 6) that every narrow operator is (KSW)-narrow. But as it was mentioned in [17] and [34] that the converse is unknown, i.e.:

Problem 1 [17]. *Is every (KSW)-narrow operator from L_1 to X narrow?*

The answer is yes if X itself can be embedded into L_1. On the other hand, the answer is evidently affirmative for the Banach spaces X which satisfy $\mathrm{Narr}\,(L_1, X) = \mathcal{L}(L_1, X)$ (see [12]).

V. M. Kadets asked a *medium* question (whether the converse to Lemma 6 is true) which possibly could help to answer (KSW)-problem. The answer to this question is affirmative and follows from the next theorem [14]:

Theorem. *Let X be any Banach space and $T \in \mathcal{L}(L_1, X)$ has the following property: for each $\varepsilon > 0$ and a measurable $A \subset [0,1]$ with $\mu(A) > 0$ there exists a measurable $B \subset A$ such that:*

$$(a) \qquad 0 < \mu(B) < \mu(A);$$

$$(b) \qquad \|Th\| < \varepsilon\|h\| \ \text{ where } h = \frac{\mu(A-B)}{\mu(B)}\chi(B) - \chi(A \setminus B).$$

Then for each $\delta \in (0,1)$, each $\varepsilon > 0$ and each measurable A $[0,1]$ with $\mu(A) > 0$ there exists a measurable $B \subset A$ such that (b) holds together with

$$(a') \qquad \mu(B) = \delta\mu(A).$$

In particular ($\delta = \frac{1}{2}$), T is narrow.

Remarks. 1. Note that the two definitions of narrow operators are incomparable (with respect to domain spaces) if we ask which one is more general: the first one concerns the operators defined on rearrangement invariant F-spaces and the second on spaces with (DP).

2. In [17] the authors called narrow operators in the old sense defined on L_1 by L_1-narrow and the word "narrow" standed for the new definition. Such a terminology takes place in a few recent papers.

It is known that in the setting of both definitions, the sum of two narrow operators in $\mathcal{L}(L_1)$ is narrow (in [27] this was stated for the old definition but the proof was incorrect; later this was proved in [34] and [14]).

Problem 2 [14]. *Is it true that for any Banach space X the sum of two narrow operators from $\mathcal{L}(L_1, X)$ is narrow (this is not known in the sense of both definitions)?*

Problem 3. *Is it true that for any Banach space X if T $\mathcal{L}(L_1, X)$ fixes no copy of l_2 then T is narrow (for the old definition this problem was posed in [27, Question 3, p.64], and for the new one in [17, Question 5]).*

REFERENCES

1. Abramovich Yu. *New classes of spaces on which compact operators satisfy the Daugavet equation* // J. Operator Theory. – 1991. – V.25. – P.331–345.
2. Alspach D. E. *Small into isomorphisms on L_p spaces* // Ill. J. Math. – 1983. – V.27, No 2. – P.300–314.
3. Benyamini Y., Lin P.-K. *An operator on L^p without best compact approximation* // Israel J. Math. – 1985. – V.51, No 4. – P.298–304.
4. Даугавет И. К. *Об одном свойстве вполне непрерывных операторов в пространстве C* // Успехи мат. наук. – 1963. – Т.18, №5. – С.157–158.
5. Dor L. E. *On projections in L^1* // Ann. Math. – 1975. – V.102, No 3. – P.463–474.
6. Franchetti C. *Lower bounds for the norms of projections with small kernels* // Bull. Austral. Math. Soc. – 1992. – V.45. – P.507–511.
7. Franchetti C., Semenov E. M. *A Hilbert space characterization among function spaces* // Anal. Math. – 1995. – V.21. – P.85–93.
8. Godefroy G., Kalton N. J., Li D. *Operators between subspaces and quotients of L^1* // Indiana Univ. Math. J. – 2000. – V.9, No 1. – P.245–286.
9. Holub J. R. *A property of weakly compact operators on $C([0,1])$* // Proc. Amer. Math. Soc. – 1986. – V.97. – P.396–398.
10. Holub J. R. *Daugavet's equation and operators on $L^1(\mu)$* // Proc. Amer. Math. Soc. – 1987. – V.100, No 2. – P.295–300.
11. Kadets V. M. *Some remarks concerning the Daugavet equation* // Quaestiones Math. – 1996. – V.19. – P.225–235.
12. Kadets V. M., Popov M. M. *On the Liapunov convexity theorem with applications to sign-embeddings* // Укр. мат. ж. – 1992. – V.44, No 9. – P.1192–1200.
13. Кадец В. М., Попов М. М. *Свойство Даугавета для узких операторов в богатых подпространствах пространств $C[0,1]$ и $L_1[0,1]$* // Алгебра и анализ. – 1996. – Т.8, вып.4. – С.43–62.
14. Kadets V. M., Popov M. M. *Some stability theorems about narrow operators on L_1 and $C(K)$* // Mat. fiz., anal., geom. – 2003. – V.10, No 1. – P.49–60.
15. Kadets V. M., Shvidkoy R. V. *The Daugavet property for pairs of Banach spaces* // Mat. fiz., anal., geom. – 1999. – V.6, No 3/4. – P.253–263.
16. Kadets V. M., Shvidkoy R. V., Sirotkin G. G., Werner D. *Banach spaces with the Daugavet property* // Trans. Amer. Math. Soc. – 2000. – V.352. No 2. – P.855–873.
17. Kadets V. M., Shvidkoy R. V., Werner D. *Narrow operators and rich subspaces of Banach spaces with the Daugavet property* // Stud. Math. – 2001. – V.147, No 3. – P.269–298.
18. Kalton N. J. *The endomorphisms of L_p $(0 \le p \le 1)$* // Indiana Univ. Math. J. – 1978. – V.27, No 3. – P.353–381.
19. Kalton N. J., Randrianantoanina B. *Surjective isometries of rearrangement-invariant spaces* // Quart. J. Math. Oxford. – 1994. – V.45. – P.301–327.
20. Kalton N. J., Peck N. T., Roberts J. W. An F-space sampler. – Cambridge etc.: Cambridge Univ. Press, 1984. – 154 pp.
21. Kwapień S. *On the form of a linear operator in the space of all measurable functions* // Bull. Acad. Polon. Sci. – 1973. – V.21. – P.951–954.
22. Lindenstrauss J., Tzafriri L. Classical Banach Spaces I. – Berlin etc.: Springer-Verlag, 1977. – 188 pp.
23. Lindenstrauss J., Tzafriri L. Classical Banach Spaces II. – Berlin etc.: Springer-Verlag, 1979. – 243 pp.

24. Лозановский Г. Я. *О почти интегральных операторах в KB-пространствах* // Вестн. Ленингр. ун-та. Сер. мат. мех. астроном. – 1966. – Т.21, №7. – С.35–44.

25. Oikhberg T. *The Daugavet property of C^*-algebras and non-commutative L_p-spaces* // Positivity. – 2002. – V.6. – P.59–73.

26. Oikhberg T. *Spaces of operators, the ψ-Daugavet property, and numerical indices* // Preprint.

27. Plichko A. M., Popov M. M. *Symmetric function spaces on atomless probability spaces* // Dissertationes Mathematicae. – 1990. – V.306. – P.1–85.

28. Попов М. М. *О нормах проекторов в $L_p(\mu)$ с "малыми" ядрами* // Функц. анализ и его прил. – 1987. – Т.21, вып. 2. – С.84–85.

29. Popov M. M., Randrianantoanina B. *A pseudo-Daugavet property for narrow projections in Lorentz spaces* // Ill. J. Math. – 2003. (to appear).

30. Randrianantoanina B. *Contractive projections in nonatomic function spaces* // Proc. Amer. Math. Soc. – 1995. – V.123. – P.1747–1750.

31. Randrianantoanina B. *A Daugavet type property in real rearrangement invariant function spaces* // Preprint.

32. Shvidkoy R. *Geometric aspects of the Daugavet property* // J.Funct. Anal. – 2000. – V.176, No 2. – P.198–212.

33. Shvidkoy R. *The largest linear space of operators satisfying the Daugavet equation in L_1* // Proc. Amer. Math. Soc. – 2001. – V.130, No 3. – P.773–777.

34. Shvidkoy R. Operators and integrals in Banach spaces. – Dissertation, Univ. of Missouri-Columbia, 2001, 125 pp.

35. Wojtaszczyk P. *Some remarks on the Daugavet equation* // Proc. Amer. Math. Soc. – 1992. – V.115. – P.1047–1052.

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