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DIFFERENTIAL-SYMBOL METHOD OF SOLVING THE NONLOCAL BOUNDARY VALUE PROBLEM IN THE CLASS OF NON-UNIQUENESS OF ITS SOLUTION

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We propose a method of solving the nonlocal boundary-value problem for nonhomogeneous PDE of the first order in time and, in general, of infinite order in spatial variables in the case when the solution uniqueness conditions are violated. We specify the way of constructing a particular solution of the problem in the class of non-uniqueness.

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Предложен метод решения нелокальной краевой задачи для неоднородного дифференциального уравнения первого порядка по времени и, вообще говоря, бесконечного порядка по пространственным переменным в случае невыполнения условий единственности решения. Указан метод построения решения задачи в классе неединственности.

The problems with nonlocal boundary conditions have been studied in many scientific investigations. This is caused by a great practical significance of these problems [1, 5] as well as theoretical investigations considering as wide as possible class of boundary conditions [3]. The problems with nonlocal boundary conditions in a selected variable for PDEs are mostly ill-posed problems [2, 6].

The present work is a continuation of our previous investigations and deals with constructing the differential-symbol method of solving this problem in the case when its solution uniqueness conditions are violated.

In the domain of variables $t \in (0,h)$, $x \in \mathbb{R}^s$, we shall examine the nonlocal boundary value problem

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right] U(t,x) = f(t,x), \tag{1}$$

$$U(0,x) + \mu U(h,x) = \varphi(x), \tag{2}$$

where $\mu \in \mathbb{R}$, $h \in \mathbb{R}$, $h > 0$, $s \in \mathbb{N}$, $f(t,x)$ and $\varphi(x)$ are given smooth functions with certain restrictions which will be specified below. Here $a\left(\frac{\partial}{\partial x}\right)$ is a differential expression, in general,

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of infinite order with constant coefficients, the symbol of this expression is an analytical in $\mathbb{R}^s$ function $a(\nu) \not\equiv 0$ whose univalent analytical continuation into $\mathbb{C}^s$ is an entire function.

Problem (1), (2) for $f(t,x)=0$, $\varphi(x)=0$, i.e.

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right] U(t,x) = 0, \tag{3}$$

$$U(0,x) + \mu U(h,x) = 0, \tag{4}$$

when $\mu = 0$, is a Cauchy problem and, in the case of existence of a solution of problem (3), (4), has only trivial solution $U(t,x) \equiv 0$. If $\mu \neq 0$ then problem (3), (4) may have also non-trivial solutions, e.g. if $\mu = -1$, $a\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x_1^2}$ then an arbitrary constant is a solution of problem (3), (4). The set of solutions of problem (3), (4), or, in other words, *the null space* of problem (1), (2), has been specified in [4].

Obviously, the solution of problem (1), (2) can be represented as a sum of an arbitrary solution of problem (3), (4), a particular solution of problem (3), (2) and the particular solution of problem (1), (4). The existence and uniqueness class of solution of problem (3), (2) as well as the formula for this solution have been also proposed in [4]. We shall investigate problem (1), (4) in the class of non-uniqueness of its solution.

1. One-dimensional case ($s = 1$)

Let $\eta(\nu) = 1 + \mu \exp[a(\nu)h]$, $P = \{\nu \in \mathbb{C} : \ \eta(\nu) = 0\}$. Also, we shall introduce the class $K_{M,L}$, for $M \subseteq \mathbb{C}$, $L \subseteq \mathbb{C}$, of quasipolynomials of the form

$$f(t,x) = \sum_{j=1}^{m} \exp[\beta_j t + \alpha_j x] Q_{n_j}(t,x), \tag{5}$$

where $m \in \mathbb{N}$, $\beta_j \in M$, $\alpha_j \in L$, $Q_{n_j}(t,x)$ is a polynomial of two variables t and x of total degree $n_j \in \mathbb{Z}_+$, for $j = \overline{1,m}$.

One can prove that the solution of problem (1), (4) can be found by the formula

$$U\left(t,x\right) = f\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\right) \left\{\Phi(\lambda,\nu,t) \exp\left[\nu x\right]\right\}\Big|_{\lambda=\nu=0}, \tag{6}$$

where

$$\Phi(\lambda,\nu,t) = \frac{\eta(\nu)\exp[\lambda t] - (1+\mu\exp[\lambda h])\exp[a(\nu)t]}{(\lambda - a(\nu))\eta(\nu)}, \tag{7}$$

moreover, it exists and is unique in the class $K_{\mathbb{C},\mathbb{C}\setminus P}$, if $f\left(t,x\right) \in K_{\mathbb{C},\mathbb{C}\setminus P}$. Suppose that $f(t,x)$ in equation (1) has the form

$$f(t,x) = \exp[\beta t + \alpha x] Q(t,x), \tag{8}$$

where $\beta \in \mathbb{C}$, $Q(t,x)$ is an arbitrary polynomial of variables t and x, $\alpha \in P$, p_α is a multiplicity of the zero α of the function $\eta(\nu)$. Formula (6), is obviously inapplicable for $f(t,x)$ of the form (8). So, we shall indicate the formula for a particular solution of problem (1), (4), which will now be determined up to solutions of problem (3), (4).

Lemma 1. *Let*

$$\begin{aligned} &R(\alpha,\nu,t,x)=\exp[a(\alpha)t+\alpha x]\sum_{k=0}^{p_\alpha-1}\frac{(\nu-\alpha)^k}{k!}x^k\,,\\ &F(\lambda,\nu,t)=\frac{\exp[\lambda t]-\exp[a(\nu)t]}{\lambda-a(\nu)}\,,\\ &Z(\lambda,\nu,t,x)=F(\lambda,\nu,t)\exp[\nu x]+\\ &+\mu F(\lambda,\nu,t-h)\exp[a(\nu)h+\lambda h+\nu x]+\mu F(\lambda,\nu,h)R(\alpha,\nu,t,x)\,. \end{aligned} \tag{9}$$

Then the function

$$G(\lambda,\nu,t,x)=\frac{Z(\lambda,\nu,t,x)}{\eta(\nu)} \tag{10}$$

is entire with respect to λ and analytic in the neighborhood of $\nu=\alpha$.

Proof. The fact that $G(\lambda,\nu,t,x)$ is an entire function in λ follows from the form of function (10) and from the fact that function (9) is entire in λ. Let us prove the second part of the lemma. We will show that the numerator of the function $G(\lambda,\nu,t,x)$ and its derivatives up to the order $(p_\alpha-1)$ inclusively with respect to the parameter ν vanish at $\nu=\alpha$. From the equalities $\eta^{(j)}(\alpha)=0$, $j=\overline{0,p_\alpha-1}$, one can easy obtain the following equalities: $\mu\exp[a(\alpha)h]=-1$, $a^{(j)}(\alpha)=0$, $j=\overline{1,p_\alpha-1}$, and from this it follows that

$$\left.\frac{\partial^j F(\lambda,\nu,t)}{\partial\nu^j}\right|_{\nu=\alpha}=0,\quad j=\overline{1,p_\alpha-1}.$$

Let us compute the values of the function $Z(\lambda,\nu,t,x)$ and its derivatives at $\nu=\alpha$ using the previous equalities. For $m=\overline{0,p_\alpha-1}$, we have:

$$\begin{aligned} &\left.\frac{\partial^m Z(\lambda,\nu,t,x)}{\partial\nu^m}\right|_{\nu=\alpha}=\exp[\alpha x]\sum_{j=0}^{m}C_m^j x^{m-j}\left.\frac{\partial^j F(\lambda,\nu,t)}{\partial\nu^j}\right|_{\nu=\alpha}-\\ &-\exp[\lambda h+\alpha x]\sum_{j=0}^{m}C_m^j x^{m-j}\left.\frac{\partial^j F(\lambda,\nu,t-h)}{\partial\nu^j}\right|_{\nu=\alpha}+\\ &+\mu\exp[a(\alpha)t+\alpha x]\sum_{j=0}^{m}C_m^j x^{m-j}\left.\frac{\partial^j F(\lambda,\nu,h)}{\partial\nu^j}\right|_{\nu=\alpha}=\\ &=\exp[\alpha x]\sum_{j=0}^{m}C_m^j x^{m-j}\left[\left.\frac{\partial^j F(\lambda,\nu,t)}{\partial\nu^j}\right|_{\nu=\alpha}-\exp[\lambda h]\left.\frac{\partial^j F(\lambda,\nu,t-h)}{\partial\nu^j}\right|_{\nu=\alpha}+\right.\\ &\left.+\mu\exp[a(\alpha)t]\left.\frac{\partial^j F(\lambda,\nu,h)}{\partial\nu^j}\right|_{\nu=\alpha}\right]=\\ &=\exp[\alpha x]\sum_{j=1}^{m}C_m^j x^{m-j}\left[\left.\frac{\partial^j F(\lambda,\nu,t)}{\partial\nu^j}\right|_{\nu=\alpha}-\exp[\lambda h]\left.\frac{\partial^j F(\lambda,\nu,t-h)}{\partial\nu^j}\right|_{\nu=\alpha}-\right.\\ &\left.-\exp[a(\alpha)(t-h)]\left.\frac{\partial^j F(\lambda,\nu,h)}{\partial\nu^j}\right|_{\nu=\alpha}\right]+\\ &+\exp[\alpha x]x^m\big[F(\lambda,\nu,t)-\exp[\lambda h]F(\lambda,\nu,t-h)-\exp[a(\alpha)(t-h)]F(\lambda,\nu,h)\big]=0\,. \end{aligned}$$

□

Theorem 2. *Let $f(t,x)$ be a function of the form (8), where $\beta \in \mathbb{C}$, $Q(t,x)$ is a polynomial of variables t and x, $\alpha \in P$, p_α is the multiplicity of the zero α of $\eta(\nu)$. Then a particular solution of problem (1), (4) can be computed with the formula*

$$U(t,x) = f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\Big\{G(\lambda,\nu,t,x)\Big\}\Big|_{\lambda=\nu=0},$$

i.e.

$$U(t,x) = Q\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\Big\{G(\lambda,\nu,t,x)\Big\}\Big|_{\substack{\lambda=\beta\\ \nu=\alpha}}, \tag{11}$$

where $G(\lambda,\nu,t,x)$ is function (10).

Proof. Let us show that $G(\lambda,\nu,t,x)$ is a solution of the equation

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right] G(\lambda,\nu,t,x) = \exp[\lambda t + \nu x] \tag{12}$$

and satisfies the condition

$$G(\lambda,\nu,0,x) + \mu G(\lambda,\nu,h,x) = 0\,. \tag{13}$$

In fact, equalities (12) and (13) follow from the equalities given below:

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]\Big\{F(\lambda,\nu,t)\exp[\nu x]\Big\} = \exp[\lambda t + \nu x]\,,$$
$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]\Big\{F(\lambda,\nu,t-h)\exp[\nu x]\Big\} = \exp[\lambda(t-h) + \nu x]\,,$$
$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]\Big\{R(\alpha,\nu,t,x)\Big\} = 0\,, \quad F(\lambda,\nu,0) = 0\,,$$
$$F(\lambda,\nu,-h)\exp[a(\nu)h + \lambda h] = -F(\lambda,\nu,h)\,, \quad 1 + \mu\exp[a(\alpha)h] = 0\,.$$

Now we shall prove that function (11) determines a solution of equation (1). Using Lemma 1, we have

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right] f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\Big\{G(\lambda,\nu,t,x)\Big\}\Big|_{\substack{\lambda=0\\ \nu=0}} =$$
$$= \left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right] Q\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\Big\{G(\lambda,\nu,t,x)\Big\}\Big|_{\substack{\lambda=\beta\\ \nu=\alpha}}$$
$$= Q\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right] G(\lambda,\nu,t,x)\right\}\Big|_{\substack{\lambda=\beta\\ \nu=\alpha}} =$$
$$= Q\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\Big\{\exp[\lambda t + \nu x]\Big\}\Big|_{\substack{\lambda=\beta\\ \nu=\alpha}} = Q(t,x)\exp[\alpha t + \beta x] = f(t,x)\,.$$

The realization of condition (4) follows from Lemma 1 and from equality (13). □

Example 3. In the domain $t \in (0,1)$, $x \in \mathbb{R}$, find the solution of the problem

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] U(t,x) = t\,, \quad U(0,x) - U(1,x) = 0.$$

The function $f(t,x)$ can be represented in the form (8), where $\beta = 0$, $\alpha = 0$, $Q(t,x) = t$. In the given case, $a(\nu) = \nu^2$, $\mu = -1$, $\eta(\nu) = 1 - \exp[\nu^2]$, $0 \in P$, moreover $p_0 = 2$.

Using formula (11), we find

$$U(t,x) = \frac{\partial}{\partial \lambda}\Big\{G(\lambda,\nu,t,x)\Big\}\Big|_{\substack{\lambda=0\\ \nu=0}} = \lim_{\substack{\lambda\to 0\\ \nu\to 0}} \frac{\partial G(\lambda,\nu,t,x)}{\partial \lambda},$$

where

$$G(\lambda,\nu,t,x) = \frac{1}{1-\exp[\nu^2]}\Big[\frac{\exp[\lambda t] - \exp[\nu^2 t]}{\lambda - \nu^2}\exp[\nu x] -$$
$$-\frac{\exp[\lambda t] - \exp[\nu^2 (t-1)]}{\lambda - \nu^2}\exp[\nu^2 + \lambda + \nu x] - \frac{\exp[\lambda] - \exp[\nu^2]}{\lambda - \nu^2}(1+\nu x)\Big]\,.$$

By Lemma 1 this limit exists, so it could be computed in such a way:

$$\lim_{\substack{\lambda\to 0\\ \nu\to 0}} \frac{\partial G(\lambda,\nu,t,x)}{\partial \lambda} = \lim_{\nu\to 0} G'_\lambda(0,\nu,t,x)\,.$$

Having computed the last limit using the L'Hospital rule, we obtain

$$U(t,x) = -\frac{x^2}{4} + \frac{t^2}{2} - \frac{t}{2}. \tag{14}$$

Remark 4. From Example 3, one can see that finding a particular solution of problem (3) is connected with an evaluation of indeterminate form of the type $\frac{0}{0}$. Formula (11) is obtained as a result of "improving" formula (6) with the element of the null space, namely $R(\alpha,\nu,t,x)$. It turns out that a particular solution of the problem could be found without using elements of the null space, just formally applying the L'Hospital rule. E.g., a particular solution of problem (3) can be found as follows:

$$U(t,x) = \frac{\partial}{\partial \lambda}\left\{\frac{(F(\lambda,\nu,t) - F(\lambda,\nu,t-1)\exp[\nu^2+\lambda])\exp[\nu x]}{1-\exp[\nu^2]}\right\}\Big|_{\substack{\lambda=0\\ \nu=0}} =$$
$$= -\frac{x^2}{4} + \frac{t^2}{2} - \frac{t}{2} - \frac{1}{6}\,.$$

As we can see, the obtained solution of problem (3) differs from solution (14) by the constant which is an element of the null space.

Such a way of finding a solution is equivalent to the formal applying the L'Hospital rule in formula (6). We shall prove that when formally applying the L'Hospital rule in formula (6), where $\Phi(\lambda,\nu,t)$ is defined by formula (7), a particular solution of problem (1), (4) is obtained.

Let $f(t,x)$ have the form (8), moreover, the degree of the polynomial $Q(t,x)$ in the variable x is equal to n. We shall denote

$$\sigma(\lambda,\nu,t,x) = \eta(\nu)\Phi(\lambda,\nu,t)\exp[\nu x]. \tag{15}$$

Then formula (6) will have the form

$$U(t,x) = Q\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{\frac{\sigma(\lambda,\nu,t,x)}{\eta(\nu)}\right\}\Bigg|_{\substack{\lambda=\beta\\ \nu=\alpha}}.$$

It is easy to see that the expression obtained as a result of action of the differential polynomial $Q\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)$ onto the expression in braces, can be represented in the form

$$Q\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{\frac{\sigma(\lambda,\nu,t,x)}{\eta(\nu)}\right\} = \frac{\rho(\lambda,\nu,t,x)}{[\eta(\nu)]^{n+1}}. \tag{16}$$

Also, one can easy make sure that α is a zero of the function $[\eta(\nu)]^{n+1}$ of the multiplicity $(n+1)p_\alpha$.

Theorem 5. *Let $f(t,x)$ have the form (8), where $\beta\in\mathbb{C}$, $\alpha\in P$, p_α is a multiplicity of the zero α of the function $\eta(\nu)$, $Q(t,x)$ a polynomial of degree n in x. Then, provided that expression (16) holds, a particular solution of problem (1), (4) can be computed with the formula*

$$U(t,x) = \frac{\left.\dfrac{\partial^{(n+1)p_\alpha}}{\partial\nu^{(n+1)p_\alpha}}\rho(\beta,\nu,t,x)\right|_{\nu=\alpha}}{\left.\dfrac{d^{(n+1)p_\alpha}}{d\nu^{(n+1)p_\alpha}}[\eta(\nu)]^{n+1}\right|_{\nu=\alpha}}. \tag{17}$$

Proof. It follows from formulas (9), (15) and (16) that the function $\rho(\lambda,\nu,t,x)$ is entire in λ and ν. Let us show that function (17) is a solution of equation (1). It is easy to see that

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]\sigma(\lambda,\nu,t,x) = \exp[\lambda t + \nu x]\eta(\nu). \tag{18}$$

Having substituted function (17) into equation (1) and taking into account equalities (18) and (16), we obtain

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]U(t,x) = \frac{\left.\dfrac{\partial^{(n+1)p_\alpha}}{\partial\nu^{(n+1)p_\alpha}}\left\{[\eta(\nu)]^{n+1}Q(t,x)\exp[\beta t+\nu x]\right\}\right|_{\nu=\alpha}}{\left.\dfrac{d^{(n+1)p_\alpha}}{d\nu^{(n+1)p_\alpha}}[\eta(\nu)]^{n+1}\right|_{\nu=\alpha}} =$$

$$= Q(t,x)\exp[\beta t]\frac{\left.\dfrac{\partial^{(n+1)p_\alpha}}{\partial\nu^{(n+1)p_\alpha}}\left\{[\eta(\nu)]^{n+1}\exp[\nu x]\right\}\right|_{\nu=\alpha}}{\left.\dfrac{d^{(n+1)p_\alpha}}{d\nu^{(n+1)p_\alpha}}[\eta(\nu)]^{n+1}\right|_{\nu=\alpha}} =$$

$$= Q(t,x)\exp[\beta t]\frac{\displaystyle\sum_{k=0}^{(n+1)p_\alpha} C^k_{(n+1)p_\alpha}\left.\frac{d^k}{d\nu^k}[\eta(\nu)]^{n+1}\right|_{\nu=\alpha} x^{(n+1)p_\alpha-k}\exp[\alpha x]}{\left.\dfrac{d^{(n+1)p_\alpha}}{d\nu^{(n+1)p_\alpha}}[\eta(\nu)]^{n+1}\right|_{\nu=\alpha}} =$$

$$= \exp[\beta t+\alpha x]Q(t,x).$$

Thus, function (16) is a solution of the nonhomogeneous equation (1). Besides, it satisfies the homogeneous nonlocal condition (4), which follows from the equality $\sigma(\lambda,\nu,0,x) + \mu\sigma(\lambda,\nu,h,x) = 0$. □

Remark 6. If $f(t,x)$ is an arbitrary function from the class $K_{\mathbb{C},P}$, i.e. it has the form (5), where $\beta_j \in \mathbb{C}$, $\alpha_j \in P$, p_{α_j} is a multiplicity of the zero α_j of the function $\eta(\nu)$, $Q_{n_j}(t,x)$ is a polynomial of variables t and x, for $j=\overline{1,m}$, then, using formula (11), one can find particular solutions $U_j(t,x)$ for $f_j(t,x)=\exp[\beta_j t+\alpha_j x]Q_{n_j}(t,x)$ and then, by the superposition principle of solution of linear differential equation's, we shall find $U(t,x)=\sum\limits_{j=1}^{m} U_j(t,x)$.

2. Multidimensional case ($s \geq 2$)

Let $j,n,r \in \mathbb{Z}_+^s$, $x=(x_1,x_2,\dots,x_s)\in\mathbb{R}^s$, $\nu=(\nu_1,\nu_2,\dots,\nu_s)\in\mathbb{C}^s$. Then

$$x^r=\prod_{i=1}^{s} x_i^{r_i}\,; \quad \nu\cdot x=\sum_{i=1}^{s}\nu_i x_i\,; \quad r!=\prod_{i=1}^{s} r_i!\,; \quad \mathbf{O}=(0,0,\dots,0)\,;$$

$$C_n^r=\frac{n!}{r!\,(n-r)!}\,; \quad |r|=\sum_{i=1}^{s} r_i\,; \quad \frac{\partial^r}{\partial\nu^r}=\frac{\partial^{|r|}}{\partial\nu_1^{r_1}\partial\nu_2^{r_2}\dots\partial\nu_s^{r_s}}\,;$$

$$P=\{\nu\in\mathbb{C}^s:\ \ \eta(\nu)\equiv 1+\mu\exp[a(\nu)h]=0\}\,. \tag{19}$$

For $\alpha\in P$, we shall introduce the following sets:

$$\Omega_1(\alpha)=\left\{\omega\in\mathbb{Z}_+^s:\ \left.\frac{\partial^\omega\eta}{\partial\nu^\omega}\right|_{\nu=\alpha}\neq 0\right\};$$
$$\Omega_2(\alpha)=\left\{\tilde{\omega}\in\mathbb{Z}_+^s:\ \ \tilde{\omega}=\omega+r,\ \omega\in\Omega_1(\alpha)\,,\ r\in\mathbb{Z}_+^s,\ r\neq\mathbf{O}\right\};$$
$$\Omega(\alpha)=\Omega_1(\alpha)\cup\Omega_2(\alpha)\ .$$

We shall also consider the set $K_{M,L}$, for $M\subseteq\mathbb{C}$, $L\subseteq\mathbb{C}^s$, of quasipolynomials of several variables

$$f(t,x)=\sum_{j=1}^{m}\exp\left[\beta_j t+\alpha_j\cdot x\right]Q_{n_j}(t,x),$$

where $m\in\mathbb{N}$, $\beta_j\in M$, $\alpha_j\in L$, $Q_{n_j}(t,x)$ are polynomials of variables $t,x_1,x_2,\dots,x_s$ of total degree $n_j\in\mathbb{Z}_+$, $j=\overline{1,m}$.

The formal solution of problem (1), (4) is

$$U(t,x)=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left\{\Phi(\lambda,\nu,t)\exp\left[\nu\cdot x\right]\right\}\Big|_{\substack{\lambda=0\\ \nu=\mathbf{O}}}\,, \tag{20}$$

where $\Phi(\lambda,\nu,t)$ is function (7), $\nu=(\nu_1,\nu_2,\dots,\nu_s)$. We shall consider the case when $f(t,x)\in K_{\mathbb{C},P}$. Formula (20) in this case obviously becomes inapplicable. As in the case $s=1$, the solution of problem (1), (4) exists, but is not unique. We shall indicate the formulas for constructing a particular solution of problem (1), (4) in the case when $f(t,x)\in K_{\mathbb{C},P}$. Similarly as in the case $s=1$, we shall write down the function

$$\sigma(t,x,\lambda,\nu)=\eta(\nu)\Phi(\lambda,\nu,t)\exp[\nu\cdot x]. \tag{21}$$

Obviously, condition (16) holds, where n is the degree of the polynomial $Q(t,x)$ in the set of variables $x_1,x_2,\dots,x_s$.

Theorem 7. *Let $f(t,x)$ be a quasipolynomial*

$$f(t,x)=\exp[\beta t+\alpha\cdot x]Q(t,x),$$

where $\beta\in\mathbb{C}$, $\alpha\in P$, $p_\alpha=\min\limits_{r\in\Omega(\alpha)}|r|$ and $r_0\in\Omega(\alpha)$ is one of the vectors realizing the minimum, i.e. $|r_0|=p_\alpha$, n is the degree of the polynomial $Q(t,x)$ in the set of variables $x_1,x_2,\dots,x_s$. Then, provided that condition (16) holds, a particular solution of problem (1), (4) can be found by the formula

$$U(t,x)=\frac{\left.\dfrac{\partial^{(n+1)r_0}}{\partial\nu^{r_0}}\rho(\beta,\nu,t,x)\right|_{\nu=\alpha}}{\left.\dfrac{d^{(n+1)r_0}}{d\nu^{r_0}}[\eta(\nu)]^{n+1}\right|_{\nu=\alpha}}. \tag{22}$$

Proof. The proof is similar to that of Theorem 5. □

Example 8. In the domain $t\in(0,1)$, $(x,y)\in\mathbb{R}^2$, find a solution of the problem

$$\left[\frac{\partial}{\partial t}-\frac{\partial^4}{\partial x\partial y^3}\right]U(t,x,y)=\exp[t]xy\,,\quad U(0,x,y)-U(1,x,y)=0\,.$$

For the given problem, we have $h=1$, $\mu=-1$, $a(\nu)=\nu_1\nu_2^3$, $P=\{\nu\in\mathbb{C}^2:\nu_1\nu_2^3=2\pi ki,k\in\mathbb{Z}\}$. The set P contains, in particular, the vector $\nu=(0,0)$. For this vector, we can construct the set $\Omega(0,0)=\{(1+k,3+m),\ k,m\in\mathbb{Z}_+\}$. Having applied formula (22), where $r_0=(1,3)$, we obtain the solution of the problem:

$$U(t,x,y)=(1-e)\left(yxt+\frac{1}{48}y^4x^2+\frac{1}{2}yx\right)+\exp[t]yx.$$

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