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NORMAL BALL STRUCTURES

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A ball structure is a triple $\mathbb{B} = (X, P, B)$, where X, P are nonempty sets and, for all $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of $X, x \in B(x, \alpha)$, which is called the ball of radius α around x. We introduce the class of normal ball structures as an asymptotic counterpart of normal topological spaces. The part of continuous functions in this situation is played by the slowly oscillation functions. We describe the ball analogues of pseudocompactness and discreteness and define a corona of a normal ball structure which is a generalization of the Higson corona of a proper metric space.

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Шаровая структура — это тройка $\mathbb{B} = (X, P, B)$, где X, P — непустые множества и для всех $x \in X$, $\alpha \in P$, $B(x, \alpha)$ — подмножество из $X, x \in B(x, \alpha)$, которое называется шаром радиуса α с центром в x. Класс нормальных шаровых структур определяется как асимптотический двойник класса нормальных топологических пространств. Роль непрерывных функций в данной ситуации играют медленно осциллирующие функции. Вводятся шаровые аналоги псевдокомпактности и дискретности, а также определяется корона нормальной шаровой структуры как обобщение короны Хигсона собственного метрического пространства.

Following [4,5], by a *ball structure* we mean a triple $\mathbb{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called the *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathbb{B}, P$ is called the *set of radiuses*.

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be ball structures, $f\colon X_1 \longrightarrow X_2$. We say that f is a $\succ$*-mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that

$$B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$$

for every $x \in X_1$. If there exists a surjective $\succ$-mapping $f\colon X_1 \longrightarrow X_2$, we write $\mathbb{B}_1 \succ \mathbb{B}_2$.

A mapping $f\colon X_1 \longrightarrow X_2$ is called a $\prec$*-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X$. If there exists an injective $\prec$-mapping $f\colon X_1 \longrightarrow X_2$, we write $\mathbb{B}_1 \prec \mathbb{B}_2$.

A bijection $f\colon X_1 \longrightarrow X_2$ is called an *isomorphism* between $\mathbb{B}_1$ and $\mathbb{B}_2$ if f is a $\succ$-mapping and f is a $\prec$-mapping.

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Let $\mathbb{B}_1 = (X, P_1, B_1)$, $\mathbb{B}_2 = (X, P_2, B_2)$ be ball structures with common support X. We say that $\mathbb{B}_1 \subseteq \mathbb{B}_2$ if the identity mapping id: $X \longrightarrow X$ is a $\prec$-mapping of $\mathbb{B}_1$ to $\mathbb{B}_2$. If $\mathbb{B}_1 \subseteq \mathbb{B}_2$ and $\mathbb{B}_2 \subseteq \mathbb{B}_1$, we write $\mathbb{B}_1 = \mathbb{B}_2$.

A property $\mathcal{P}$ of ball structures is called a *ball property* if a ball structure $\mathbb{B}$ has a property $\mathcal{P}$ provided that $\mathbb{B}$ is isomorphic to some ball structure with property $\mathcal{P}$.

Let $\mathbb{B} = (X, P, B)$ be a ball structure. For any $x \in X$, $\alpha \in P$ put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}.$$

A ball structure $\mathbb{B}^* = (X, P, B)$ is called *dual* to $\mathbb{B}$. Note that $\mathbb{B}^{**} = \mathbb{B}$.

A ball structure $\mathbb{B}$ is called *symmetric* if $\mathbb{B} = \mathbb{B}^*$.

A ball structure $\mathbb{B} = (X, P, B)$ is called *multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma(\alpha, \beta) \in P$ such that

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$$

for every $x \in X$. Here

$$B(Y, \alpha) = \bigcup_{y \in Y} B(y, \alpha),$$

for any $Y \subseteq X$, $\alpha \in P$.

A ball structure $\mathbb{B}$ is called *uniform* if $\mathbb{B}$ is symmetric and multiplicative. By [4], symmetricity and multiplicativity are ball properties so uniformity is a ball property. Formally, the notion of a uniform ball structure is an asymptotic duplicate of the notion of a uniform topological space. It is well known [1] that every uniform topological space can be approximated by metrizable spaces. Now we describe a ball analogue of such an approximation.

Let (X, d) be a metric space, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Given any $x \in X$, $r \in \mathbb{R}^+$, put

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

The ball structure $(X, \mathbb{R}^+, B_d)$ is denoted by $\mathbb{B}(X, d)$. We say that a ball structure $\mathbb{B}$ is *metrizable* if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, d)$ for some metric space (X, d). Clearly, every metrizable ball structure is uniform. To formulate a ball characterization of metrizability we need some more definitions.

Let $\mathbb{B} = (X, P, B)$ be a ball structure, $x, y \in X$, We say that x, y are *connected* if there exists $\alpha \in P$ such that $x \in B(y, \alpha)$, $y \in B(x, \alpha)$. A subset $Y \subseteq X$ is called *connected* if any two elements from Y are connected. A ball structure $\mathbb{B}$ is called *connected* if its support is connected. If a ball structure is uniform, then connectedness is an equivalence relation on its support. Hence, the support of every uniform ball structure disintegrates to connected components. Note also that connectedness is a ball property [4].

For an arbitrary ball structure $\mathbb{B} = (X, P, B)$ we define a preodering $\leq$ on the set P by the rule

$$\alpha \leq \beta \text{ if and only if } B(x, \alpha) \subseteq B(x, \beta)$$

for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leq \beta$. A *cofinality* $\mathrm{cf}\mathbb{B}$ of $\mathbb{B}$ is the minimal cardinality of cofinal subsets of P. Note that cofinality is a ball property [4].

By [4], for every ball structure $\mathbb{B}$ the following statements are equivalent

(i) $\mathbb{B}$ is metrizable;

(ii) $\mathbb{B} = \mathbb{B}(X, d)$ for some metric space (X, d);

(iii) $\mathbb{B}$ is uniform, connected and $\mathrm{cf}\mathbb{B} \leq \aleph_0$.

To approximate the uniform ball structures by means of metrizable ball structures we use the following two constructions.

Let $\{\mathbb{B}_\lambda = (X_\lambda, P, B_\lambda) : \alpha \in I\}$ be a family of ball structures with pairwise disjoint supports and common set of radiuses, $X = \bigcup_{\lambda \in I} X_\lambda$. For every $x \in X, x \in X_\lambda$ and every $\alpha \in P$, put $B(x, \alpha) = B_\lambda(x, \alpha)$. The ball structure $\mathbb{B} = (X, P, B)$ is called a *disjoint union* of the family $\{\mathbb{B}_\lambda : \lambda \in I\}$. Clearly, every uniform ball structure is a disjoint union of its connected components. A ball structure is called *pseudometrizable* if it is a disjoint union of metrizable ball structures.

Let $\{\mathbb{B}_\lambda = (X, P_\lambda, B_\lambda) : \lambda \in I\}$ be a family of ball structures with common support. Suppose that, for any $\lambda_1, \lambda_2 \in I$, there exists $\lambda \in I$ such that $\mathbb{B}_{\lambda_1} \subseteq \mathbb{B}_\lambda$, $\mathbb{B}_{\lambda_2} \subseteq \mathbb{B}_\lambda$. For every $\lambda \in I$, choose a copy $P'_\lambda = f_\lambda(P_\lambda)$ such that the family $\{P'_\lambda : \lambda \in I\}$ is disjoint. Put $P = \bigcup_{\lambda \in I} P'_\lambda$. For any $x \in X$, $\beta \in P$, $\beta \in P'_\lambda$, put $B(x, \beta) = B_\lambda(x, f_\lambda^{-1}(\beta))$. The ball structure $\mathbb{B} = (X, P, B)$ is called the *inductive limit* of the family $\{\mathbb{B}_\lambda : \lambda \in I\}$. Clearly, $\mathbb{B}_\lambda \subseteq \mathbb{B}$ for every $\lambda \in I$. If every $\mathbb{B}_\lambda$ is uniform, then $\mathbb{B}$ is uniform.

Using metrizability criterion, it is easy to show that every uniform ball structure is the inductive limit of some family of pseudometrizable ball structures.

Let $\mathbb{B} = (X, P, B)$ be ball structures. A subset $Y \subseteq X$ is called *bounded* if there exist $x \in X$, $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$. We say that $\mathbb{B}$ is bounded if its support is bounded. Let $\mathbb{B}$ be a connected uniform ball structure, $x_0 \in X$, $Y \subseteq X$. Then Y is bounded if and only if there exists $\alpha \in P$ such that $Y \subseteq B(x_0, \alpha)$. It follows that the union of any finite family of bounded subset of connected uniform ball structure is bounded.

Let $\mathbb{B} = (X, P, B)$ be a ball structure. We say that subsets Y, Z of X are *asymptotically disjoint* (and write $Y \perp Z$) if, for every $\alpha \in P$, there exists a bounded subset $U_\alpha \subseteq X$ such that

$$B(Y \setminus U_\alpha, \alpha) \cap B(Z \setminus U_\alpha, \alpha) = \varnothing.$$

We say that subsets Y, Z of X are *asymptotically separated* (and write $Y \amalg Z$) if , for every $\alpha \in P$, there exists a bounded subset $U_\alpha \subseteq X$ such that

$$B(Y \setminus U_\alpha, \alpha) \cap B(Z \setminus U_\beta, \beta) = \varnothing$$

for all $\alpha, \beta \in P$.

A uniform ball structure is called *normal* if $Y \perp Z$ implies $Y \amalg Z$ for all subsets Y, Z of X.

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be isomorphic ball structures, $f \colon X_1 \longrightarrow X_2$ be an isomorphism, $Y \subseteq X_1$, $Z \subseteq X_1$. It is a routine verification that $Y \perp Z$ implies $f(Y) \perp f(Z)$ and $Y \amalg Z$ implies $f(Y) \amalg f(Z)$. Hence, normality is a ball property.

The part of continuous function on the stage of ball structures is played by the slowly oscillating functions.

Let $\mathbb{B} = (X, P, B)$ be a ball structure. A function $f \colon X \longrightarrow \mathbb{R}$ is called *slowly oscillating* if, for any $\varepsilon > 0$, $\alpha \in P$, there exists a bounded subset $U \subseteq X$ such that

$$\operatorname{diam} f(B(x, \alpha)) \leq \varepsilon$$

for every $x \in X \setminus U$, where $\operatorname{diam} A = \sup_{a,b \in A} |a - b|$.

In §1 we introduce some concrete classes of normal ball structures and give examples of uniform ball structures which are not normal. In §2 we prove the counterparts of Urysohn lemma and Tietze-Urysohn theorem for normal ball structures. In §3 we define the ball analogues of pseudocompactness and discreteness. In §4 we describe a corona of a uniform ball structure and show that in the case of normal ball structure this construction is a generalization of the Higson corona (see the survey [2]) of a proper metric space.

§1 Examples

A connected uniform ball structure $\mathbb{B} = (X, P, B)$ is called *ordinal* if there exists a cofinal well-ordered by $\leq$ subset of P. Clearly, every metrizable ball structure is ordinal.

Proposition 1.1. *Every ordinal ball structure is normal.*

Proof. We may suppose that P is well-ordered, $\mathrm{cf}\, P = |P|$ and $B(x, \alpha) = B^*(x, \alpha)$ for all $x \in X, \alpha \in P$. Let $Y \subseteq X$, $Z \subseteq X$ and $Y \perp Z$. Take the minimal element $\alpha_0 \in P$ and choose a bounded subset U_{α_0} such that

$$B(Y \backslash U_{\alpha_0}, \alpha_0) \cap B(Z \backslash U_{\alpha_0}, \alpha_0).$$

Suppose that for some $\alpha \in P$ we have constructed a family $\{U_\beta : \beta < \alpha\}$ of bounded subsets such that

$$U_\gamma \subseteq U_\beta, \quad B(Y \backslash U_\gamma, \gamma) \cap B(Z \backslash U_\beta, \beta) = \varnothing$$

for all $\gamma \leq \beta < \alpha$. Since $\mathbb{B}$ is connected and $\mathrm{cf}\, B = |B|$, the subset $U = \bigcup_{\beta<\alpha} U_\beta$ is bounded. Choose a bounded subset U_α such that

$$B(B(U, \alpha), \alpha) \subseteq U_\alpha, \quad B(Y \backslash U_\alpha, \alpha) \cap B(Z \backslash U_\alpha, \alpha) = \varnothing.$$

Then $B(Y \backslash U_\beta, \beta) \cap B(Z \backslash U_\alpha, \alpha) = \varnothing$, $B(Y \backslash U_\alpha, \alpha) \bigcap B(Z \backslash U_\beta, \beta) = \varnothing$ for all $\beta \leq \alpha$. Hence, $Y \amalg Z$ and $\mathbb{B}$ is normal. $\square$

Let X be a set and let $\mathcal{P}$ be a family of partitions of X. For any $x, y \in X$ and $P \in \mathcal{P}$, denote by $B(x, P)$ the set $\{y \in X : x, y$ are in the same cell of $P\}$. The ball structure $(X, \mathcal{P}, B)$ is denoted by $\mathbb{B}(x, \mathcal{P})$. Clearly, $\mathbb{B}(x, \mathcal{P})$ is symmetric. Given any $P_1, P_2 \in \mathcal{P}$, we say that P_2 is an enlargement of P_1 if $B(x, P_1) \subseteq B(x, P_2)$ for every $x \in X$. A ball structure $\mathbb{B}(X, \mathcal{P})$ is multiplicative if and only if, for any $P_1, P_2 \in \mathcal{P}$, there exists $P \in \mathcal{P}$ such that P is an enlargement of P_1 and P_2.

A ball structure $\mathbb{B}$ is called *cellular* if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, \mathcal{P})$ for some set X and some family $\mathcal{P}$ of partitions of X. Given any ball structure $\mathbb{B} = (X, P, B)$, $x, y \in X$ and $\alpha \in P$, we say that x, y are *α-path connected* if there exists a sequence $x_0, x_1, ..., x_n$, $x_0 = x$, $x_n = y$ such that $x_{i+1} \in B(x_i, \alpha)$, $x_i \in B(x_{i+1}, \alpha)$ for every $i \in \{0, 1, ..., n-1\}$. For any $x \in X$, $\alpha \in P$, put

$$B^{\square}(x, \alpha) = \{y \in X : x, y \text{ are } \alpha\text{-path connected}\}.$$

The ball structure $\mathbb{B}^{\square}(X, P, B^{\square})$ is called the *cellularization* of $\mathbb{B}$. By [4] a ball structure $\mathbb{B}$ is cellular if and only if $\mathbb{B} = \mathbb{B}^{\square}$. A metrizable ball structure $\mathbb{B}$ is cellular if and only if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, d)$ for some non-Archimedean metric space.

Example 1.1. Let X be a set and let φ be a filter on X. For any $x \in X$, $F \in \varphi$ put

$$B(x, F) = \begin{cases} X \backslash F, & \text{if } x \notin F; \\ \{x\}, & \text{if } x \in F; \end{cases}$$

Clearly, the ball structure (X, φ, B) is uniform and cellular. It is denoted by $\mathbb{B}(X, \varphi)$. Note that $\mathbb{B}(X, \varphi)$ is connected if and only if $\bigcap \varphi = \varnothing$. Let Y, Z be subsets of X. Observe that $Y \perp Z$ if and only if there exists $F \in \varphi$ such that $(Y \cap Z) \subseteq X \setminus F$. It follows that $\mathbb{B}(X, \varphi)$ is normal.

Let G be a group with the identity e and let $\mathcal{F}$ be a family of subsets of G such that $e \in F$ for every $F \in \mathcal{F}$. Given any $g \in G$, $F \in \mathcal{F}$, put

$$B_l(g, F) = Fg, \quad B_r(g, F) = gF.$$

The ball structures $(G, \mathcal{F}, B_l)$, $(G, \mathcal{F}, B_r)$ are denoted by $\mathbb{B}_l(G, \mathcal{F})$, $\mathbb{B}_r(G, \mathcal{F})$, respectively. A family $\mathcal{F}$ is called *symmetric* if, for every $F \in \mathcal{F}$, there exists $F' \in \mathcal{F}$ such that $F^{-1} \subseteq F'$. A family $\mathcal{F}$ is called *multiplicative* if $F_1F_2 \in \mathcal{F}$ for any $F_1, F_2 \in \mathcal{F}$. Clearly, $\mathbb{B}_l(G, \mathcal{F})$, $\mathbb{B}_r(G, \mathcal{F})$ are uniform for every symmetric multiplicative family $\mathcal{F}$. In this case the mapping $G \longrightarrow G$ defined by $g \longmapsto g^{-1}$ is an isomorphism between $\mathbb{B}_l(G, \mathcal{F})$ and $\mathbb{B}_r(G, \mathcal{F})$.

For every infinite group G and every infinite cardinal $\alpha \leq |G|$, denote by $\mathcal{F}_\alpha$ the family of all subsets of G of cardinality $< \alpha$, containing e. The connected uniform ball structures $\mathbb{B}_l(G, \mathcal{F}_\alpha)$, $\mathbb{B}_r(G, \mathcal{F}_\alpha)$ are denoted by $\mathbb{B}_l(G, \alpha)$, $\mathbb{B}_r(G, \alpha)$. If $\alpha > \aleph_0$, $\mathbb{B}_l(G, \alpha)$, $\mathbb{B}_r(G, \alpha)$ are cellular. The ball structures $\mathbb{B}_l(G, |G|)$, $\mathbb{B}_r(G, |G|)$ are ordinal. We shall write $\mathbb{B}_l(G)$, $\mathbb{B}_r(G)$ instead of $\mathbb{B}_l(G, \aleph_0)$, $\mathbb{B}_r(G, \aleph_0)$. Note that $\mathbb{B}_l(G)$=$\mathbb{B}_r(G)$ if and only if $\{x^{-1}gx : x \in G\}$ are finite for every $g \in G$. In this case we write $\mathbb{B}(G)$ instead of $\mathbb{B}_l(G)$ and $\mathbb{B}_r(G)$.

Example 1.2. Let $G = \bigoplus_{\alpha < \omega_1} G_\alpha$ be a direct sum of nonzero cyclic groups. Let g_α be a generator of G_α. We show that $\mathbb{B}(G)$ is not normal. Put $x = \{g_\alpha : \alpha < \omega_1\}$ and partition X onto two subsets Y, Z such that $|Y| = |Z| = |X|$. Take an arbitrary finite subset $F \subset G$ and choose $\alpha_1, \alpha_2, ..., \alpha_n$ such that $\mathrm{pr}_\alpha g = 0$ for all $g \in F$, $\alpha \notin \{\alpha_1, \alpha_2, ..., \alpha_n\}$. Then $y + F \cap z + F = \varnothing$ for all $y \in Y \setminus \{g_{\alpha_1}, g_{\alpha_2}, ..., g_{\alpha_n}\}$, $z \in Z \setminus \{g_{\alpha_1}, g_{\alpha_2}, ..., g_{\alpha_n}\}$. Hence, $Y \perp Z$. Assume that $Y \amalg Z$. For every $g \in G$, choose $Y_g \subseteq Y$, $Z_g \subseteq Z$ such that $Y \setminus Y_g$, $Z \setminus Z_g$ are finite and

$$g + Y_g \cap g' + Z_{g'} = \varnothing$$

for all $g, g' \in G$. Put

$$Y(Z) = \bigcup_{z \in Z} (z + Y_z)$$

and note that $Y(Z) \cap (y + Z_y) = \varnothing$ for every $y \in Y$. Fix an arbitrary countable subset $\{z_n : n \in \omega\}$ of Z and put $Y' = \bigcap_{n \in \omega} Y_{z_n}$. Since Y is uncountable and $Y \setminus Y_{z_n}$ is finite for every $n \in \omega$, we have $Y' \neq \varnothing$ and $\{z_n : n \in \omega\} + Y' \subseteq Y(Z)$. Take an arbitrary element $y \in Y'$. Since Z_y contains all but finitely many elements of $\{z_n : n \in \omega\}$, there exists $m \in \omega$ such that $z_m \in Z_y$. Then $y + z_m \in Y(Z)$ and $z_m + y \in y + Z_y$. Hence, $Y(Z) \cap (y + Z_y) \neq \varnothing$, a contradiction.

Let G be an arbitrary uncountable Abelian group. It is well known that G has a subgroup which is a direct sum of uncountably many cyclic groups. In view of Example 1.2, in order to show that $\mathbb{B}(G)$ is not normal, it suffices to check that normality is inherited by substructures.

Let $\mathbb{B} = (X, P, B)$ be a ball structure, $Y \subseteq X$. For any $y \in Y$, $\alpha \in P$ put $B_Y(y, \alpha) = B(y, \alpha) \cap Y$. The ball structure $\mathbb{B}_Y = (Y, P, B_Y)$ is called a *substructure* of $\mathbb{B}$. If $\mathbb{B}$ is uniform, then $\mathbb{B}_Y$ is uniform. If $\mathbb{B}$ is connected, then $\mathbb{B}_Y$ is connected.

Proposition 1.2. *Every substructure of a normal ball structure* $\mathbb{B} = (X, P, B)$ *is normal.*

Proof. Let $Y \subseteq X$, $Y_0 \subseteq Y$, $Y_1 \subseteq Y$ and Y_0, Y_1 asymptotically disjoint in $\mathbb{B}_Y$. For every $\alpha \in P$, choose a bounded subset U_α of X such that

$$B_Y(Y_0 \setminus U_\alpha, \alpha) \cap B_Y(Y_1 \setminus U_\alpha, \alpha) = \varnothing.$$

Since $\mathbb{B}$ is a multiplicative, there exists a mapping $\gamma\colon P \times P \longrightarrow P$ such that

$$B(B(x,\alpha),\beta) \subseteq B(x, \gamma(\alpha,\beta))$$

for all $x \in X$, $\alpha, \beta \in P$. Since $\mathbb{B}$ is symmetric, we may suppose that $B(x,\alpha) = B^*(x,\alpha)$ for all $x \in X$, $\alpha \in P$. Then

$$B(Y_0 \backslash U_{\gamma(\alpha,\alpha)}, \alpha) \cap B(Y_1 \backslash U_{\gamma(\alpha,\alpha)}, \alpha) = \varnothing$$

for every $\alpha \in P$, so Y_0 and Y_1 are asymptotically disjoint in $\mathbb{B}$. Since $\mathbb{B}$ is normal, Y_0 and Y_1 are asymptotically separated in $\mathbb{B}$. Hence, Y_0, Y_1 are asymptotically separated in $\mathbb{B}_Y$ and $\mathbb{B}$ is normal. □

In §2 we show (Lemma 2.3) that a disjoint union of an arbitrary collection of normal ball structures is normal. Hence a uniform ball structure is normal if and only if every its connected component is normal.

§2 Slowly oscillating functions

Theorem 2.1. *Let $\mathbb{B} = (X, P, B)$ be a normal ball structure, Y_0, Y_1 be disjoint subsets of X such that $Y_0 \perp Y_1$. Then there exists a slowly oscillating function $f\colon X \longrightarrow \mathbb{R}$ such that $f|_{Y_0} \equiv \varnothing$, $f|_{Y_1} \equiv 1$.*

In order to prove this asymptotic counterpart of Urysohn lemma we need some auxiliary results.

Let $\mathbb{B}(X, P, B)$ be a ball structure, $Y \subseteq X$ and let $\{U_\alpha : \alpha \in P\}$ be a family of bounded subsets of X. The set

$$\hat{Y} = \bigcup_{\alpha \in P} B(Y \setminus U_\alpha, \alpha)$$

is called a *pyramid* with the core Y determined by the family $\{U_\alpha : \alpha \in P\}$.

Lemma 2.1. *Let $\mathbb{B} = (X, P, B)$ be a uniform ball structure, Y, Z be subsets of X. Then the following statements are equivalent*

(i) $X \perp Z$;

(ii) there exists a pyramid $\hat{Y}$ with the core Y such that $\hat{Y} \cap Z = \varnothing$.

Proof. *(i)* $\Longrightarrow$ *(ii)*. For every $\alpha \in P$, choose a bounded subset V_α such that $B(Y \setminus V_\alpha, \alpha) \cap B(Z \setminus V_\alpha, \alpha) = \varnothing$. Put $U_\alpha = B^*(V_\alpha, \alpha)$. Since $\mathbb{B}$ is uniform, U_α is bounded. Denote by $\hat{Y}$ the pyramid with the core Y determined by the family $\{U_\alpha : \alpha \in P\}$. Then $\hat{Y} \cap Z = \varnothing$.

(ii) $\Longrightarrow$ *(i)*. Since $\mathbb{B}$ is multiplicative, there exists a mapping $\gamma\colon P \times P \longrightarrow P$ such that $B(B(x,\alpha),\beta) \subseteq B(x, \gamma(\alpha,\beta))$ for every $x \in X$. Since $\mathbb{B}$ is symmetric, we may suppose that $B(x,\alpha) = B^*(x,\alpha)$ for all $x \in X$, $\alpha \in P$. Let a family $\{U_\alpha : \alpha \in P\}$ of bounded subsets determine $\hat{Y}$. Since $\hat{Y} \cap Z = \varnothing$, we have

$$B(Y \backslash U_{\gamma(\alpha,\alpha)}, \gamma(\alpha,\alpha)) \cap Z = \varnothing$$

for every $\alpha \in P$. It follows that

$$B(Y\backslash U_{\gamma(\alpha,\alpha)}, \alpha) \cap B(Z, \alpha) = \varnothing$$

for every $\alpha \in P$. Hence, $Y \perp Z$. □

Lemma 2.2. *Let $\mathbb{B} = (X, P, B)$ be a normal connected ball structure, Y, Z be subsets of X. Then the following statements are equivalent*

(i) $Y \perp Z$;

(ii) there exist pyramids $\hat{Y}$, $\hat{Z}$ with the cores Y, Z such that $\hat{Y} \cap \hat{Z} = \varnothing$ and $\hat{Y} \perp \hat{Z}$.

Proof. $(i) \Longrightarrow (ii)$ By normality of $\mathbb{B}$, $Y \amalg Z$. For every $\alpha \in P$, pick a bounded subset U_α such that

$$B(Y\backslash U_\alpha, \alpha) \cap B(Z\backslash U_\beta, \beta) = \varnothing$$

for all $\alpha, \beta \in P$. Denote by Y_1, Z_1 the pyramids with the cores Y, Z determined by the family $\{U_\alpha : \alpha \in P\}$. Clearly, $Y_1 \cap Z_1 = \varnothing$. By Lemma 2.1, $Y \perp (X\backslash Y_1)$, $Z \perp (X\backslash Z_1)$. By normality of $\mathbb{B}$, $Y \amalg (X\backslash Y_1)$, $Z \amalg (X\backslash Z_1)$. Repeating the arguments, we construct the pyramids $\hat{Y}$, $\hat{Z}$ with the cores Y, Z such that $\hat{Y} \perp (X\backslash Y_1)$, $\hat{Z} \perp (X\backslash Z_1)$. Since $\mathbb{B}$ is connected, the union of any two bounded subsets of X is bounded. It follows that $\hat{Y}$, $\hat{Z}$ can be chosen so that $\hat{Y} \subseteq Y_1$, $\hat{Z} \subseteq Z_1$. Then $\hat{Y} \cap \hat{Z} = \varnothing$ and $\hat{Y} \perp \hat{Z}$.

$(ii) \Longrightarrow (i)$. Let the pyramids $\hat{Y}, \hat{Z}$ be determined by the families $\{U_\alpha : \alpha \in P\}$, $\{V_\alpha : \alpha \in P\}$ of bounded subsets. Since $\hat{Y} \cap \hat{Z} = \varnothing$, we have

$$B(Y\backslash U_\alpha, \alpha) \cap B(Z\backslash V_\beta, \beta) = \varnothing$$

for all $\alpha, \beta \in P$. Put $W_\alpha = U_\alpha \cup V_\alpha$. Since $\mathbb{B}$ is connected, W_α is bounded and

$$B(Y\backslash W_\alpha, \alpha) \cap B(Z\backslash W_\alpha, \alpha) = \varnothing$$

for every $\alpha \in P$. Hence, $Y \perp Z$. □

Lemma 2.3. *Let $\mathbb{B}(X, P, B)$ be a disjoint union of the family $\{\mathbb{B}_\lambda = (X_\lambda, P, B_\lambda) : \lambda \in I\}$ of connected uniform ball structures and let Y, Z be subsets of X such that $Y \perp Z$. Then there exists $\lambda_0 \in I$ such that, for every $\lambda \in I$, $\lambda \neq \lambda_0$, $Y \cap X_\lambda \neq \varnothing$ implies $Z \cap Z_\lambda = \varnothing$.*

Proof. Suppose the contrary and choose $\lambda_0, \lambda_1 \in I$, $\lambda_0 \neq \lambda_1$ such that

$$Y \cap X_{\lambda_0} \neq \varnothing, \quad Z \cap X_{\lambda_0} \neq \varnothing, \quad Y \cap X_{\lambda_1} \neq \varnothing, \quad Z \cap X_{\lambda_1} \neq \varnothing.$$

Put $Y_0 = Y \cap X_{\lambda_0}$, $Z_0 = Z \cap X_{\lambda_0}$, $Y_1 = Y \cap X_{\lambda_1}$, $Z_1 = Z \cap Z_{\lambda_1}$. Take any elements $y_0 \in Y_0$, $z_0 \in Z_0$, $y_1 \in Y_1$, $z_1 \in Z_1$. Since $\mathbb{B}_{\lambda_0}$, $\mathbb{B}_{\lambda_1}$ are connected and uniform, there exists $\alpha \in P$ such that

$$z_0 \in B(y_0, \alpha), \quad z_1 \in B(y_1, \alpha).$$

Choose an arbitrary bounded subset U of X and note that U is contained in some connected component X_λ of X. Hence, at least one of the following two statements holds:

$$B(Y_0\backslash U, \alpha) \cap (Z_0\backslash U) \neq \varnothing, \quad B(Y_1\backslash U, \alpha) \cap (Z_1\backslash U) \neq \varnothing.$$

Thus, Y and Z are not asymptotically disjoint. □

Proof of Theorem 2.1. Since every uniform ball structure is a disjoint union of its connected components and each connected component of normal ball structure is normal, in view of Lemma 2.3 we may suppose that $\mathbb{B}$ is connected. Applying Lemma 2.2, choose the pyramids $\hat{Y}_0$, $\hat{Y}_1$ with the cores Y_0, Y_1 such that $\hat{Y}_0 \cap \hat{Y}_1 = \varnothing$, $\hat{Y}_0 \cap Y_1 = \varnothing$, $\hat{Y}_0 \perp \hat{Y}_1$. Put $Z(0) = Y_0$, $Z(\frac{1}{2}) = Y_0 \cup \hat{Y}_0$, $Z(1) = X \backslash Y_1$. Then

$$Z(0) \subseteq Z\Big(\frac{1}{2}\Big) \subseteq Z(1), \quad Z(0) \perp \Big(X \backslash Z\Big(\frac{1}{2}\Big)\Big), \quad Z\Big(\frac{1}{2}\Big) \perp (X \backslash Z(1)).$$

Assume that we have chosen the family $Z(0)$, $Z(\frac{1}{2^n})$, ..., $Z(\frac{2^n-1}{2^n})$, $Z(1)$ such that

$$Z\Big(\frac{i}{2^n}\Big) \subseteq Z\Big(\frac{i+1}{2^n}\Big), \quad Z(\frac{i}{2^n}) \perp (X \backslash Z(\frac{i+1}{2^n}))$$

for every $i \in \{0, 1, ..., 2^n - 1\}$.. Apply Lemma 2.2 to each pair $Z(\frac{i}{2^n})$, $X \backslash Z(\frac{i+1}{2^n})$ and choose a subset $Z(\frac{2i+1}{2^{n+1}})$ of X such that

$$Z\Big(\frac{i}{2^n}\Big) \subseteq Z\Big(\frac{2i+1}{2^{n+1}}\Big) \subseteq Z\Big(\frac{i+1}{2^n}\Big),$$

$$Z\Big(\frac{i}{2^n}\Big) \perp \Big(X \backslash Z\Big(\frac{2i+1}{2^{n+1}}\Big)\Big), \quad Z\Big(\frac{2i+1}{2^{n+1}}\Big) \perp \Big(X \backslash Z\Big(\frac{2i+1}{2^n}\Big)\Big).$$

Thus, for every natural number n and every $i \in \{0, 1, ..., 2^n\}$ we have defined the subset $Z(\frac{i}{2^n})$ of X.

If $x \notin Z(1)$, put $f(x) = 1$. If $x \in Z(1)$, put

$$f(x) = \inf\Big\{\frac{i}{2^n} : x \in Z(\frac{i}{2^n}), \quad n \in \mathbb{N}, \quad i \in \{0, 1, ..., 2^n - 1\}\Big\}.$$

Clearly, $f\,|_{Y_0} \equiv 0$, $f\,|_{Y_1} \equiv 1$. Show that f is slowly oscillating.

Fix any $\alpha \in P$, $\varepsilon > 0$ and choose a natural number n such that $\frac{3}{2^n} < \varepsilon$. Since $Z(\frac{i}{2^n}) \perp (X \backslash Z(\frac{i+1}{2^n}))$, $i \in \{0, 1, ..., 2^n - 1\}$, there exists a bounded subset U_i such that

$$B(Z(\frac{i}{2^n}) \backslash U_i, \alpha) \subseteq Z(\frac{i+1}{2^n}), \quad B^*(Z(\frac{i}{2^n}) \backslash U_i, \alpha) \subseteq Z(\frac{i+1}{2^n}).$$

Put $U = U_0 \cup U_1 \cup \cdots \cup U_{2^n-1}$. Since $\mathbb{B}$ is connected, U is bounded. Take an arbitrary element $x \in X \backslash U$. If $x \in Z(\frac{i+1}{2^n}) \backslash Z(\frac{i}{2^n})$, then $\frac{i}{2^n} \leq f(x) \leq \frac{i+1}{2^n}$. It follows that $\operatorname{diam} f(B(x, \alpha)) \leq \frac{3}{2^n}$, so f is slowly oscillating. □

Let $\mathbb{B} = (X, P, B)$ be a ball structure, $Y \subseteq X$. We say that a function $f\colon Y \longrightarrow \mathbb{R}$ is *slowly oscillating* if, for all $\alpha \in P$, there exists a bounded subset $U \subseteq X$ such that $\operatorname{diam} f(B(y, \alpha) \cap Y) < \varepsilon$ for every $y \in Y \setminus U$.

Theorem 2.2 *For every uniform ball structure $\mathbb{B} = (X, P, B)$, the following two statements are equivalent*

(i) $\mathbb{B}$ is normal;

(ii) for every subset $Y \subseteq X$ and every bounded slowly oscillating function $f\colon Y \longrightarrow \mathbb{R}$, there exists a bounded slowly oscillating function $g\colon X \longrightarrow \mathbb{R}$ such that $g|_Y = f$.

Proof. $(i) \Longrightarrow (ii)$. Using the arguments from the standard proof [1, p.91] of Tietze-Urysohn theorem with Theorem 2.1 instead of Urysohn lemma, we can construct a sequence $\langle g_n \rangle_{n\in\omega}$ of slowly oscillating functions $g_n \colon X \longrightarrow \mathbb{R}$ such that, for every $\varepsilon > 0$, there exists a natural number N such that

$$|g_n(x) - g_m(x)| < \varepsilon$$

for all $x \in X$, $n, m > N$ and the sequence $\langle g_n(y) \rangle_{n\in\omega}$ converges to $f(y)$ for every $y \in Y$. For every $x \in X$, put $g(x) = \lim_{n\longrightarrow\infty} g_n(x)$. Clearly, g is an extension of f. Prove that g is slowly oscillating.

For every $\varepsilon > 0$, choose a natural number $n = n(\varepsilon)$ such that

$$|g(x) - g_n(x)| < \frac{\varepsilon}{3}$$

for every $x \in X$. Since g_n is slowly oscillating, for every $\alpha \in P$, there exists a bounded subset U_α such that

$$\text{diam } g_n(B(x, \alpha)) < \frac{\varepsilon}{3}$$

for every $x \in X \setminus U_\alpha$. Now let $x \in X \setminus U_\alpha$, $y \in B(x, \alpha)$. Then

$$|g(x) - g(y)| \le |g(x) - g_n(x)| + |g_n(x) - g_n(y)| + |g_n(y) - g(y)| < \varepsilon.$$

Hence, diam $g(B(x, \alpha)) < 2\varepsilon$ for every $x \in X \backslash U_\alpha$ and g is slowly oscillating. If g is unbounded, choose $a, b \in \mathbb{R}$ such that $f(Y) \subseteq [a, b]$. For every $x \in X$ put

$$g'(x) = \begin{cases} g(x), & g(x) \in [a, b]; \\ b, & g(x) > b; \\ a, & g(x) < a. \end{cases}$$

Then g' is a bounded slowly oscillating extension of f.

$(ii) \Longrightarrow (i)$. In view of Lemma 2.3 we may suppose that $\mathbb{B}$ is connected. Let Y, Z be subsets of X, $Y \perp Z$. Note that $Z \cap Y$ is bounded, put $Z' = Z \backslash Y$ and note that $Y \perp Z'$, $Y \cap Z' = \varnothing$. Define $f \colon Y \cup Z' \longrightarrow \mathbb{R}$ by the rule $f\,|_Y \equiv 0$, $f\,|_{Z'} \equiv 1$. For every $\alpha \in P$, pick a bounded subset U_α of X such that

$$B(Y \backslash U_\alpha, \alpha) \cap B(Z' \backslash U_\alpha, \alpha) = \varnothing.$$

It follows that diam $f(B(x, \alpha) \cap (Y \cup Z')) = 0$ for every $x \in (Y \cup Z') \setminus U_\alpha$, so f is slowly oscillating. Take a slowly oscillating extension $g \colon X \longrightarrow \mathbb{R}$ of f. For every $\alpha \in P$, choose a bounded subset V_α of X such that diam $g(B(x, \alpha)) < \frac{1}{2}$ for every $x \in X \backslash V_\alpha$. Then

$$B(Y \backslash V_\alpha, \alpha) \cap B(Z' \backslash V_\beta, \beta) = \varnothing$$

for all $\alpha, \beta \in P$. Hence, $Y \amalg Z'$. Since $\mathbb{B}$ is connected and $Y \cap Z$ is bounded, $Y \amalg Z$ and B is normal. □

By Tietze-Urysohn theorem, every (not necessarily bounded) continuous function defined on a closed subset of a normal topological space X has a continuous extension onto X. We show (Example 2.1) that the asymptotic variant of this theorem is not valid for all metrizable ball structures, but it holds (Example 2.2) for every cellular metrizable ball structure.

Example 2.1. Let $\mathbb{B}(\mathbb{R}, d)$ be a ball structure of the metric space $(\mathbb{R}, d)$, $d(x, y) = |x - y|$. Put $y_0 = 0$ and define the subset $Y = \{y_n : n \in \omega\}$ of $\mathbb{R}$ by the rule $y_{n+1} = y_n + n$. It is easy to see that every function defined on Y is slowly oscillating. Put $f(y_n) = y_n$ for every $n \in \omega$. Suppose that there exists a slowly oscillating extension $g \colon \mathbb{R} \longrightarrow \mathbb{R}$ of f. Choose a real number $r > 0$ such that

$$\operatorname{diam} g(B_d(x, 1)) < \frac{1}{2}$$

for every $x \in \mathbb{R}$, $|x| > r$. Then $|f(y) - f(x)| < \frac{1}{2}|x - y|$ for all positive integers y, x with $x > r$, $y > r$. Take an arbitrary n with $y_n > r$. Then $|f(y_{n+1}) - f(y_n)| = n$, $y_{n+1} - y_n = n$, a contradiction.

Example 2.2. Let (X, d) be a non-Archimedean metric space, $Y \subseteq X$, $f \colon Y \longrightarrow \mathbb{R}$ be a slowly oscillating function. For every $n \in \omega$, put $Y_n = B_d(Y, n)$. Describe an extension $g \colon X \longrightarrow \mathbb{R}$ of f onto $X = \bigcup_{n \in \omega} Y_n$. Put $g(x) = f(x)$ for every $x \in Y_0$. Since $\mathbb{B}(X, d)$ is cellular, there exists a subset $C_1 \subseteq Y$ such that the family $\{B_d(c, 1) : c \in C_1\}$ forms a partition of Y_1. For every $x \in B_d(c, 1) \backslash Y_0$, $c \in C_1$, put $g(x) = f(c)$. Since $\mathbb{B}(X, d)$ is cellular, there exists a subset $C_2 \subseteq Y$ such that the family $\{B_d(c, 2) : c \in C_2\}$ forms a partition of Y_2. For every $x \in B_d(c, 2) \setminus Y_1$, $c \in C_2$, put $g(x) = f(c)$ and so on. After ω steps we get $g \colon X \longrightarrow \mathbb{R}$.

Take an arbitrary $x \in X$ and choose the minimal number k such that $x \in Y_k$. If $n \in \omega$ and $n \geq k$, then

$$\operatorname{diam} g(B_d(x, n)) = \operatorname{diam}\ f(B_d(x, n) \cap Y).$$

If $n \in \omega$ and $n < k$, then $\operatorname{diam} g(B_d(x, n)) = 0$. Since f is slowly oscillating, these observations show that g is slowly oscillating.

The next two examples give the explicit constructions of separating slowly oscillating functions for some special normal ball structures.

Example 2.3. Let (X, d) be a metric space, $Y_0, Y_1 \subseteq X$, $Y_0 \cap Y_1 = \varnothing$, $Y_0 \perp Y_1$. Put $f(x) = 0$ for every $x \in Y_0$ and $f(x) = 1$ for every $x \in Y_1$. Note that $\operatorname{cl} Y_0 \cap \operatorname{cl} Y_1$ is bounded and put $f(x) = 0$ for every $x \in (\operatorname{cl}\ Y_0 \cap \operatorname{cl} Y_1) \backslash (Y_0 \cup Y_1)$. Take an arbitrary element $x \notin Y_0 \cup Y_1 \cup (\operatorname{cl} Y_0 \cap \operatorname{cl} Y_1)$ and put

$$f(x) = \frac{d(x, Y_0)}{d(x, Y_0) + d(x, Y_1)},$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$. It is a routine verification that $f \colon X \longrightarrow [0, 1]$ is slowly oscillating.

Example 2.4. Let $\mathbb{B} = (X, P, B)$ be a cellular ordinal ball structure, $Y_0 \subseteq X$, $Y_1 \subseteq X$, $Y_0 \cap Y_1 = \varnothing$, $Y_0 \perp Y_1$. We construct a slowly oscillating function $f \colon X \longrightarrow \{0, 1\}$ such that $f|_{Y_0} \equiv 0$, $f|_{Y_1} \equiv 1$. We may suppose that P is well ordered and $B^{\square}(x, \alpha) = B(x, \alpha)$ for every $\alpha \in P$. Fix an increasing family $\{U_\alpha : \alpha \in P\}$ of bounded subsets of X such that $B(Y_0 \backslash U_\alpha, \alpha) \cap Y_1 = \varnothing$ for every $\alpha \in P$. Put

$$f(x) = \begin{cases} 0, & y \in Y_0 \cup \bigcup_{\alpha \in P} U_\alpha; \\ 1, & \text{otherwise.} \end{cases}$$

We omit the verification that f is slowly oscillating.

§3 Pseudoboundedness and pseudodiscreteness

A ball structure $\mathbb{B} = (X, P, B)$ is called *pseudobounded* if, for every slowly oscillating function $f : X \longrightarrow \mathbb{R}$, there exists a bounded subset U of X such that the restriction $f|_{X\setminus U}$ is bounded. Clearly, every bounded ball structure is pseudobounded.

Proposition 3.1. *Every pseudobounded ordinal ball structure $\mathbb{B} = (X, P, B)$ is bounded.*

Proof. We may suppose that P is well ordered. If $\operatorname{cf} P \leq \aleph_0$, then $\mathbb{B}$ is metrizable and we may suppose that $\mathbb{B} = \mathbb{B}(X, d)$, where d is a metric on X . Fix an arbitrary $x_0 \in X$ and for every $x \in X$ put $f(x) = \sqrt{d(x, x_0)}$. It is easy to check that f is slowly oscillating. Choose $r > 0$, $c > 0$ such that $f(x) \leq c$ for every x, $d(x, x_0) \geq r$. Hence, $\mathbb{B}$ is bounded.

If $\operatorname{cf} P > \aleph_0$, we assume that $\mathbb{B}$ is unbounded and show that $\mathbb{B}$ is not pseudobounded. In this case $\mathbb{B}$ is cellular and we may suppose that $B^{\square}(x, \alpha) = B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. Choose an increasing family $\{U_\alpha : \alpha \in P\}$ of bounded subsets of X such that $B(U_\alpha, \alpha) = U_\alpha$. Note that $X = \bigcup_{\alpha\in P} U_\alpha$, take an arbitrary element $x \in X$ and choose the minimal ordinal α with $x \in U_\alpha$. If $\alpha < \omega$, put $f(x) = \alpha$. If α is a limit ordinal, put $f(\alpha) = 0$. If $\alpha > \omega$ and α is not a limit ordinal, choose a limit ordinal β and a natural number n such that $\alpha = \beta + n$. Put $f(x) = n$. Thus, we have defined $f : X \longrightarrow \omega$. Now take an arbitrary $\alpha \in P$ and observe that $f(B(x, \alpha)) = \{f(x)\}$ for every $x \notin U_\alpha$. Thus, f is slowly oscillating and f is bounded on every subset Y such that $X \setminus Y$ is bounded. □

Consider an arbitrary uncountable group G. By Proposition 3.1, $\mathbb{B}_l(G, |G|)$ is not pseudobounded. By Proposition 1.1, $\mathbb{B}_l(G, |G|)$ is normal. Clearly, every countable subset of G is bounded. Let us compare these observations with the standard topological statement: a normal topological space is pseudocompact if and only if it is countably compact.

Example 3.1. Let G be an uncountable Abelian group. We prove that $\mathbb{B}(G)$ is pseudobounded. Suppose the contrary and fix an unbounded slowly oscillating function $f : G \longrightarrow \mathbb{R}$. Choose a countable subgroup H of G such that $f(H)$ is an unbounded subset of $\mathbb{R}$. Let $H = \{h_n : n \in \omega\}$, $h_0 = 0$, $H_n = \{h_k : k \leq n\}$. For every $n \in \omega$, choose a finite subset $F_n \subset G$ such that

$$\operatorname{diam} f(x + H_n) \leq \frac{1}{n}$$

for every $x \in G \setminus F_n$. Denote by G' the smallest subgroup of G containing H and $\bigcup_{n\in\omega} F_n$. Fix an arbitrary element $g \in G\setminus G'$. Then f is constant on the coset $g + H$. On the other hand, there exists a finite subset F of G such that $f(\{x, x + g\}) < 1$ for every $x \in G \setminus F$. In particular, $|f(h) - f(h + g)| < 1$ for every $h \in H \setminus F$. Since $f(h + g) = f(g)$ for every $h \in G'$, we conclude that $f(H)$ is bounded, which contradicts to the choice of H.

Let $\mathbb{B} = (X, P, B)$ be a ball structure. A subset $Y \subseteq X$ is called *pseudodiscrete* if every function $f : Y \longrightarrow \mathbb{R}$ is slow oscillating. A ball structure $\mathbb{B}$ is called pseudodiscrete if its support is pseudodiscrete.

Proposition 3.2. *For every uniform ball structure $\mathbb{B} = (X, P, B)$ and every subset $Y \subseteq X$ the following statements are equivalent*

- (i) *Y is pseudodiscrete;*
- (ii) *every bounded function $f : Y \longrightarrow X$ is slowly oscillating;*
- (iii) *for every $\alpha \in P$, there exists a bounded subset U_α of X such that $Y \cap B(y, \alpha) = \{y\}$ for every $y \in Y \setminus U_\alpha$.*

Proof. $(i) \Longrightarrow (ii)$. Obvious.

$(ii) \Longrightarrow (iii)$. By Zorn Lemma, there exists a subset $Z \subseteq Y$ such that the family $\{B(z,\alpha) : z \in Z\}$ is disjoint and for every $y \in Y$, there exists $z \in Z$ such that $B(z,\alpha) \cap B(y,\alpha) \neq \varnothing$. Define a mapping $f : Y \longrightarrow \{0,1\}$ by the rule

$$f(y) = \begin{cases} 0, & y \in \bigcup_{z \in Z} B(z,\alpha); \\ 1, & \text{otherwise}. \end{cases}$$

Since f is slowly oscillating, $f^{-1}(1)$ is bounded. Denote by Z_0 the set of all $z \in Z$ such that $|B(z,\alpha)| > 1$. Suppose that Z_0 is unbounded. For every $z \in Z_0$ take $z' \in B(z,\alpha)$, $z' \neq z$. Put $h(z') = 0$ and extend h onto Y arbitrarily. Clearly, h is not slowly oscillating. Hence, Z_0 is bounded. Put $U_\alpha = f^{-1}(1) \cup Z_0$ and obtain (iii).

$(iii) \Longrightarrow (i)$. Let $f : Y \longrightarrow \mathbb{R}$ be an arbitrary mapping. Then $\operatorname{diam} f(B(y,\alpha) \cap Y) = 0$ for every $y \in Y \setminus U_\alpha$, so f is slowly oscillating. □

Every ball structure $\mathbb{B}(X,\varphi)$ defined in Example 1.1 satisfies (iii) of Proposition 3.2, so $\mathbb{B}(X,\varphi)$ is pseudodiscrete.

Example 3.2. Let X be a set and let φ be a free ultrafilter on X. We show that $\mathbb{B}(X,\varphi)$ is pseudobounded if and only if φ is countably complete (i.e. φ is closed under countable intersections).

Let φ be countably complete, $f : X \longrightarrow \mathbb{R}$ be an arbitrary mapping. Partition $\mathbb{R}$ onto ω subsets $\{A_n : n \in \omega\}$ of diameter < 1. Choose $n \in \omega$ such that $f^{-1}(A_n) \in \varphi$. Then f is bounded on the subset $f^{-1}(A_n)$ and $X \setminus f^{-1}(A_n)$ is a bounded subset of $\mathbb{B}(X,\varphi)$, so $\mathbb{B}(X,\varphi)$ is pseudobounded.

Now assume that $\mathbb{B}(X,\varphi)$ is pseudobounded, but φ is not countably complete. Choose a family $\{F_n : n \in \omega\}$ of subsets of X such that $F_0 = X$, $F_n \in \varphi$, $F_{n+1} \subset F_n$ and $\bigcap_{n \in \omega} F_n = \varnothing$. For every $x \in X$, choose the minimal number n such that $x \notin F_n$ and put $f(x) = n$. Since $\mathbb{B}(X,\varphi)$ is pseudobounded and pseudodiscrete, there exists a subset $F \in \varphi$ such that $f\,|_F$ is bounded. Pick $m \in \omega$ such that $f(x) < m$ for every $x \in F$. Then $F \cap F_{m+1} = \varnothing$, $F, F_m \in \varphi$, a contradiction.

§4 Coronas of ball structures

Fix a ball structure $\mathbb{B} = (X,P,B)$, endow X with the discrete topology and consider the Stone-Čech compactification βX of X. We take the points of βX to be the ultrafilters on X with the points of X identified with the principal ultrafilters. Denote by $X^\#$ the set of all ultrafilters r on X such that every $R \in r$ is unbounded in $\mathbb{B}$. Given any $r, q \in X^\#$, we write $r \parallel q$ if there exists $\alpha \in P$ such that $B(R,\alpha) \in q$ for every $R \in r$.

Lemma 4.1. *If $\mathbb{B}$ is a uniform ball structure, then $\parallel$ is an equivalence on $X^\#$.*

Proof. Let $r, q, s \in X^\#$, $r \parallel q$, $q \parallel s$. Suppose that q is not parallel to r. Then, for every $\alpha \in P$, there exists $Q_\alpha \in q$ such that $B(Q_\alpha,\alpha) \notin r$. Put $R_\alpha = X \setminus B(Q,\alpha)$. Then $B^*(R,\alpha) \cap Q = \varnothing$, so r is not parallel to q.

Since $r \parallel q$, $q \parallel s$, there exist $\alpha, \beta \in P$ such that $B(R,\alpha) \in q$, $B(Q,\beta) \in s$ for all $R \in r$, $Q \in q$. Choose $\gamma(\alpha,\beta) \in P$ such that $B(B(x,\alpha),\beta) \subseteq B(x,\gamma(\alpha,\beta))$ for every $x \in X$. Then $B(R,\gamma(\alpha,\beta)) \in s$ for every $R \in r$, so $r \parallel s$. □

Denote by $\sim$ the minimal (by inclusion) closed (in $X^\# \times X^\#$) equivalence on $X^\#$ such that $\|\subseteq\sim$. The compact Hausdorff space $X^\#/\sim$ is called the corona of $\mathbb{B}$, it is denoted by $\nu(\mathbb{B})$.

Let K be a compact Hausdorff space. For every mapping $f\colon X \longrightarrow K$, denote by f^β the Stone-Čech extension of f onto βX.

Let $p, q \in X^\#$. We say that $p \approx q$ if $f^\beta(p) = f^\beta(q)$ for every slowly oscillating mapping $f\colon X \longrightarrow [0,1]$. Clearly, $\approx$ is a closed equivalence on $X^\#$.

Lemma 4.2. *Let $\mathbb{B} = (X, P, B)$ be a normal ball structure, r, $q \in X^\#$. Then the following statements are equivalent*

(i) $r \approx q$;

(ii) for every $R \in r$, $Q \in q$, there exists $\alpha \in P$ such that $B(R,\alpha) \bigcap B(Q,\alpha)$ is unbounded.

(iii) $r \sim q$.

Proof. (i) $\Longrightarrow$ (ii) Suppose the contrary. Then there exist $R \in r$, $Q \in q$ such that $R \perp Q$. By Theorem 2.1, there exists a slowly oscillating function $f\colon X \longrightarrow [0,1]$ such that $f|_R \equiv 0$, $f|_Q \equiv 1$. Clearly, $f^\beta(r) = 0$, $f^\beta(q) = 1$ so $f^\beta(r) \neq f^\beta(q)$.

(ii) $\Rightarrow$ (iii) Put $S = B(R,\alpha) \cap B(Q,\alpha)$. For every $x \in S$, pick $f(x) \in R$, $g(x) \in Q$ such that $f(x) \in B(x,\alpha)$, $g(x) \in B(x,\alpha)$. Fix any ultrafilter s from $S^\#$ and note that $s \parallel f^\beta(s)$, $s \parallel g^\beta(s)$, so $f^\beta(s) \parallel g^\beta(x)$. It follows that for all neighborhoods $R^\#$ and $Q^\#$, there exist $f^\beta(s) \in R^\#$, $g^\beta(s) \in Q^\#$ such that $f^\beta(x) \parallel g^\beta(s)$. Hence, $r \sim q$.

(iii) $\Rightarrow$ (i) Since $\approx$ is a closed equivalence, it suffices to show that $r \parallel q$ implies $f^\beta(r) = f^\beta(q)$ for every slowly oscillating function $f\colon X \longrightarrow [0,1]$. Suppose the contrary. Then there exists a slowly oscillating function $f\colon X \longrightarrow [0,1]$ such that $f^\beta(r) \neq f^\beta(q)$. Let $f^\beta(r) = a$, $f^\beta(q) = b$, $|b - a| = d$. Put

$$U_a = \Big\{t \in [0,1] : |a-t| < \frac{d}{4}\Big\}, \quad U_b = \Big\{t \in [0,1] : |b-t| < \frac{d}{4}\Big\}.$$

Choose $R \in r$, $Q \in q$ with $f(P) \subseteq U_a$, $f(Q) \subseteq U_b$. Since $r \parallel q$, there exists $\alpha \in P$ such that $B(R,\alpha) \cap Q$ is unbounded. Put $R' = \{x \in R : B(x,\alpha) \cap Q \neq \varnothing\}$ and note that R' is unbounded. Since f is slowly oscillating there exists $R'' \subseteq R$ such that $\operatorname{diam} f(B(x,\alpha)) < \frac{d}{4}$ for every $x \in R''$. But then $B(R'',\alpha) \cap Q = \varnothing$, a contradiction with $R'' \subseteq R'$. □

Lemma 4.3. *Let $\mathbb{B} = (X, P, B)$ be a cellular ordinal ball structure, $r, q \in X^\#$. Then the following statement are equivalent*

(i) $r \sim q$;

(ii) $f^\beta(r) = f^\beta(q)$ *for every slowly oscillating function* $f\colon X \longrightarrow \{0,1\}$.

Proof. Apply Lemma 4.2 and Example 2.4 instead of Theorem 2.1. □

Let $\mathbb{B} = (X, P, B)$ be a normal ball structure. Given any $r \in X^\#$, denote by $[r] = \{q \in X^\# : r \sim q\}$. By Lemma 4.2, $q \in [r]$ if and only if R, Q are not asymptotically disjoint for all $R \in r$, $Q \in q$. To describe the topology of $\nu(\mathbb{B})$, given any $r \in X^\#$, $Q \subseteq X$, we write $r \perp Q$ if and only if there exists $R \in r$ such that $R \perp Q$. Then the family $\{[r] : r \perp Q\}$, where Q runs over all subsets of X, forms a base for open subsets of $\nu(\mathbb{B})$. Using Lemma 2.2, we get the following description of neighborhoods of points in $\nu(\mathbb{B})$. Let $r \in X^\#$, $R \in r$ and let $\hat{R}$ be a pyramid with the core R. Then $\{[q] : \hat{R} \in q\}$ is a neighborhood of $[r]$, and every neighborhood of $[r]$ contains a neighborhood of this form. By Lemma 4.2, a normal ball structure $\mathbb{B} = (X, P, B)$ is pseudodiscrete if and only if $X^{\#} = \nu(\mathbb{B})$. By Lemma 4.3, $\nu(\mathbb{B})$ is zero-dimensional for every cellular ordinal ball structure.

We conclude the paper with one application of coronas to the semigroups of free ultrafilters on groups.

Let G be a discrete group, βG be the Stone-Čech compactification of G. Following [3], we extend the multiplication from G to βG. Given any $r, q \in \beta G$ and $A \subseteq G$, put

$$A \in rq \text{ if and only if } \{g \in G : g^{-1}A \in q\} \in r.$$

This multiplication on βG is associative, so βG is a semigroup and $G^* = \beta G \setminus G$ is a subsemigroup of βG. By [3, Section 6.3], for every countable group G, G^* has 2^c pairwise disjoint closed left ideals in βG.

Theorem 4.1. *For every countable group G, G^* is the union of 2^c pairwise disjoint closed left ideals in βG.*

Proof. Consider the ball structure $\mathbb{B} = \mathbb{B}_l(G)$ and note that $G^{\#} = G^*$. Take any $r, q \in G^*$ and observe that $r \parallel q$ if and only if $r = gq$ for some element $g \in G$. Since βG is a right topological semigroup, it follows that every element $[r] \in \nu(\mathbb{B})$ is a closed left ideal in βG. Clearly, $|\nu(\mathbb{B})| \leq 2^c$. To show that $|\nu(\mathbb{B})| \leq 2^c$, fix an increasing family $\{G_n : n \in \omega\}$ of finite subsets of G such that $G = \bigcup_{n\in\omega} G_n$. Then choose inductively a sequence $\langle x_n \rangle_{n\in\omega}$ in G such that $G_n x_n \cap G_m x_m = \varnothing$ for all $n \neq m$. Take any $r, q \in G^*$ such that $r \neq q$ and $\{x_n : n \in \omega\} \in r$, $\{x_n : n \in \omega\} \in q$. By Lemma 4.2, $[r] \neq [q]$, so $|\nu(\mathbb{B})| \geq 2^c$. □

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