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ON RELATIONS BETWEEN THE ABSCISSA OF CONVERGENCE AND THE ABSCISSA OF ABSOLUTE CONVERGENCE OF RANDOM DIRICHLET SERIES

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We establish some relations between the abscissa of convergence and the abscissa of absolute convergence of random Dirichlet series and prove their exactness.

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Установлены некоторые соотношения между абсциссой сходимости и абсциссой абсолютной сходимости случайного ряда Дирихле и доказана их точность.

1. Introduction

Let $\lambda = (\lambda_n)$ be a sequence of nonnegative numbers increasing to $+\infty$,

$$\tau(\lambda) = \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n} < +\infty,$$

and $S(\lambda)$ be the class of Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \qquad s = \sigma + it. \tag{1}$$

For series (1) let $\sigma_c(F)$ be the abscissa of convergence, $\sigma_a(F)$ be the abscissa of absolute convergence, and $\beta(F) = \sup\{\sigma \in \mathbb{R} : \ |a_n|e^{\sigma\lambda_n} = o(1), \ n \to +\infty\}$ ($\beta(F) = -\infty$, if $|a_n|e^{\sigma\lambda_n} \neq o(1)$, $n \to +\infty$, for every $\sigma \in \mathbb{R}$).

It is known (see for example [1,2,3]) that $\sigma_a(F) \leq \sigma_c(F) \leq \sigma_a(F) + \tau(\lambda)$ and these inequalities are exact. Moreover, the next simple statement is valid. For convenience, we give it with proof, although it is known.

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Theorem A.

(i) For each Dirichlet series $F \in S(\lambda)$,

$$\sigma_a(F) \le \sigma_c(F) \le \beta(F) \le \sigma_a(F) + \tau(\lambda). \tag{2}$$

(ii) For every $a, b, c \in [-\infty, +\infty]$ such that

$$a \le c \le b \le a + \tau(\lambda),$$

there exists a Dirichlet series $F \in S(\lambda)$ such that $\sigma_a(F) = a$, $\sigma_c(F) = c$ and $\beta(F) = b$.

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space and (ξ_n) be a sequence of independent random variables in this space. For a random variable ξ let $\mathrm{M}\,\xi = \int_\Omega \mathrm{Re}\,\xi(\omega)P(d\omega) + i\int_\Omega \mathrm{Im}\,\xi(\omega)P(d\omega)$ be its mean value and $D\xi = \mathrm{M}(|\xi - \mathrm{M}\,\xi|^2)$ be its dispersion.

We consider the random Dirichlet series

$$F_\omega(s) = \sum_{n=0}^{\infty} \xi_n e^{s\lambda_n}. \tag{3}$$

It follows from the zero-one law that $\sigma_a(F_\omega)$, $\sigma_c(F_\omega)$ and $\beta(F_\omega)$ are constants from $[-\infty, +\infty]$ a.s. These constants will be denoted by $\hat\sigma_a(F_\omega)$, $\hat\sigma_c(F_\omega)$ and $\hat\beta(F_\omega)$, respectively.

Consider the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \varepsilon_n n^s, \tag{4}$$

where $\varepsilon_n \in \{-1, 1\}$ for $n \ge 1$. As it is easily seen, $\sigma_a(F) = -1$, $\lambda_n = \ln(n+1)$ for $n \ge 0$, and $\tau(\lambda) = 1$. If $\varepsilon_n = 1$ for $n \ge 1$, we obtain $\sigma_c = \sigma_a$. If $\varepsilon_n = (-1)^n$ for $n \ge 1$, we have $\sigma_c = \sigma_a + \tau(\lambda) = 0$. But, it follows from the next assertion that for the "majority" of Dirichlet series (4), $\sigma_c(F) = \sigma_a(F) + \frac{\tau(\lambda)}{2}$.

Theorem B [4]. *Let (ξ_n) be a Rademacher sequence, i.e. each ξ_n takes the values $+1$ and -1 with equal probability $\frac{1}{2}$, and let $\lambda_n = \ln(n+1)$ for every $n \ge 0$. Then for Dirichlet series (3) we obtain $\hat\sigma_c(F_\omega) = -\frac{1}{2}$.*

Since for the Dirichlet series from Theorem B we have

$$\mathrm{M}(\xi_n e^{s\lambda_n}) = 0, \qquad \sum_{n=0}^{\infty} D(\xi_n e^{s\lambda_n}) = \sum_{n=0}^{\infty} e^{2\sigma\lambda_n},$$

and the last series converges for $\sigma < -\frac{1}{2}$ and diverges for $\sigma \ge -\frac{1}{2}$, Theorem B is an obvious corollary from

Theorem C [5, p.54]. *Let (η_n) be a sequence of independent random variables and*

$$\eta_n'(\omega) = \begin{cases} \eta_n(\omega), & \text{if } |\eta_n(\omega)| \le 1; \\ \frac{\eta_n(\omega)}{|\eta_n(\omega)|}, & \text{if } |\eta_n(\omega)| > 1. \end{cases}$$

The series $\sum_{n=0}^{\infty} \eta_n$ converges a.s. if and only if the series $\sum_{n=0}^{\infty} \mathrm{M}\,\eta_n'$ and $\sum_{n=0}^{\infty} D\eta_n'$ converge.

In this paper, using Theorem C, we generalize Theorem B. Let

$$G(s)=\sum_{n=0}^{\infty} a_n^2 e^{2s\lambda_n}, \tag{5}$$

and

$$F_\omega(s)=\sum_{n=0}^{\infty} \eta_n a_n e^{s\lambda_n}, \tag{6}$$

where a_n are coefficients of series (1) and (η_n) is a sequence of independent random variables.

Theorem 1. *Let (η_n) be a sequence of independent random variables such that $c_1 \le |\eta_n(\omega)| \le c_2$ a.s., where c_1 and c_2 are positive constants, and $\mathrm{M}\,\eta_n = 0$ for every $n \ge 0$.*

(i) For random Dirichlet series (6) we have

$$\hat\sigma_a(F_\omega) \le \frac{\hat\sigma_a(F_\omega)+\hat\beta(F_\omega)}{2} \le \hat\sigma_c(F_\omega) \le \min\left\{\hat\sigma_a(F_\omega)+\frac{\tau(\lambda)}{2}, \hat\beta(F_\omega)\right\}. \tag{7}$$

(ii) For every $a, b, c \in [-\infty, +\infty]$ such that

$$a \le \frac{a+b}{2} \le c \le \min\left\{a+\frac{\tau(\lambda)}{2}, b\right\}, \tag{8}$$

there exists a sequence (a_n) of complex numbers such that for random Dirichlet series (6) we have $\hat\sigma_a(F_\omega)=a$, $\hat\sigma_c(F_\omega)=c$, and $\hat\beta(F_\omega)=b$.

Theorem 1 follows immediately from Theorem C and

Theorem 2. *(i) For Dirichlet series (1) and (5) we have*

$$\sigma_a(F) \le \frac{\sigma_a(F)+\beta(F)}{2} \le \sigma_a(G) \le \min\left\{\sigma_a(F)+\frac{\tau(\lambda)}{2}, \beta(F)\right\}. \tag{9}$$

(ii) For every $a, b, c \in [-\infty, +\infty]$ such that inequalities (8) are valid, there exists a sequence (a_n) of complex numbers such that for Dirichlet series (1) and (5) we have $\sigma_a(F)=a$, $\sigma_a(G)=c$, and $\beta(F)=b$.

We apply Theorem 1 to the proofs of next assertions.

Theorem 3. *(i) If ξ_n are symmetric independent random variables, then for random Dirichlet series (3) inequalities (7) are valid.*

(ii) For every $a, b, c \in [-\infty, +\infty]$ such that inequalities (8) are valid, there exists a sequence (ξ_n) of symmetric independent random variables such that for random Dirichlet series (3) we have $\hat\sigma_a(F_\omega)=a$, $\hat\sigma_c(F_\omega)=c$, and $\hat\beta(F_\omega)=b$.

Theorem 4. *Let (ξ_n) an arbitrary sequence of independent random variables. Then, for random Dirichlet series (3), either inequalities (7) are valid or there exists a usual Dirichlet series (1) with $\sigma_c(F)=\hat\sigma_c(F_\omega)$, $\sigma_a(F)=\hat\sigma_a(F_\omega)$ and $\beta(F)=\hat\beta(F_\omega)$ such that, for series $F'_\omega = F_\omega - F$,*

$$\hat\sigma_a(F'_\omega) \le \frac{\hat\sigma_a(F'_\omega)+\hat\beta(F'_\omega)}{2} \le \hat\sigma_c(F'_\omega) \le \min\left\{\hat\sigma_a(F'_\omega)+\frac{\tau(\lambda)}{2}, \hat\beta(F'_\omega)\right\}.$$

The proofs of Theorems 3 and 4 are similar to the proof of the next J.-P. Kahane's theorem [5, p. 62–64].

Theorem D [5]. *Let* $-\infty < \sigma_c(F_\omega) < +\infty$.

(i) If the variables ξ_n *are symmetric, then the straight line* $\{s : \operatorname{Re} s = \sigma_c(F_\omega)\}$ *a.s. is the natural boundary for series (3).*

(ii) In general case, either the straight line $\{s : \operatorname{Re} s = \sigma_c(F_\omega)\}$ *is a.s. the natural boundary for* F_ω *or there exists a usual Dirichlet series (1) with* $\sigma_c(F) = \hat\sigma_c(F_\omega)$ *such that the straight line* $\{s : \operatorname{Re} s = \sigma_c(F_\omega - F)\}$ *is a.s. the natural boundary for* $F_\omega - F$ *and* $\hat\sigma_c(F_\omega - F) > \hat\sigma_c(F_\omega)$.

We note that Theorem B generalizes the well known results of Ryll-Nardzewski [6].

Theorems A and 2 are trivial in the cases $\sigma_a(F) = \pm\infty$ or $a = \pm\infty$. In the sequel, we assume that $-\infty < \sigma_a(F) < +\infty$ and $-\infty < a < +\infty$.

2. Proof of Theorem A

(i) The first and second inequalities of (2) are obvious. The third inequality follows from inequalities (9), which we prove below.

(ii) We use next

Lemma [7]. *In every sequence* λ *such that* $\tau(\lambda) \geq \tau^* \geq 0$, *there exists a subsequence* λ^* *such that* $\tau(\lambda^*) = \tau^*$.

Let $\tau^* = b - a$. Then $\tau(\lambda) \geq \tau^* \geq 0$. We select from the sequence λ a subsequence $\lambda^* = (\lambda_k^*)$ such that $\tau(\lambda^*) = \tau^*$. Put

$$F_1(s) = \sum_{k=0}^{\infty} (-1)^n e^{\sigma\lambda_k^*}, \qquad F_2(s) = \frac{1}{2}\sum_{k=0}^{\infty} e^{(a-c)\lambda_k^*} e^{\sigma\lambda_k^*}, \qquad F_3(s) = F_1(s) + F_2(s).$$

Then we obtain consequently that $\sigma_a(F_1) = -\tau^*$, $\sigma_c(F_1) = 0$, $\sigma_a(F_2) = \sigma_c(F_2) = -\tau^* - a + c$, $\sigma_a(F_3) = -\tau^*$, $\sigma_c(F_3) = -\tau^* - a + c$, $\beta(F_3) = 0$. Hence, if $F(s) = F_3(s - a - \tau^*)$, we have $\sigma_a(F_4) = a$, $\sigma_c(F_4) = c$, $\beta(F_3) = a + \tau^* = b$ and $F \in S(\lambda^*) \subset S(\lambda)$. Theorem A is proved.

3. Proof of Theorem 2

(i) Let

$$\sigma < \frac{\sigma_a(F) + \beta(F)}{2}, \qquad \varepsilon = \frac{\sigma_a(F) + \beta(F) - 2\sigma}{2}.$$

Then $\varepsilon > 0$ and $2\sigma - \beta(F) + \varepsilon = \sigma_a(F) - \varepsilon < \sigma_a(F)$. Hence,

$$\sum_{n=0}^{\infty} |a_n| e^{(2\sigma - \beta(F) + \varepsilon)\lambda_n} < \infty. \tag{10}$$

Furthermore,

$$|a_n| e^{(\beta(F) - \varepsilon)\lambda_n} \leq 1, \qquad n \geq n_0. \tag{11}$$

Using (10) and (11) we obtain

$$\sum_{n=n_0}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} = \sum_{n=n_0}^{\infty} |a_n| e^{(\beta(F)-\varepsilon)\lambda_n} |a_n| e^{(2\sigma-\beta(F)+\varepsilon)\lambda_n} \le \sum_{n=n_0}^{\infty} |a_n| e^{(2\sigma-\beta(F)+\varepsilon)\lambda_n} < \infty,$$

i.e., series (5) converges. Hence, the second inequality of (9) is proved (the first one is obvious).

We prove the third inequality. Assume, contrary to this inequality, that

$$\delta = \frac{1}{3}\left(\sigma_a(G) - \sigma_a(F) - \frac{\tau(\lambda)}{2}\right) > 0.$$

(The inequality $\sigma_a(G) \le \beta(F)$ is obvious.)

Since $\ln n \le (\tau + \delta)\lambda_n$ for each $n \ge n_1$, we have

$$\sum_{n=n_1}^{\infty} e^{-(\tau+2\delta)\lambda_n} \le \sum_{n=n_1}^{\infty} n^{-\frac{\tau+2\delta}{\tau+\delta}} < \infty. \tag{12}$$

Furthermore,

$$\sum_{n=0}^{\infty} |a_n|^2 e^{(2(\sigma_a(G)-\delta)\lambda_n} < \infty. \tag{13}$$

Thus, by the Cauchy–Bunyakovsky inequality, (12) and (13), we obtain

$$\sum_{n=0}^{\infty} |a_n| e^{(\sigma_a(F)+\delta)\lambda_n} = \sum_{n=0}^{\infty} |a_n| e^{(\sigma_a(G)-\delta)\lambda_n} e^{-\frac{\tau(\lambda)+2\delta}{2}\lambda_n} \le$$

$$\le \left(\sum_{n=0}^{\infty} |a_n|^2 e^{2(\sigma_a(G)-\delta)\lambda_n}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} e^{-(\tau(\lambda)+2\delta)\lambda_n}\right)^{\frac{1}{2}} < \infty,$$

that contradicts to the definition of $\sigma_a(F)$.

(ii) First we consider the case

$$a \le b \le a + \frac{\tau(\lambda)}{2}.$$

Let $\tau' = 2(c - a)$, $\gamma = a + b - 2c$. Then, $\gamma \le 0$ and $0 \le \tau' \le \tau$.

Now, by Lemma, we select from the sequence λ a subsequence $\lambda' = (\lambda'_k)$ such that $\tau(\lambda') = \tau'$. Let (k_p) be a increasing sequence of nonnegative integers such that for the series

$$F_1(s) = \sum_{p=0}^{\infty} e^{s\lambda'_{k_p}}$$

we have $\sigma_a(F_1) = 0$.

Set $a_k = 1 + e^{\gamma\lambda'_k}$ if $k = k_p$ and $a_k = e^{\gamma\lambda'_k}$ in the other cases. Consider the Dirichlet series

$$F_2(s)=\sum_{k=0}^{\infty}a_ke^{s\lambda_k'}=\sum_{p=0}^{\infty}e^{s\lambda_{k_p}'}+\sum_{k=0}^{\infty}e^{(s+\gamma)\lambda_k'},$$
$$G_2(s)=\sum_{k=0}^{\infty}a_k^2e^{2s\lambda_k'}=\sum_{p=0}^{\infty}(1+2e^{\gamma\lambda_{k_p}'})e^{2s\lambda_{k_p}'}+\sum_{k=0}^{\infty}e^{2(s+\gamma)\lambda_k'}.$$

It is clear that $\beta(F_2)=0$, $\sigma_a(F_2)=-\tau'-\gamma=a-b$, and $\sigma_a(G_2)=-\frac{\tau'}{2}-\gamma=c-b$. Hence, for the series $F(s)=F_2(s-b)$ and $G(s)=G_2(s-b)$ we have $\beta(F)=b$, $\sigma_a(F)=a$, and $\sigma_a(G_2)=c$.

Next, we assume that
$$a+\frac{\tau(\lambda)}{2}\le b\le a+\tau(\lambda).$$
Set $\tau''=2(b-c)$. Then, $0\le\tau''\le\tau$. Let $\lambda''=(\lambda_k'')$ be a subsequence of λ such that $\tau(\lambda'')=\tau''$. Consider the Dirichlet series

$$F_1(s)=\sum_{n=0}^{\infty}e^{-(a+\tau)\lambda_n}e^{s\lambda_n},\qquad G_1(s)=\sum_{n=0}^{\infty}e^{-2(a+\tau)\lambda_n}e^{2s\lambda_n},$$
$$F_2(s)=\sum_{n=0}^{\infty}e^{-(2c-b+\tau'')\lambda_n''}e^{s\lambda_n''},\qquad G_2(s)=\sum_{n=0}^{\infty}e^{-2(2c-b+\tau'')\lambda_n''}e^{2s\lambda_n''}.$$

For this series we have

$$\sigma_a(F_1)=-\tau+a+\tau=a,\sigma_a(G_1)=-\frac{\tau}{2}+a+\tau=a+\frac{\tau}{2},\beta(F_1)=a+\tau,$$
$$\sigma_a(F_2)=-\tau''+2c-b+\tau''=2c-b\ge a+b-b=a,$$
$$\sigma_a(G_2)=-\frac{\tau''}{2}+2c-b+\tau''=c,\beta(F_2)=2c-b+\tau''=b.$$

Let
$$F(s)=\sum_{n=0}^{\infty}a_ne^{s\lambda_n}=F_1(s)+F_2(s),\quad G(s)=\sum_{n=0}^{\infty}a_n^2e^{2s\lambda_n}.$$
Since $a_n\ge 0$ for $n\ge 0$, we obtain $\sigma_a(F)=\min\{\sigma_a(F_1),\sigma_a(F_2)\}=a$, $\sigma_a(G)=\min\{\sigma_a(G_1),\sigma_a(G_2)\}=c$, and $\beta(F)=\min\{\beta(F_1),\beta(F_2)\}=b$. The proof of Theorem 2 is complete.

4. Proof of Theorem 3

(i) Let (ξ_n) be a sequence of independent symmetric random variables in a probability space $(\Omega,\mathcal{A},\mathrm{P})$ and (ε_n) be a Rademacher sequence in a probability space $(\Omega',\mathcal{A}',\mathrm{P}')$. Consider series (3) and the series

$$F_{(\omega',\omega)}(s)=\sum_{n=0}^{\infty}\varepsilon_n\xi_ne^{s\lambda_n}$$

in the probability space $(\Omega'\times\Omega,\mathcal{A}'\times\mathcal{A},\mathrm{P}'\times\mathrm{P})$. Note that $\hat\beta(F_{(\omega',\omega)})=\hat\beta(F_\omega)$ and $\hat\sigma_a(F_{(\omega',\omega)})=\hat\sigma_a(F_\omega)$, by Fubini's theorem, and $\hat\sigma_c(F_{(\omega',\omega)})=\hat\sigma_(F_\omega)$, by the reduction principle [1, p. 20].

For every fixed $\omega \in \Omega$ and for almost all $\omega' \in \Omega'$, by Theorem 2,

$$\sigma_a(F_{(\omega',\omega)}) \le \frac{\sigma_a(F_{(\omega',\omega)}) + \beta(F_{(\omega',\omega)})}{2} \le \sigma_c(F_{(\omega',\omega)}) \le$$
$$\le \min\left\{\sigma_a(F_{(\omega',\omega)}) + \frac{\tau(\lambda)}{2}, \beta(F_{(\omega',\omega)})\right\}. \qquad (14)$$

By Fubini's theorem, inequalities (14) are fulfilled a.s. in $\Omega' \times \Omega$, and this implies (7).

(ii) This part of Theorem 3 follows from (ii) of Theorem 1 in the case $(\eta_n) = (\varepsilon_n)$, where (ε_n) is a Rademacher sequence.

5. Proof of Theorem 4

Let (ξ_n) be a sequence of independent random variables in a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ and $\eta_n(\omega', \omega) = \xi_n(\omega) - \xi_n(\omega')$ for every $n \ge 0$ and $(\omega', \omega) \in \Omega \times \Omega$. Then (η_n) is a sequence of independent symmetric random variables in the probability space $(\Omega \times \Omega, \mathcal{A} \times \mathcal{A}, \mathrm{P} \times \mathrm{P})$. Consider the series

$$F_{(\omega',\omega)}(s) = F_\omega(s) - F_{\omega'}(s) = \sum_{n=0}^{\infty} (\xi_n(\omega) - \xi_n(\omega'))e^{s\lambda_n}.$$

Applying (i) of Theorem 3, we see that for this series inequalities (14) hold a.s. in $\Omega \times \Omega$. By converse Fubini's theorem, there exists a fixed $\omega' \in \Omega$ such that for almost all $\omega \in \Omega$ and for the series $F(s) = F_{\omega'}(s)$ the relations $\sigma_a(F) = \sigma_a(F_\omega)$, $\sigma_c(F) = \sigma_c(F_\omega)$, $\beta(F) = \beta(F_\omega)$ and (14) are valid. This completes the proof of Theorem 4.

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