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SOME CONDITIONS IMPLYING BLO, VMO, BOUNDEDNESS AND CONTINUITY

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In this paper we consider some local characteristics of functions. Namely, we give conditions implying BLO, VMO, boundedness and continuity for functions which are integrable on a cube.

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Рассматриваются некоторые локальные характеристики функций. Именно, приводятся условия, влекущие BLO, VMO ограниченность для функций, интегрируемых в кубе.

1. Definitions and results

Let $f \in L(Q_0)$, where $Q_0 \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes. For any measurable set $E \subset Q_0$, we denote by $|E|$ its Lebesgue measure and by χ_E its characteristic function. The nondecreasing and equimeasurable rearrangement of f is defined by

$$f_*(t) = \inf_{|E|=t} \sup_{x\in E} f(x), \quad 0 < t \le |Q_0|.$$

Denote $f_{**}(t) = t^{-1}\int_0^t f_*(s)ds, \quad 0 < t \le |Q_0|$. Obviously, $f_{**}(t) \le f_*(t)$, $t \in (0;|Q_0|]$, and $f_{**}(|Q_0|) = f_{Q_0} \equiv |Q_0|^{-1}\int_{Q_0} f(x)dx$. Moreover, it is easy to see that f_{**} is nondecreasing and absolutely continuous on $[\varepsilon;|Q_0|]$ for any $\varepsilon \in (0;|Q_0|)$.

Let φ be a nonnegative, locally integrable function on $\mathbb{R}$ and put

$$J_\varepsilon(\varphi, f, r) = \int_\varepsilon^r \varphi(f_{**}(t))\frac{f_*(t) - f_{**}(t)}{t}dt, \quad 0 < \varepsilon < r \le |Q_0|.$$

In the proof of Lemma 2.1 we will show that $J_\varepsilon(\varphi, f, r)$ exists for any $0 < \varepsilon \le r \le |Q_0|$. Hence, the limit

$$J(\varphi, f, r) = \lim_{\varepsilon\to+0} J_\varepsilon(\varphi, f, r) = \int_0^r \varphi(f_{**}(t))\frac{f_*(t) - f_{**}(t)}{t}dt$$

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exists for every $r \in (0; |Q_0|]$ (possibly, it equals infinity).

M. Franciosi noted in [1, Theorem 3.1] that if a nonpositive function f satisfies the condition
$$\int_0^{|Q_0|} \frac{1}{-f_{**}(t)} \cdot \frac{f_*(t) - f_{**}(t)}{t} dt < \infty,$$
than f is bounded from below on Q_0 (in [1, Theorem 3.1] for nonnegative f this result was formulated in terms of nonincreasing rearrangement, but it is more convenient for us to use f_*). The following theorem makes this result more precise.

Theorem 1.1. *Suppose φ is a nonnegative, locally integrable function on $\mathbb{R}$ which satisfies the condition*
$$\int_{-\infty}^{0} \varphi(u)du = \infty. \tag{1}$$
Then for $f \in L(Q_0)$ the inequality
$$J(\varphi, f, |Q_0|) < \infty \tag{2}$$
holds if and only if f is bounded from below on Q_0.

The next criterion follows immediately from Theorem 1.1.

Corollary 1.2. *Assume that φ is a nonnegative, locally integrable function on $\mathbb{R}$ and* (1) *holds. Then a function $f \in L(Q_0)$ is bounded on Q_0 if and only if f satisfies conditions* (2) *and*
$$J(\varphi, -f, |Q_0|) < \infty. \tag{3}$$

Let $L(f, Q) = f_Q - \operatorname{ess\,inf}_{x \in Q} f(x)$ denote the low oscillation of f over a cube $Q \subset Q_0$. One says that $f \in BLO$ if $\sup_{Q \subset Q_0} L(f, Q) < \infty$, where the supremum is taken over all cubes $Q \subset Q_0$ (see [2]). It is clear that only the functions f from BLO are bounded from below on Q_0. The following theorem gives an estimate of the low oscillation of f.

Theorem 1.3. *Suppose φ is a measurable function on $\mathbb{R}$ which satisfies the condition*
$$0 < \alpha(\mathbb{R}) \equiv \operatorname*{ess\,inf}_{u \in \mathbb{R}} \varphi(u) \leq \beta(\mathbb{R}) \equiv \operatorname*{ess\,sup}_{u \in \mathbb{R}} \varphi(u) < \infty.$$
Then for $f \in L(Q_0)$ the inequality
$$\sup_{Q \subset Q_0} J(\varphi, f\chi_Q, |Q|) < \infty \tag{4}$$
holds if and only if $f \in BLO$.

Let $\Omega(f, Q) = |f - f_Q|_Q$ denote the mean oscillation of f over the cube $Q \subset Q_0$. One says that $f \in VMO$ if
$$\eta(f, h) = \sup_{|Q| \leq h} \Omega(f, Q) \to 0, \quad h \to +0,$$
where the supremum is taken over all cubes $Q \subset Q_0$ with measure not exceeding $h > 0$ (see [3]).

Let $\bar{f}(x) = \chi_{[0;1/2]}(x)$, $0 \le x \le 1$, then

$$\eta(\bar{f}, 2\delta) \ge \Omega\left(\bar{f}, \left[\frac{1}{2} - \delta; \frac{1}{2} + \delta\right]\right) = \frac{1}{2}$$

for each $\delta \in (0; 1/2)$. Hence $\bar{f} \notin VMO$. On the other hand,

$$\int_0^1 \frac{\bar{f}_*(t) - \bar{f}_{**}(t)}{t} dt = \int_{\frac{1}{2}}^1 \frac{\bar{f}_*(t) - \bar{f}_{**}(t)}{t} dt < \infty.$$

So we see that the last inequality is an insufficient condition for $\bar{f}$ to be in VMO (compare with [1, Theorem 4.1]). Using Theorem 1.3 we give a sufficient condition that a function belong to VMO.

Corollary 1.4. *Assume φ is a locally integrable function on $\mathbb{R}$ which satisfies $\alpha(\mathbb{R}) \equiv \operatorname{ess\,inf}_{u \in \mathbb{R}} \varphi(u) > 0$. If for $f \in L(Q_0)$ the equality*

$$\lim_{h \to +0} \sup_{|Q| \le h} J(\varphi, f\chi_Q, |Q|) = 0 \tag{5}$$

holds, then $f \in VMO$.

We emphasize that there exists a bounded function belonging to VMO which is nonequivalent to any continuous function (see [4, chapter IX, § 2, p. 374]). The following theorem gives a criterion for f to be equivalent to some continuous function.

Theorem 1.5. *Suppose φ is a locally integrable function on $\mathbb{R}$ which satisfies $\alpha(I) \equiv \operatorname{ess\,inf}_{u \in I} \varphi(u) > 0$ for every segment $I \subset \mathbb{R}$. Then a function $f \in L(Q_0)$ is equivalent to a continuous function on Q_0 if and only if f satisfies conditions* (5) *and*

$$\lim_{h \to +0} \sup_{|Q| \le h} J(\varphi, -f\chi_Q, |Q|) = 0. \tag{6}$$

Notice that the condition on φ in Theorem 1.5 is in general weaker than the condition in Corollary 1.4.

2. Proof of the main results

The proof of Theorems 1.1, 1.3 and 1.5 are based on the following lemma.

Lemma 2.1. *Suppose φ is a nonnegative, locally integrable function on $\mathbb{R}$. Then for every $g \in L(Q_0)$ the equality*

$$J(\varphi, g, r) = \int_{g_{**}(+0)}^{g_{**}(r)} \varphi(u) du, \quad 0 < r \le |Q_0| \tag{7}$$

holds.

Proof. Fix $r \in (0; |Q_0|]$. For almost every $t \in (0; r)$, we have

$$(g_{**})'(t) = -\frac{1}{t^2}\int_0^t g_*(s)ds + \frac{1}{t}g_*(t) = \frac{g_*(t) - g_{**}(t)}{t}.$$

Whence

$$J_\varepsilon(\varphi, g, r) = \int_\varepsilon^r \varphi(g_{**}(t))(g_{**})'(t)dt = \int_{g_{**}(\varepsilon)}^{g_{**}(r)} \varphi(u)du$$

for each $\varepsilon \in (0; r)$ (see [5, chapter VII, § 3, p. 320]). Taking limit as $\varepsilon \to +0$ in the last equality we obtain (7). □

Proof of Theorem 1.1. Combining (2) and (7) we conclude that

$$J(\varphi, f, |Q_0|) = \int_{f_{**}(+0)}^{f_{Q_0}} \varphi(u)du < \infty.$$

Thus by (1) we see that $f_{**}(+0) > -\infty$. In other words, f is bounded from below.

Conversely, let $f_{**}(+0) > -\infty$, then from (7) we get

$$J(\varphi, f, |Q_0|) = \int_{f_{**}(+0)}^{f_{Q_0}} \varphi(u)du < \infty,$$

since φ is locally integrable. □

Remark 2.2. If $J(\varphi, f, r_0) < \infty$ for some $r_0 \in (0; |Q_0|]$, then $J(\varphi, f, r) < \infty$ for any $r \in (0; |Q_0|]$. Therefore we may assume that f satisfies $J(\varphi, f, r_0) < \infty$ instead of (2).

Proof of Corollary 1.2. If f is bounded on Q_0, then f and $-f$ are bounded from below. So, by Theorem 1.1, (2) and (3) hold.

Conversely, by Theorem 1.1, if (2) and (3) hold, then f and $-f$ are bounded from below on Q_0. Consequently, f is bounded. □

Proof of Theorem 1.3. For every cube $Q \subset Q_0$, equality (7) gives

$$J(\varphi, f\chi_Q, |Q|) = \int_{(f\chi_Q)_{**}(+0)}^{(f\chi_Q)_{**}(|Q|)} \varphi(u)du \geq \alpha(\mathbb{R})L(f, Q) \tag{8}$$

and

$$J(\varphi, f\chi_Q, |Q|) \leq \beta(\mathbb{R})L(f, Q).$$

Whence we deduce that

$$\alpha(\mathbb{R}) \sup_{Q\subset Q_0} L(f, Q) \leq \sup_{Q\subset Q_0} J(\varphi, f\chi_Q, |Q|) \leq \beta(\mathbb{R}) \sup_{Q\subset Q_0} L(f, Q).$$

□

Proof of Corollary 1.4. For any cube $Q \subset Q_0$, note that

$$\Omega(f, Q) = \frac{2}{|Q|}\int_{\{x\in Q: f(x)<f_Q\}} (f_Q - f(x))dx \leq 2L(f, Q)$$

and from (8) observe that

$$\Omega(f,Q) \le \frac{2}{\alpha(\mathbb{R})} J(\varphi, f\chi_Q, |Q|).$$

Hence

$$\eta(f,h) \le \frac{2}{\alpha(\mathbb{R})} \sup_{|Q|\le h} J(\varphi, f\chi_Q, |Q|), \quad h>0.$$

Thus by (5) we obtain that $f \in VMO$. □

Proof of Theorem 1.5. Let f be equivalent to some continuous function. Without loss of generality, assume that f is continuous on Q_0. For an arbitrary cube $Q \subset Q_0$, we denote $m(f,Q) = \inf_{t\in Q} f(t)$, $M(f,Q) = \sup_{t\in Q} f(t)$.

Fix $\varepsilon > 0$ and choose $\delta > 0$ such that the inequality

$$\int_e \varphi(u)du \le \varepsilon \tag{9}$$

holds for every measurable set $e \subset (m(f,Q_0); M(f,Q_0))$ with $|e| \le \delta$. By uniform continuity of f on Q_0, there exists $\delta_1 > 0$ such that

$$M(f,Q) - m(f,Q) \le \delta, \quad |Q| \le \delta_1. \tag{10}$$

Using (7), we obtain

$$J(\varphi, f\chi_Q, |Q|) = \int_{m(f,Q)}^{f_Q} \varphi(u)du \le \int_{m(f,Q)}^{M(f,Q)} \varphi(u)du.$$

Therefore, on account of (9) and (10), $J(\varphi, f\chi_Q, |Q|) \le \varepsilon$ for any cube $Q \subset Q_0$ with $|Q| \le \delta_1$. Hence (5) holds. By continuity of the function $-f$, we have (6).

Now let f satisfy conditions (5) and (6). By Corollary 1.2, f is bounded on Q_0. Put $I_0 = (\operatorname{ess\,inf}_{t\in Q_0} f(t); \operatorname{ess\,sup}_{t\in Q_0} f(t))$. Replacing f by $-f$ in (8) we observe that

$$J(\varphi, -f\chi_Q, |Q|) \ge \alpha(I_0) L(-f,Q) = \alpha(I_0)(\operatorname*{ess\,sup}_{x\in Q} f(x) - f_Q) \tag{11}$$

for any cube $Q \subset Q_0$. Thus adding (8) and (11) we have

$$\operatorname*{ess\,sup}_{t\in Q} f(t) - \operatorname*{ess\,inf}_{t\in Q} f(t) \le \frac{1}{\alpha(I_0)} (J(\varphi, f\chi_Q, |Q|) + J(\varphi, -f\chi_Q, |Q|))$$

for each cube $Q \subset Q_0$. From (5) and (6) we get that the right-hand side of last inequality tends to zero as $|Q| \to 0$. Consequently, f is equivalent to some continuous function on Q_0. □

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