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ON THE RADICAL OF $\mathcal{FP}^+(S_n)$

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We prove that the semigroups $\mathfrak{B}_n$ of all binary relations and the factor-power $\mathcal{FP}^+(S_n)$ of the symmetric group can be asymptotically approximated by nilpotent semigroups. Further, we show that almost all elements of these semigroups satisfy the equation $x^2 = 0$, where 0 denotes the full binary relation. All these facts are obtained from a careful study of the radical $\mathfrak{R}_n$ in $\mathcal{FP}^+(S_n)$. Along this study we also derive some corollaries for doubly stochastic matrices.

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Доказано, что полугруппы $\mathfrak{B}_n$ всех бинарных отношений и фактор-степень $\mathcal{FP}^+(S_n)$ симметрической группы могут быть асиптотически приближены нильпотентными полугруппами. Далее, показано, что почти все элементы этих полугрупп удовлетворяют уравнению $x^2 = 0$, где 0 обозначает полное бинарное отношение. Все эти факты получены путем аккуратного изучения радикала $\mathfrak{R}_n$ в $\mathcal{FP}^+(S_n)$. В процессе этого изучения выводятся некоторые следствия для дважды стохастических матриц.

1. Introduction and the main result

Let T be a semigroup of (possibly partial) transformations on the set M. The Boolean $\mathcal{B}(T)$ inherits from T the natural structure of a semigroup. Moreover, the equivalence relation $\sim$ on $\mathcal{B}(T)$, defined as follows: $A \sim B$ if and only if $m^A = m^B$ for all $m \in M$, appears to be a congruence on $\mathcal{B}(T)$. The corresponding quotient $\mathcal{B}(T)/_\sim$ is usually denoted by $\mathcal{FP}(T, M)$. The equivalence class of the empty set is always the isolated zero in the semigroup $\mathcal{FP}(T, M)$, so we can form the semigroup $\mathcal{FP}^+(T, M) = \mathcal{FP}(T, M) \setminus \{\varnothing\}$, which is called *the factor power of* (T, M). Extending the action of the semigroup T on M to the Boolean $\mathcal{B}(M)$, we get a natural action of $\mathcal{FP}^+(T, M)$ on $\mathcal{B}(M)$.

This construction appeared first in [2] and was later studied in [3, 4, 7, 8] with the special emphasize on the case, when $T = S_n$, the full finite symmetric group with the natural action on the set $N = \{1, 2, \dots, n\}$. In particular, it was shown that $\mathcal{FP}^+(S_n)$ asymptotically approximates the semigroup $\mathfrak{B}_n$ of all binary relations on N, moreover, $\mathcal{FP}^+(S_n)$ has a very nice inner description inside $\mathfrak{B}_n$. The semigroup $\mathcal{FP}^+(S_n)$ appears naturally also as the quotient of the semigroup Ω_n of doubly stochastic real matrices. Several classes of subsemigroups, automorphisms and Green relations for $\mathcal{FP}^+(S_n)$ have been already described.

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In the present paper we continue the study of the semigroup $\mathcal{FP}^+(S_n)$ with the emphasize on its *radical* $\mathfrak{R}_n$, which is the intersection of all maximal nilpotent subsemigroups in $\mathcal{FP}^+(S_n)$ (see [4]). Alternatively, $\mathfrak{R}_n$ consists of all $\tau \in \mathcal{FP}^+(S_n)$, such that $|\tau(A)| > |A|$ for every proper subset $A \subset N$. The semigroup $\mathfrak{R}_n$ is nilpotent of nilpotency degree n ([4, Theorem 6]). The main result of the present paper is the following theorem, which seems to be quite surprising:

Theorem 1. *$\mathfrak{R}_n$ asymptotically approximates $\mathcal{FP}^+(S_n)$ and $\mathfrak{B}_n$, that is,*

$$\lim_{n\to\infty} \frac{|\mathfrak{R}_n|}{|\mathcal{FP}^+(S_n)|} = \lim_{n\to\infty} \frac{|\mathfrak{R}_n|}{|\mathfrak{B}_n|} = 1.$$

In other words, the semigroup $\mathfrak{B}_n$ is asymptotically approximated by a nilpotent semigroup. If one recalls, see [9, 12], that an arbitrary semigroup is isomorphic to a transitive semigroup of binary relations, then our results seem to be quite parallel with and somehow explain the main result in [11], where it was obtained that 99% of all semigroups of order 8 are nilpotent.

The paper is organized as follows: we prove our main theorem in Section 2. In Section 3 we discuss some corollaries, analogous results and other applications of the radical $\mathfrak{R}_n$. In particular, we derive a result about a kind of "row-column symmetric" behavior of doubly stochastic matrices. Finally, we formulate several conjectures and open problems in Section 4.

2. Proof of the main result

We have a natural chain of embeddings $\mathfrak{R}_n \subset \mathcal{FP}^+(S_n) \subset \mathfrak{B}_n$. As we have already mentioned in the Introduction, [3, Theorem 6] states that the semigroup $\mathcal{FP}^+(S_n)$ asymptotically approximates $\mathfrak{B}_n$. Hence, it is enough to prove that the radical $\mathfrak{R}_n$ asymptotically approximates $\mathcal{FP}^+(S_n)$. Actually, we are not going to calculate the necessary limit but rather would like to show that

$$\lim_{n\to\infty} \frac{|\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n|}{|\mathcal{FP}^+(S_n)|} = 0,$$

which would clearly imply the necessary statement.

Our aim now is to find an effective upper bound for the number of elements in $\mathcal{FP}^+(S_n)$ which do not belong to the radical. From the definition of the radical, an element, τ, belongs to $\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$ if and only if there exists $A \subset N$, $A \neq \varnothing, N$, such that $|\tau(A)| = |A|$. To get a convenient formula, we even will leave $\mathcal{FP}^+(S_n)$ and make the estimate in $\mathfrak{B}_n$. In the next lemma for $A \subset N$ and $\phi \in \mathfrak{B}_n$ we denote

$$\phi(A) = \{b \in N \ : \ \text{there exists } a \in A \text{ such that } a\phi b\}.$$

We call a binary relation, $\phi \in \mathfrak{B}_n$, *strongly invariant* provided that there exist $A, B \subset N$, $0 < |A| = |B| < n$, satisfying $\phi(A) \subset B$ and $\phi(\overline{A}) \subset \overline{B}$.

Lemma 1. *The number of strongly invariant binary relations $\phi \in \mathfrak{B}_n$ is less than or equal*

$$d_n = \sum_{i=1}^{n-1} \binom{n}{i}^2 2^{i^2+(n-i)^2}.$$

Proof. We denote $i = |A|$. Then we can choose A in $\binom{n}{i}$ different ways and B in $\binom{n}{i}$ different ways. Every binary relation ϕ, which we count, is, by assumptions, a subset of $(A \times B) \cup (\overline{A} \times \overline{B})$. But $|(A \times B) \cup (\overline{A} \times \overline{B})| = i^2 + (n-i)^2$ and to complete the proof one has to apply the multiplication rule and sum up over all i. □

Corollary 1. $|\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n| \leq d_n$.

Proof. Let $\tau \in \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$ and $A \subset N$, $A \neq \varnothing, N$, such that $|\tau(A)| = |A|$. Then we have $s(A) = A$ for every permutation $s \in \tau$ (here τ is considered as the maximal element in the corresponding equivalence class of $\mathcal{B}(S_n)$), and hence $s(\overline{A}) = \overline{A}$. This implies $\tau(\overline{A}) = \overline{A}$. Therefore τ is a strongly invariant binary relation and the statement follows from Lemma 1. □

Denote

$$d'_n = \binom{n}{1}^2 2^{1^2+(n-1)^2} + \binom{n}{n-1}^2 2^{(n-1)^2+1^2} = 2n^2 2^{n^2-2n+2}$$

and $d''_n = d_n - d'_n$.

Now we can compute the limit $\lim\limits_{n\to\infty} \frac{d_n}{2^{n^2}}$, using $\binom{n}{k} \leq 2^n$ and the fact that

$$i^2 + (n-i)^2 \leq 2^2 + (n-2)^2 = n^2 - 4n + 8$$

for all $n \geq 4$ and $2 \leq i \leq n-2$:

$$0 \leq \lim_{n\to\infty} \frac{d_n}{2^{n^2}} = \lim_{n\to\infty} \frac{d'_n}{2^{n^2}} + \lim_{n\to\infty} \frac{d''_n}{2^{n^2}} \leq$$

$$\leq 2 \lim_{n\to\infty} \frac{n^2 2^{n^2-2n+2}}{2^{n^2}} + \lim_{n\to\infty} \frac{\sum\limits_{i=2}^{n-2} \binom{n}{i}^2 2^{i^2+(n-i)^2}}{2^{n^2}} \leq$$

$$\leq \lim_{n\to\infty} n^2 2^{3-2n} + \lim_{n\to\infty} n 2^{2n} 2^{8-4n} = 0.$$

Hence

$$0 \leq \lim_{n\to\infty} \frac{|\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n|}{|\mathcal{FP}^+(S_n)|} \leq \lim_{n\to\infty} \frac{d_n |\mathfrak{B}_n|}{|\mathcal{FP}^+(S_n)||\mathfrak{B}_n|} = \lim_{n\to\infty} \frac{d_n}{2^{n^2}} \lim_{n\to\infty} \frac{|\mathfrak{B}_n|}{|\mathcal{FP}^+(S_n)|} = 0.$$

This completes the proof of Theorem 1.

3. Corollaries, applications and some other facts for the radical $\mathfrak{R}_n$

Let $B_n \subset A_n$, $n \in \mathbb{N}$, be two families of sets. We say that *almost all* elements of A_n belong to B_n if $|B_n|/|A_n| \to 1$, $n \to \infty$ (we refer the reader to [6] for the corresponding terminology in graph theory). The following statement contains several direct corollaries from Theorem 1.

Corollary 2. 1. *$\mathfrak{B}_n$ is asymptotically approximated by a nilpotent semigroup.*

2. *$\mathcal{FP}^+(S_n)$ is asymptotically approximated by a nilpotent semigroup.*
3. *Almost all elements in $\mathfrak{B}_n$ are nilpotent with respect to the full binary relation 0.*
4. *Almost all elements in $\mathcal{FP}^+(S_n)$ are nilpotent.*
5. *Almost all elements in $\mathcal{FP}^+(S_n)$ are contained in the radical.*
6. *The transitive closure of almost all oriented graphs coincides with the full oriented graph (with loops).*
7. *Almost all oriented graphs are connected.*
8. *Almost all oriented graphs are strongly connected.*

Quite amazing property of the radical $\mathfrak{R}_n$ obtained above is a good motivation to study $\mathfrak{R}_n$ in more details. We start this study with one more asymptotic property of the radical. We recall that the element 0 in $\mathcal{FP}^+(S_n)$ corresponds to the full binary relation $N \times N$ in $\mathfrak{B}_n$.

Proposition 1. *Almost all elements of $\mathfrak{R}_n$ (and hence of $\mathcal{FP}^+(S_n)$ and $\mathfrak{B}_n$) satisfy the equation $x^2 = 0$.*

Proof. Because of Theorem 1 and [3, Theorem 6], it is enough to prove the statement for $\mathfrak{B}_n$. We try to estimate the number of those binary relations ϕ for which $\phi^2 \neq 0$. If $\phi^2 \neq 0$, then there exist $x, y \in N$ (possibly $x = y$) such that ϕ^2 does not contain (x, y). First we assume $x \neq y$. That ϕ^2 does not contain (x, y) means that for every $z \in N$ the elements (x, z) and (z, y) cannot be contained in ϕ together, that is either both (x, z) and (z, y) do not belong to ϕ or (x, z) does not belong to ϕ or (z, y) does not belong to ϕ. Hence, the number of such ϕ does not exceed $3^n \cdot 2^{n^2-2n}$.

If $x = y$, then ϕ cannot contain (x, x) and for every $z \in N$ either both (x, z) and (z, x) do not belong to ϕ or (x, z) does not belong to ϕ or (z, x) does not belong to ϕ. Hence, the number of such ϕ does not exceed $3^{n-1} \cdot 2^{n^2-2n+1}$. Therefore the number of all binary relations ϕ, satisfying $\phi^2 \neq 0$, does not exceed $\binom{n}{2} \cdot 3^n \cdot 2^{n^2-2n} + n \cdot 3^{n-1} \cdot 2^{n^2-2n+1}$. Dividing by $|\mathfrak{B}_n| = 2^{n^2}$ we get

$$\frac{n(n-1) \cdot 3^n + n \cdot 3^{n-1} \cdot 4}{2^{2n+1}} = \frac{3^n(n(n-1) + \frac{4}{3}n)}{2^{2n+1}}.$$

Since $3^3 < 2^5$, we can rewrite and estimate the last expression in the following way:

$$\frac{3^n(n(n-1) + \frac{4}{3}n)}{2^{2n+1}} = \frac{3^n}{2^{5n/3}} \cdot \frac{(n(n-1) + \frac{4}{3}n)}{2 \cdot 2^{n/3}} < \frac{(n(n-1) + \frac{4}{3}n)}{2 \cdot 2^{n/3}} \to 0, \quad n \to \infty.$$

□

Since $\mathfrak{R}_n$ is a two-sided ideal of $\mathcal{FP}^+(S_n)$, we can consider the corresponding Rees congruence $\rho = \rho_{\mathfrak{R}_n}$ and the quotient $\mathcal{FP}^+(S_n)_\rho$ modulo this congruence. The semigroup $\mathcal{FP}^+(S_n)_\rho$ is quite small in comparison with $\mathcal{FP}^+(S_n)$. Indeed, from Theorem 1 it follows immediately that

$$\lim_{n\to\infty} \frac{|\mathcal{FP}^+(S_n)_\rho|}{|\mathcal{FP}^+(S_n)|} = 0.$$

The natural embedding $S_n \subset S_{n+1}$, which fixes $n+1$, extends to the natural embedding $\mathcal{FP}^+(S_n) \subset \mathcal{FP}^+(S_{n+1})$. It is obvious that the image of $\mathcal{FP}^+(S_n)$ under this embedding does not intersect $\mathfrak{R}_{n+1}$ and hence $|\mathcal{FP}^+(S_n)_\rho| > |\mathcal{FP}^+(S_n)|$. Hence the semigroup $\mathcal{FP}^+(S_n)_\rho$ grows quicker than the radical of the previous factor power. At the same time most of properties can be easily translated from $\mathcal{FP}^+(S_n)$ to $\mathcal{FP}^+(S_n)_\rho$ (using the identification of non-zero elements from $\mathcal{FP}^+(S_n)_\rho$ with the corresponding elements in $\mathcal{FP}^+(S_n)$). In particular, we get the following.

Theorem 2. *1. The idempotents in $\mathcal{FP}^+(S_n)_\rho$ are equivalence relations.*

2. *$\mathcal{FP}^+(S_n)_\rho$ and $\mathcal{FP}^+(S_n)$ have the same maximal subgroups.*
3. *The Green relations on $\mathcal{FP}^+(S_n)_\rho$ are induced from $\mathcal{FP}^+(S_n)$.*
4. *There is a natural bijection between the nilpotent subsemigroups in $\mathcal{FP}^+(S_n)_\rho$ and the nilpotent subsemigroups in $\mathcal{FP}^+(S_n)$ which contain $\mathfrak{R}_n$. This bijection preserves inclusion, in particular, for $n > 2$ there is a natural bijection between the maximal nilpotent subsemigroups in $\mathcal{FP}^+(S_n)_\rho$ and the maximal nilpotent subsemigroups in $\mathcal{FP}^+(S_n)$.*
5. *All automorphisms of $\mathcal{FP}^+(S_n)_\rho$ are inner.*

Proof. The first statement follows from the fact that $\mathfrak{R}_n$ is nilpotent and hence contains exactly one idempotent, which is 0. The second statement follows from the observation that all elements of $\mathfrak{R}_n$ are nilpotent and hence $\mathfrak{R}_n$ does not intersect any maximal subgroup corresponding to a non-zero idempotent.

The third statement follows from the fact that $\mathcal{FP}^+(S_n)_\rho$ is a Rees factor of $\mathcal{FP}^+(S_n)$.

The forth statement follows from the definition of $\mathfrak{R}_n$ as the intersection of all maximal nilpotent subsemigroups of $\mathcal{FP}^+(S_n)$.

The last statement can be obtained *mutatis mutandis* from [7] if one remarks that all arguments used in [7] can actually be applied to the set $\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. □

We refer the reader to [8] and [4] for the description of Green relations and maximal nilpotent subsemigroups of $\mathcal{FP}^+(S_n)$ respectively. By Theorem 2, the same statements hold for the semigroup $\mathcal{FP}^+(S_n)_\rho$ as well.

We would like to finish this section with one more property of the radical and an application to the doubly stochastic matrices. Recall that there exists the canonical anti-involution $*$ on $\mathcal{FP}^+(S_n)$, defined for the elements $\tau \in \mathcal{FP}^+(S_n)$, $\tau = \overline{A} \in \mathcal{B}(S_n)/_\sim$, as follows:

$$(\overline{A})^* = \overline{\{a^{-1} : a \in A\}}.$$

Lemma 2. $\mathfrak{R}_n^* = \mathfrak{R}_n$.

Proof. Since $*$ is an anti-involution, it is bijective on $\mathcal{FP}^+(S_n)$, and hence it is enough to prove that $(\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n)^* = \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. Let $\tau = \overline{A} \in \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. Then there exists a proper subset, $T \subset N$, such that $|\overline{A}(T)| = |T|$. Set $T' = \tau(T)$. Then for every $a \in A$ we have $a(T) = T'$ and hence $a^{-1}(T') = T$. Therefore $(\overline{A})^*(T') = T$. In particular, $|(\overline{A})^*(T')| = |T'|$ and thus $(\overline{A})^* \in \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. □

Recall that an $n \times n$ real matrix, M, is called *doubly stochastic* provided that it has non-negative entries and in every row and in every column the sum of all elements is equal to 1. The set of all doubly stochastic matrices is a semigroup under the usual matrix multiplication. We say that two doubly stochastic matrices are equivalent if their entries are equal to zero simultaneously. It was shown in [3, Theorem 2] that this equivalence is, in fact, a congruence and that the corresponding quotient semigroup is isomorphic to $\mathcal{FP}^+(S_n)$. We say that a doubly stochastic matrix, M, *satisfies the C-condition*, if for every positive integer k and every $1 \leq j \leq n$ the number of zeros in the j-th column in the matrix M^k is either zero or strictly less than the number of zeros in the j-th column in the matrix M^{k-1}. Analogously we define *the R-condition* with respect to the rows of M.

Lemma 3. *A doubly stochastic matrix, M, belongs to the equivalence class corresponding to some element in $\mathfrak{R}_n$, if and only if M satisfies the C-condition.*

Proof. This follows from the fact that the natural action of $\mathcal{FP}^+(S_n)$ on N is coordinated with the natural action of doubly stochastic matrices on vectors from $\mathbb{R}^n_+$. □

Corollary 3. *A doubly stochastic matrix, M, satisfies the C-condition if and only if it satisfies the R-condition.*

Proof. According to Lemma 3, the matrix M satisfies the C-condition if and only if it belongs to the equivalence class corresponding to an element from $\mathfrak{R}_n$. The anti-involution $*$ naturally extends to the transposition of doubly stochastic matrices. The transposition of matrices interchange rows and columns and preserves classes which correspond to $\mathfrak{R}_n$, by Lemma 2. Now the statement follows from Lemma 3, applied to M^t. □

4. Problems and conjectures

We would like to finish the paper with a list of several problems and conjectures related to the radical $\mathfrak{R}_n$ and the results of this paper. The first conjecture is very natural and is motivated by Theorem 1 and the results from [11]. For a positive integer, n, let us denote by a_n the number of isomorphism classes of semigroups with n elements, and by b_n the number or the isomorphism classes of nilpotent semigroups with n elements. We also denote by a'_n the total number of semigroups on N, and by b'_n the total number of nilpotent semigroups on N of nilpotency degree at most 3. Clearly $b_n < a_n$ and $b'_n < a'_n$.

Conjecture 1. $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = 1$. *In other words, almost all finite semigroups are nilpotent.*

Conjecture 1 is an "up-to-isomorphism" version of the following conjecture, mentioned in [11] with the reference to [5], where a stronger statement (for nilpotent semigroups of nilpotency degree 3) is formulated as a theorem, but the proof is only outlined.

Conjecture 2. $\lim\limits_{n\to\infty} \dfrac{b'_n}{a'_n} = 1$.

By [11], the ratio $\tilde{b}_8/\tilde{a}_8$, where the corresponding numbers are counted up to isomorphism and anti-isomorphism, is approximately $\frac{99}{100}$.

Further, each finite semigroup S is a subsemigroup in some $\mathfrak{B}_n$. One can hardly expect that the classical representation of S in $\mathfrak{B}_n$ will be related to the radical. However, as it was shown in [9], every finite semigroup is isomorphic to a transitive semigroup of binary relations on a finite set. Further the transitive semigroups are usually contained in the radical and thus it is natural to expect that almost all S will be represented inside $\mathfrak{N}_n$ and therefore will be nilpotent.

The positive answer to Conjecture 1 would clearly imply the positive answer to the following statement, which however can be viewed as a separate claim.

Conjecture 3. *Let b''_n denote the number of the isomorphism classes of semigroups with n elements having exactly 1 idempotent. Then $\lim\limits_{n\to\infty}\frac{b''_n}{a_n}=1$.*

The next problem is a natural continuation of the previous one. Conjecture 1 states that almost all semigroups are nilpotent and a motivation for this is that $\mathfrak{B}_n$ contains a very big nilpotent subsemigroup $\mathfrak{N}_n$. It is then natural to expect that nilpotent subsemigroups can be represented inside $\mathfrak{N}_n$.

Problem 1. *Which finite nilpotent subsemigroups embed into $\mathfrak{N}_n$ for some n?*

The following conjecture represents our point of view on this problem, however, we do not have so much evidence for it as in the case of Conjecture 1.

Conjecture 4. *All finite nilpotent subsemigroups embed into $\mathfrak{N}_n$.*

Now we formulate several problems and conjectures about $\mathfrak{N}_n$. The semigroup $\mathfrak{N}_n$ is nilpotent and hence it has trivial (1-element) classes for all Green relations. However, inside $\mathcal{FP}^+(S_n)$ the picture is quite different. Of course, $\mathfrak{N}_n$, being a two-sided ideal, is still closed under Green relations, but now it is possible that even $\mathcal{H}$-classes in $\mathfrak{N}_n$ are quite big. Here are the Green classes for the semigroup $\mathcal{FP}^+(S_3)$ (we set $N=\{1,2,3\}$, $\mathcal{H}^i_{k,l}=\mathcal{L}^i_k\cap\mathcal{R}^i_l$, and remark that $\mathcal{J}=\mathcal{D}$):

$$\mathcal{D}^1=\mathcal{R}^1_1=\mathcal{L}^1_1=\mathcal{H}^1_{1,1}=\left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\};$$

$\mathcal{D}^2$:

	$\mathcal{R}^2_1$	$\mathcal{R}^2_2$	$\mathcal{R}^2_2$
$\mathcal{L}^2_1$	$\left\{\begin{pmatrix}1&2&3\\1,2&1,2&3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,2&3&1,2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\3&1,2&1,2\end{pmatrix}\right\}$
$\mathcal{L}^2_2$	$\left\{\begin{pmatrix}1&2&3\\2,3&2,3&1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\2,3&1&2,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1&2,3&2,3\end{pmatrix}\right\}$
$\mathcal{L}^2_3$	$\left\{\begin{pmatrix}1&2&3\\1,3&1,3&2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,3&2&1,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\2&1,3&1,3\end{pmatrix}\right\}$

$\mathcal{D}^3$:

	$\mathcal{R}^3_1$	$\mathcal{R}^3_2$	$\mathcal{R}^3_2$
$\mathcal{L}^3_1$	$\left\{\begin{pmatrix}1&2&3\\1,2&N&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,2&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&N&1,2\end{pmatrix}\right\}$
$\mathcal{L}^3_2$	$\left\{\begin{pmatrix}1&2&3\\2,3&N&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&2,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&N&2,3\end{pmatrix}\right\}$
$\mathcal{L}^3_3$	$\left\{\begin{pmatrix}1&2&3\\1,3&N&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&N&1,3\end{pmatrix}\right\}$

$$\mathcal{D}^4=\mathcal{R}_1^4=\mathcal{L}_1^4=\mathcal{H}_{1,1}^4=\left\{\begin{pmatrix}1&2&3\\1,2&2,3&3,1\end{pmatrix},\begin{pmatrix}1&2&3\\2,3&1,2&3,1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1&2&3\\3,1&2,3&1,2\end{pmatrix},\begin{pmatrix}1&2&3\\1,2&3,1&2,3\end{pmatrix},\begin{pmatrix}1&2&3\\2,3&3,1&1,2\end{pmatrix},\begin{pmatrix}1&2&3\\3,1&1,2&2,3\end{pmatrix}\right\};$$

$\mathcal{D}^5$:

	$\mathcal{R}_1^5$	$\mathcal{R}_2^5$	$\mathcal{R}_2^5$
$\mathcal{L}_1^5$	$\left\{\begin{pmatrix}1&2&3\\1,2&2,3&N\end{pmatrix},\begin{pmatrix}1&2&3\\2,3&1,2&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,2&N&2,3\end{pmatrix},\begin{pmatrix}1&2&3\\2,3&N&1,2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,2&2,3\end{pmatrix},\begin{pmatrix}1&2&3\\N&2,3&1,2\end{pmatrix}\right\}$
$\mathcal{L}_2^5$	$\left\{\begin{pmatrix}1&2&3\\1,3&1,2&N\end{pmatrix},\begin{pmatrix}1&2&3\\1,2&1,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,3&N&1,2\end{pmatrix},\begin{pmatrix}1&2&3\\1,2&N&1,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,3&1,2\end{pmatrix},\begin{pmatrix}1&2&3\\N&1,2&1,3\end{pmatrix}\right\}$
$\mathcal{L}_3^5$	$\left\{\begin{pmatrix}1&2&3\\1,3&2,3&N\end{pmatrix},\begin{pmatrix}1&2&3\\2,3&1,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,3&N&2,3\end{pmatrix},\begin{pmatrix}1&2&3\\2,3&N&1,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,3&2,3\end{pmatrix},\begin{pmatrix}1&2&3\\N&2,3&1,3\end{pmatrix}\right\}$

$$\mathcal{D}^6=\mathcal{R}_1^6=\mathcal{L}_1^6=\mathcal{H}_{1,1}^6=\left\{\begin{pmatrix}1&2&3\\N&N&N\end{pmatrix}\right\}.$$

For $\mathcal{FP}^+(S_3)$ we have $\mathfrak{R}_3=\mathcal{D}^3\cup\mathcal{D}^4\cup\mathcal{D}^5\cup\mathcal{D}^6$ and we see that even the $\mathcal{H}$-classes of $\mathfrak{R}_3$, regarded as a semigroup in $\mathcal{FP}^+(S_3)$, can be non-trivial. Actually, it is easy to see that in this case we have only 10 elements in $\mathfrak{R}_3$ forming trivial $\mathcal{H}$-classes. One can also remark that all elements from $\mathfrak{R}^3$ satisfy $x^2=0$.

Problem 2. *Let c_n denote the number of those elements in $\mathfrak{R}_n$ which form trivial $\mathcal{H}$-classes in $\mathcal{FP}^+(S_n)$. Find the asymptotic $\frac{c_n}{|\mathfrak{R}_n|}$, $n\to\infty$.*

The following question is closely related to this problem.

Problem 3. *Let τ be an element of $\mathcal{FP}^+(S_n)$ (in particular, of $\mathfrak{R}_n$). Find the cardinality of the $\mathcal{H}$-class (resp. $\mathcal{L}$-class, $\mathcal{R}$-class or $\mathcal{D}$-class) containing the element τ.*

The next conjecture is natural after Proposition 1, where it was shown that almost all elements in $\mathfrak{R}_n$ satisfy $x^2=0$.

Conjecture 5. *Almost all pairs (x,y) of elements from $\mathfrak{R}_n$ satisfy $xy=0$.*

Certainly, the elements satisfying $x^2=0$ do not form a semigroup of nilpotency degree 2. Denote by f_n the maximal cardinality among all nilpotent subsemigroups in $\mathfrak{R}_n$ of nilpotency degree 2.

Problem 4. *Find the asymptotic $\frac{f_n}{|\mathfrak{R}_n|}$, $n\to\infty$.*

Our last collection of problems is related to the study of the so-called *cross-sections* for semigroups. Let S be a semigroup and $\sim$ an equivalence relation on S. A $\sim$*-cross-section* is a subsemigroup, T, of S, containing exactly one element from every equivalence class with respect to $\sim$. The most important are cross-sections with respect to the Green relations, see [1, 10] and references therein.

Problem 5. *Describe all $\mathcal{H}$- and $\mathcal{D}$-cross-sections of $\mathcal{FP}^+(S_n)$.*

Problem 6. *Denote by $\sim_{\mathcal{H}}$ and $\sim_{\mathcal{D}}$ the equivalence relations on $\mathfrak{N}_n$ obtained by restricting to $\mathfrak{N}_n$ the Green relations $\mathcal{H}$ and $\mathcal{D}$ on $\mathcal{FP}^+(S_n)$ respectively. Describe all $\sim_{\mathcal{H}}$- and $\sim_{\mathcal{D}}$-cross-sections of $\mathfrak{N}_n$.*

Connected to these problems is the following general question:

Problem 7. *Describe all congruences ρ on $\mathcal{FP}^+(S_n)$ (resp. $\mathfrak{N}_n$) and determine for which ρ there exist ρ-cross-sections (retracts) in $\mathcal{FP}^+(S_n)$ (resp. $\mathfrak{N}_n$).*

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