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ON ENTIRE DIRICHLET SERIES BOUNDED IN A STRIP

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We establish a condition on exponents of an entire Dirichlet series under which its sum $F \equiv 0$ provided that F is bounded in a strip.

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Найдено условие на показатели целого ряда Дирихле, при котором его сумма $F \equiv 0$, если только F ограничена в полосе.

For an entire function $f(z) = \sum_{n=1}^{\infty} f_n z^{\lambda_n}$ a problem of conditions on λ_n under which $f \not\equiv 0$ and f is bounded on a ray is investigated by J. Anderson and K. Binmore [1]. V. A. Martirosyan [2] investigated boundedness of such function in an angle. Analogues of Anderson-Binmore results for entire Dirichlet series

$$F(z) = \sum_{n=1}^{\infty} d_n e^{z\lambda_n}, \quad z = x + iy, \tag{1}$$

are obtained in [3–7]. Complete analogues of Martirosyan results are unknown in spite of the investigations in [6]. In the present paper we obtain counterparts of such results for Dirichlet series. The method of proof is based on Carleman's formula [8]. We note that the similar arguments is used, for example, in [3].

So, let $0 < \sigma < +\infty$, $D_\sigma = \{z : |\operatorname{Im} z| \le \sigma\}$, $0 < \lambda_n \uparrow +\infty$ and Dirichlet series (1) has the abscissa of absolute convergence $A = +\infty$. We put

$$S(r) = \sum_{\lambda_n \le r} \left(\frac{1}{\lambda_n} - \frac{\lambda_n}{r^2} \right), \quad S_0(r) = \sum_{\lambda_n \le r} \frac{1}{\lambda_n}, \quad m_F(x) = \sum_{n=1}^{\infty} |d_n| e^{x\lambda_n}.$$

The following result is main in the paper.

Theorem. *If a function $F \not\equiv 0$ is bounded in D_σ, then there exists $c_1 > 0$ such that for all $r \ge 1$ and all $\tau \ge 1$*

$$\sum_{\lambda_n - \lambda_1/2 \le r} \left(\frac{1}{\lambda_n - \lambda_1/2} - \frac{\lambda_n - \lambda_1/2}{r^2} \right) \ge \frac{\sigma}{\pi} \ln r + \frac{\tau}{2} - \frac{2}{\pi r} \left(\ln^+ m_F(\tau) - \frac{\lambda_1 \tau}{2} \right) + c_1. \tag{2}$$

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Proof. We put $\lambda_0 = \lambda_1/2$ and

$$G(z) = \int\limits_{iv-\infty}^{iv+\infty} F(w)\exp\{-w(\lambda_0+z)\}dw, \quad w = u+iv. \tag{3}$$

Clearly,

$$G(x+iy) \le e^{vy} \int\limits_{-\infty}^{+\infty} |F(u+iv)|\exp\{-u(\lambda_0+x)\}du$$

and, since $|F(u+iv)| = O(e^{u\lambda_1})\,(u\to-\infty)$ and $|F(u+iv)| = O(1)\,(u\to+\infty)$ for all v, $|v|\le\sigma$, the integral in (3) convergent absolutely and uniformly on each compact set from the strip $\{z : -\lambda_0 \le \operatorname{Re} z \le \lambda_1-\lambda_0\}$ and depends on $v\in[-\sigma,\,\sigma]$. Therefore, the function G is analytic in this strip. Moreover, choosing $v=-\sigma$ if $y>0$, and $v=\sigma$ if $y\le 0$, we have $|G(iy)| \le c_2\exp\{-\sigma|y|\}$ for all $y\in\mathbb{R}$ and some constant $c_2>0$.

For an arbitrary $\tau\in(0,\,+\infty)$ we write $G(z) = G_1(z,\tau)+G_2(z,\tau)$, where

$$G_1(z,\tau) = \int\limits_{iv-\infty}^{iv+\tau} F(w)\exp\{-w(\lambda_0+z)\}dw, \quad G_2(z,\tau) = \int\limits_{iv+\tau}^{iv+\infty} F(w)\exp\{-w(\lambda_0+z)\}dw.$$

Since for fixed $v\in[-\sigma,\,\sigma]$

$$G_1(z,\tau) = \sum_{n=1}^{\infty} d_n \int\limits_{iv-\infty}^{iv+\tau} \exp\{-w(\lambda_0+z-\lambda_n)\}dw = \sum_{n=1}^{\infty} \frac{d_n\exp\{(\tau+iv)(\lambda_n-\lambda_0-z)\}}{\lambda_n-\lambda_0-z},$$

the function $G_1(z,\tau)$ is meromorphic with poles $z_n=\lambda_n-\lambda_0$ and

$$|G_1(z,\tau)| \le \sum_{n=1}^{\infty} \frac{|d_n|\exp\{\tau(\lambda_n-\lambda_0)+vy\}}{|y|}.$$

The function $G_2(z,\tau)$ is analytic in the half-plane $\{z : \operatorname{Re} z > -\lambda_0\}$ and

$$|G_2(z,\tau)| \le c_3\exp\{-\tau(\lambda_0+x)+vy\}, \quad \operatorname{Re} z\ge 0.$$

Therefore, the function G is meromorphic in $\{z : \operatorname{Re} z > -\lambda_0\}$, $|G(iy)|\le c_2\exp\{-\sigma|y|\}$ for all $y\in\mathbb{R}$ and

$$|G(z)| \le \left(\frac{m_F(\tau)}{|y|}+\frac{c_3}{\lambda_0}\right)\exp\{-\sigma|y|-\tau(x+\lambda_0)\}, \quad \operatorname{Re} z\ge 0.$$

Hence, using Carleman's formula [8], we obtain

$$-\sum_{\lambda_n-\lambda_0\le r}\left(\frac{1}{\lambda_n-\lambda_0}-\frac{\lambda_n-\lambda_0}{r^2}\right) \le \frac{1}{2\pi}\int\limits_1^r\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|G(it)|\,dt +$$

$$+\frac{1}{2\pi}\int\limits_{-r}^{-1}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|G(-it)|\,dt + \frac{1}{\pi r}\int\limits_{-\pi/2}^{\pi/2}\ln|G(re^{i\theta})|\cos\theta d\theta \le$$

$$\le -\frac{\sigma}{\pi}\ln r - \frac{\tau}{2} + \frac{2}{\pi r}\left(\ln^+ m_F(\tau)-\lambda_0\tau\right)+c_1.$$

□

As in [3], from Theorem we obtain some corollaries.

Corollary 1. *If* $\sum\limits_{n=1}^{\infty}\lambda_n^{-2}<+\infty$,

$$\varliminf_{r\to+\infty}\left(S(r)-\frac{\sigma}{\pi}\ln r\right)=-\infty \tag{4}$$

and F is bounded in D_σ then $F\equiv 0$.

Indeed, if $F\not\equiv 0$ then by Theorem with $\tau=1$ we have

$$\sum_{\lambda_n-\lambda_0\le r}\left(\frac{1}{\lambda_n-\lambda_0}-\frac{\lambda_n-\lambda_0}{r^2}\right)\le\frac{\sigma}{\pi}\ln r+O(1),\quad r\to+\infty,\quad \lambda_0=\lambda_1/2. \tag{5}$$

But

$$\frac{1}{\lambda_n-\lambda_0}=\frac{1}{\lambda_n}+\frac{1}{\lambda_n(\lambda_n-\lambda_0)}$$

and

$$\sum_{\lambda_n-\lambda_0\le r}\frac{\lambda_n-\lambda_0}{r^2}=\sum_{\lambda_n\le\lambda_0+r}\frac{\lambda_n}{r^2}+O(1),\quad r\to+\infty,$$

because in view of the condition $\sum_{n=1}^{\infty}\lambda_n^{-2}<+\infty$ we have $n(r)/r^2\to 0\,(r\to+\infty)$, where $n(r)=\sum_{\lambda_n\le r}1$. Therefore, condition (5) is equivalent to the condition $S(r)\ge\dfrac{\sigma}{\pi}\ln r+O(1)\,(r\to+\infty)$, which contradicts to (4).

Corollary 2. *If*

$$\varliminf_{r\to+\infty}\frac{S_0(r)}{\ln r}<\frac{\sigma}{\pi} \tag{6}$$

and F is bounded in D_σ then $F\equiv 0$.

Indeed, from (6) it follows that $\sum_{n=1}^{\infty}\lambda_n^{-2}<+\infty$ and

$$\varliminf_{r\to+\infty}\frac{S(r)}{\ln r}=\varliminf_{r\to+\infty}\frac{S_0(r)}{\ln r}<\frac{\sigma}{\pi},$$

and by Corollary 1 $F\equiv 0$.

Corollary 3. *Let $\ln m_f(x)\le c_1e^{\alpha x}$ for all $x\in\mathbb{R}$ and some $c_1>0$ and $\alpha>0$. If*

$$\sum_{n=1}^{\infty}\lambda_n^{-2}<+\infty,$$

$$\varliminf_{r\to+\infty}\left(S(r)-\left(\frac{\sigma}{\pi}+\frac{1}{2\alpha}\right)\ln r\right)=-\infty \tag{7}$$

and F is bounded in D_σ then $F\equiv 0$.

Indeed, if $F \not\equiv 0$ then by Theorem, as in the proof of Corollary 1, we have

$$\sum_{\lambda_n \le \lambda_0 + r} \left(\frac{1}{\lambda_n} - \frac{\lambda_n}{r^2} \right) \ge \frac{\sigma}{\pi} \ln r - \frac{\tau}{2} - \frac{2}{\pi r} (c_1 e^{\alpha\tau} - \lambda_0 \tau) + O(1), \quad r \to +\infty,$$

whence for $\tau = \frac{1}{\alpha} \ln \left(\frac{\pi r}{4 c_1 \alpha} - \frac{\lambda_0}{c_1 \alpha} \right)$ we obtain

$$\sum_{\lambda_n \le \lambda_0 + r} \left(\frac{1}{\lambda_n} - \frac{\lambda_n}{r^2} \right) \ge \frac{\sigma}{\pi} \ln r + \frac{1}{2\alpha} \ln r + O(1), \quad r \to +\infty, \tag{8}$$

that is we have a contradiction to (7).

Corollary 4. *Let* $\ln m_f(x) \le c_1 e^{(\alpha+\varepsilon)x}$ *for all* $x \in \mathbb{R}$ *and some* $c_1 > 0$, $\alpha \ge 0$ *and* $\varepsilon > 0$. *If* $\sum_{n=1}^{\infty} \lambda_n^{-2} < +\infty$,

$$\varliminf_{r \to +\infty} \frac{S(r)}{\ln r} < \frac{\sigma}{\pi} + \frac{1}{2\alpha} \tag{9}$$

and F *is bounded in* D_σ *then* $F \equiv 0$.

Indeed, if we choose $\tau = \frac{1}{\alpha+\varepsilon} \ln \frac{\pi r + 4\lambda_0}{c_1(\alpha+\varepsilon)}$ then from (2) we obtain (8), whence a contradiction to (9) follows.

We remark that if λ_n are positive integers then in view of results from [2] conditions (4), (6), (7) and (9) are necessary in order that corresponding statements are true.

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