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T. M. Radul

A NORMAL FUNCTOR BASED ON THE HARTMAN-MYCIELSKI CONSTRUCTION

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We construct a normal functor in the category $\mathcal{C}omp$ based on the Hartman-Mycielski construction

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Построен нормальный функтор в категории $\mathcal{C}omp$, основанный на конструкции Хартмана-Мыцельского.

0. The general theory of functors acting in the category $\mathcal{C}omp$ of compact Hausdorff spaces (compacta) and continuous mappings was founded by E.V. Shchepin [1]. He distinguished some elementary properties of such functors and defined the notion of normal functor that has become very fruitful. The class of normal and closed to them functors includes many classical constructions: the hyperspace exp, the space of probability measures P, the superextension λ, the space of inclusion hyperspaces G and many other functors ([2], [3]).

Let X be a space and d a bounded by 1 admissible metric on X. By $HM(X)$ we denote the space of all maps from $[0,1]$ to the space X such that $f|[t_i,t_{i+1}] \equiv \text{const}$ for some $0 = t_0 \le \dots \le t_n = 1$ considered with the metric

$$d_{HM}(f,g) = \int_0^1 d(f(t),g(t))dt, \qquad f,g \in HM(X).$$

The construction $HM(X)$ is known as the Hartman-Mycielski construction [4].

For every $Z \in |\mathcal{C}omp|$ consider

$$HM_n(Z) = \Big\{ f \in HM(Z) \mid \text{there exist } 0 = t_1 < \dots < t_{n+1} = 1 \\ \text{with } f|[t_i,t_{i+1}) \equiv z_i \in Z, i \in \{1,\dots,n\} \Big\}.$$

Let $\mathcal{U}$ be the unique uniformity of Z. For every $U \in \mathcal{U}$ and $\varepsilon > 0$, let $\langle \alpha, U, \varepsilon \rangle = \big\{ \beta \in HM_n(Z) \mid m\{t \in [0,1) \mid (\alpha(t), \beta(t')) \notin U\} < \varepsilon \big\}$. The sets $\langle \alpha, U, \varepsilon \rangle$ form a base of a compact Hausdorff topology in $HM_n Z$.

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Given a map $f\colon X \to Y$ in $\mathcal{C}omp$, define a map $HM_nX \to HM_nY$ by the formula $HM_nF(\alpha) = f \circ \alpha$. The functor HM_n is normal in $\mathcal{C}omp$ [5, 2.5.2].

For $X \in |\mathcal{C}omp|$ we consider the space HMX with the topology described above. Generally, HMX is not compact. Zarichnyi has asked if there exist a normal functor in $\mathcal{C}omp$ which has all the functors HM_n as subfunctors [5]. The aim of this paper is to construct such a functor.

The paper is organized as follows: in Section 1 for each $X \in |\mathcal{C}omp|$ we build some compactification HX of the space HMX and show that H is a functor in $\mathcal{C}omp$, in Section 2 we prove that H is normal.

1. Let $X \in |\mathcal{C}omp|$. By CX we denote the Banach space of all continuous functions $\phi\colon X \to \mathbb{R}$ with the usual sup-norm: $\|\phi\| = \sup\{|\phi(x)| \mid x \in X\}$. We denote by I the segment $[0,1]$. We shall also use the notation $C(X,I)$ for the subspace of $C(X)$ consisting of all functions with the codomain I.

For $X \in |\mathcal{C}omp|$, let us define an uniformity on HMX. For each $\phi \in C(X)$ and $a, b \in [0,1]$ with $a < b$ we define the function $\phi_{(a,b)}\colon HMX \to \mathbb{R}$ by the formula $\phi_{(a,b)} = \frac{1}{(b-a)} \int_a^b \phi \circ \alpha(t)dt$. Put $S_{HM}(X) = \{\phi_{(a,b)} \mid \phi \in C(X) \text{ and } (a,b) \subset [0,1)\}$.

For $\phi_1, \dots, \phi_n \in S_{HM}(X)$ define a pseudometric $\rho_{\phi_1,\dots,\phi_n}$ on HMX by the formula $\rho_{\phi_1,\dots,\phi_n}(f,g) = \max\bigl\{|\phi_i(f) - \phi_i(g)| \mid i \in \{1,\dots,n\}\bigr\}$ where $f, g \in HMX$.

Theorem 1. *The family of pseudometrics* $\mathcal{P} = \{\rho_{\phi_1,\dots,\phi_n} \mid n \in \mathbb{N}$ *and* $\phi_1,\dots,\phi_n \in S_{HM}(X)\}$ *determines a totally bounded uniformity* $\mathcal{U}_{HMX}$ *on* HMX.

Proof. For $\phi_1,\dots,\phi_m \in S_{HM}(X)$ and $\psi_1,\dots,\psi_n \in S_{HM}(X)$ we have

$$\max\{\rho_{\phi_1,\dots,\phi_m}, \rho_{\psi_1,\dots,\psi_n}\} = \rho_{\phi_1,\dots,\phi_m,\psi_1,\dots,\psi_n}.$$

Consider any $f, g \in HMX$ with $f \neq g$. Then there exists an interval $(a,b) \subset [0,1)$ such that $f|(a,b) = x_1 \neq x_2 = g|(a,b)$. Choose a function $\phi \in C(X)$ such that $\phi(x_1) = 0$ and $\phi(x_2) = 1$. Then we have $\rho_{\phi_{(a,b)}}(f,g) = 1 > 0$.

Hence the family $\mathcal{P}$ defines an uniformity in the set HMX. Let us show that this uniformity generates the topology of HMX.

Consider any neighborhood $\langle f, U, \varepsilon\rangle$ of $f \in HMX$. Let $0 = t_0 < t_1 < \dots < t_k = 1$ such that $f|[t_{i-1}, t_i) = x_i$ for $x_i \in X$ and $i \in \{1,\dots,k\}$. Put $V_{x_i} = \{y \in X \mid (x_i, y) \in U\}$. Choose a function $\phi_i \in C(X,I)$ such that $\phi_i(x_i) = 0$ and $\phi_i|X \setminus V_{x_i} = 1$. It is easy to check that $V_f = \{g \in HMX \mid \rho_{\phi_{1(t_0,t_1)},\dots,\phi_{k(t_{k-1},t_k)}}(g,f) < \varepsilon\} \subset \langle f, U, \varepsilon\rangle$.

Now let us consider any $f \in HMX$ and any neighborhood V_f in the topology generated by the uniformity $\mathcal{U}_{HMX}$. We can suppose that $V_f = \{g \mid \rho_{\phi_{(a,b)}}(f,g) < \varepsilon\}$ where $\phi \in C(X)$. Define an entourage of the diagonal in $X \times X$ as $U = \{(x,y) \mid |\phi(x) - \phi(y)| < \varepsilon\}$. One can check that $\langle f, U, \varepsilon(b-a)/2 \operatorname{diam} \phi(X)\rangle \subset V_f$.

The totally boundedness can be proved using arguments from [6, 8.3.4]. □

Lemma 1. *For each compactum* X *of infinite weight we have* $w(\mathcal{U}_{HMX}) \leq w(X)$.

Proof. There exists a dense subset $\mathcal{D} \subset C(X)$ such that $|\mathcal{D}| \leq w(X)$ [2]. Put $\mathcal{D}_{HM} = \{\phi_{(a,b)} \mid \phi \in \mathcal{D} \text{ and } a, b \in [0,1] \cap \mathbb{Q}\}$. Let us consider the family of entourages $\mathcal{B} = \bigl\{\{(f,g) \in HMX \times HMX \mid \rho_{\phi_1,\dots,\phi_n}(f,g) < 1/k\} \mid \phi_1,\dots,\phi_n \in \mathcal{D}_{HM}, k \in \mathbb{N}\bigr\}$. We see that $|\mathcal{B}| \leq w(X)$.

Let us show that $\mathcal{B}$ is a base of uniformity $\mathcal{U}_{HMX}$. We have $\mathcal{B} \subset \mathcal{U}_{HMX}$. Consider any $U \in \mathcal{U}_{HMX}$. Without loss of generality we can assume that

$$U = \left\{ (f,g) \in HMX \times HMX \mid 1/(b-a) | \int_a^b \phi \circ f dt - \int_a^b \phi \circ g dt | < \varepsilon \right\}$$

for some $\phi \in C(X)$, $\varepsilon > 0$ and $a < b$, $a, b \in [0,1] \cap \mathbb{Q}$. Consider any $n \in \mathbb{N}$ with $3/n < 1/\varepsilon$ and $\psi \in \mathcal{D}$ such that $\max_{x \in X} |\phi(x) - \psi(x)| < 1/n$. Let us define $B \in \mathcal{B}$ by $B = \big\{ (f,g) \in HMX \times HMX \mid 1/(b-a) | \int_a^b \psi \circ f dt - \int_a^b \psi \circ g dt | < 1/n \big\}$. Let us show that $B \subset U$. Consider any $(f,g) \in B$. We have

$$\frac{1}{b-a} \left| \int_a^b \phi \circ f dt - \int_a^b \phi \circ g dt \right| \le \frac{1}{a-b} \left(\left| \int_a^b \phi \circ f dt - \int_a^b \psi \circ f dt \right| + \right.$$
$$\left. + \left| \int_a^b \psi \circ f dt - \int_a^b \psi \circ g dt \right| + \left| \int_a^b \psi \circ g dt - \int_a^b \phi \circ g dt \right| \right) < \frac{3}{n} < \varepsilon$$

or $(f,g) \in U$. □

For each compactum X we consider the uniform space $(HX, \mathcal{U}_{HX})$ which is the completion of $(HMX, \mathcal{U}_{HMX})$ and the topological space HX with the topology induced by the uniformity $\mathcal{U}_{HX}$. Since $\mathcal{U}_{HMX}$ is totally bounded, the space HX is compact. We see also that $w(\mathcal{U}_{HX}) \le w(\mathcal{U}_{HMX})$ [6, 8.3.12], hence $w(HX) \le w(X)$ for each compactum X of infinite weight.

Let $f \colon X \to Y$ be a continuous map. Define the map $HMf \colon HMX \to HMY$ by the formula $HMf(\alpha) = f \circ \alpha$ for $\alpha \in HMX$.

Lemma 2. *For each $\psi \in C(Y)$ and $a, b \in [0,1]$ we have $\psi_{(a,b)} \circ HMf = (\psi \circ f)_{(a,b)}$.*

Proof. Consider any $\alpha \in HMX$. We have $\psi_{(a,b)} \circ HMf(\alpha) = 1/(a-b) \int_a^b \psi \circ HMf(\alpha)(t) dt = 1/(a-b) \int_a^b \psi \circ f \circ \alpha(t) dt = (\psi \circ f)_{(a,b)}(\alpha)$. □

The following lemma is an evident corollary of Lemma 2.

Lemma 3. *For each continuous map $f \colon X \to Y$ the map*

$$HMf \colon (HMX, \mathcal{U}_{HMX}) \to (HMY, \mathcal{U}_{HMY})$$

is uniformly continuous.

Hence there exists the continuous map $Hf \colon HX \to HY$ such that $Hf | HMX = HMf$. It is easy to see that $H \colon \mathcal{C}omp \to \mathcal{C}omp$ is a covariant functor and HM_n is a subfunctor of H for each $n \in \mathbb{N}$.

2. We are going to prove that the functor H is normal. In what follows we will need some notions from the general theory of functors.

Let $F \colon \mathcal{C}omp \to \mathcal{C}omp$ be a covariant functor. A functor F is called *monomorphic (epimorphic)* if it preserves monomorphisms (epimorphisms). For a monomorphic functor F and an embedding $i \colon A \to X$ we shall identify the space $F(A)$ and the subspace $F(i)(F(A)) \subset F(X)$.

A monomorphic functor F is said to be *preimage-preserving* if for each map $f \colon X \to Y$ and each closed subset $A \subset Y$ we have $(F(f))^{-1}(F(A)) = F(f^{-1}(A))$.

For a monomorphic functor F the *intersection-preserving* property is defined as follows: $F(\bigcap\{X_\alpha \mid \alpha \in \mathcal{A}\}) = \bigcap\{F(X_\alpha) \mid \alpha \in \mathcal{A}\}$ for every family $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ of closed subsets of X.

A functor F is called *continuous* if it preserves the limits of inverse systems $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \mathcal{A}\}$ over a directed set $\mathcal{A}$.

Finally, a functor F is called *weight-preserving* if $w(X) = w(F(X))$ for every infinite $X \in \mathcal{C}omp$.

A functor F is called *normal* [1] if it is continuous, monomorphic, epimorphic, preserves weight, intersections, preimages, singletons and the empty space.

It is obvious that H preserves singletons and empty set.

Proposition 1. *H is a monomorphic functor.*

Proof. Let $f\colon A \to X$ be an embedding. Since Hf is an uniformly continuous extension of HMf, it is enough to show that $HMf\colon HMA \to HM(f)(A) = HM(f(A))$ is a homeomorphism. It is easy to see that HMf is injective. Let us show that $HMf\colon HMA \to HM(fA)$ is open. Consider any open neighborhood (α, U, ε) of $\alpha \in HMA$. Then $(f\times f)(U)$ is an open entourage of the diagonal in $fA\times fA$. We can choose an entourage of the diagonal $V \subset X\times X$ such that $V \cap (A \times A) = (f \times f)(U)$. One can check that $(HMf(\alpha), V, \varepsilon) \cap HM(fA) = HMf(\alpha, U, \varepsilon)$. □

Proposition 2. *The functor H is epimorphic.*

Proof. Let $f\colon X \to Y$ be a surjective map. Since HMY is dense in HY, it is enough to prove that $HMY \subset Hf(HX)$. Consider any $\alpha \in HMY$. Let $\{y_1, \dots, y_n\} = \alpha([0, 1))$. Since f is surjective, we can choose the function $g\colon \{y_1, \dots, y_n\} \to X$ such that $f \circ g = \mathrm{id}_{\{y_1,\dots,y_n\}}$. Define $\beta \in HMX$ by the formula $\beta(t) = g(\alpha(t))$ for $t \in [0, 1)$. It is easy to see that $Hf(\beta) = \alpha$. □

We will need some notations. For $\phi \in C(X)$ and $(a, b) \subset [0, 1)$ by $\tilde\phi_{(a,b)}\colon HX \to \mathbb{R}$ we denote the uniformly continuous extension of the function $\phi_{(a,b)}\colon HMX \to \mathbb{R}$. It follows from Lemma 2 that for each $f\colon X \to Y$, for each $\psi \in C(Y)$ we have $\widetilde{(\psi \circ f)}_{(a,b)} = \tilde\psi_{(a,b)} \circ Hf$.

Proposition 3. *H is a continuous functor.*

Proof. Let $X = \varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \mathcal{A}\}$ is an inverse system where all X_α are compact. Denote by Y the limit space of the inverse system $H(\mathcal{S}) = \{H(X_\alpha), H(\pi_\alpha^\beta), \mathcal{A}\}$ and by $\pi\colon H(X) \to Y$ the limit of the maps $H(\pi_\alpha)$, where $\pi_\alpha\colon X \to X_\alpha$ are the limit projections of the system $\mathcal{S}$.

Let us show that π is a homeomorphism. Consider any γ, $\beta \in HX$ with $\gamma \neq \beta$. Then there exists a function $\phi \in C(X)$ and $a, b \in [0, 1)$ such that $|\tilde\phi_{(a,b)}(\gamma) - \tilde\phi_{(a,b)}(\beta)| = \varepsilon > 0$. By the Stone-Weierstrass theorem there exists $\alpha \in A$ and $\psi \in C(X_\alpha)$ such that $\|\phi - \psi \circ \pi_\alpha\| < \varepsilon/8$. Choose $\gamma_1, \beta_1 \in HMX$ such that $|\tilde\phi_{(a,b)}(\gamma_1) - \tilde\phi_{(a,b)}(\gamma)| < \varepsilon/8$ and $|\tilde\phi_{(a,b)}(\beta_1) - \tilde\phi_{(a,b)}(\beta)| < \varepsilon/8$. Then we have

$$\begin{aligned}\varepsilon = |\tilde\phi_{(a,b)}(\gamma) - \tilde\phi_{(a,b)}(\beta)| &\le |\phi_{(a,b)}(\gamma_1) - \phi_{(a,b)}(\beta_1)| + \varepsilon/4 \le \\ &\le |(\psi \circ \pi_\alpha)_{(a,b)}(\gamma_1) - (\psi \circ \pi_\alpha)_{(a,b)}(\beta_1)| + \varepsilon/2 \le \\ &\le |\widetilde{(\psi \circ \pi_\alpha)}_{(a,b)}(\gamma) - \widetilde{(\psi \circ \pi_\alpha)}_{(a,b)}(\beta)| + 3\varepsilon/4 = \\ &= |\tilde\psi_{(a,b)} \circ H(\pi_\alpha)(\gamma) - \tilde\psi_{(a,b)} \circ H(\pi_\alpha)(\beta)| + 3\varepsilon/4.\end{aligned}$$

Hence, $H(\pi_\alpha)(\gamma) \neq H(\pi_\alpha)(\beta)$ and $\pi(\alpha) \neq \pi(\beta)$ We have just proved that π is an embedding.

Since the functor H is epimorphic, the map π is a surjection. □

Let A be a closed subset of a compactum X. We say that $\mu \in HX$ is *supported on A* if $\mu \in HA \subset HX$.

Lemma 4. *Let $\mu \in HX$ and let A be a closed subset of X. Then μ is supported on A iff for each ϕ, $\psi \in C(X)$ and $(a,b) \subset [0,1)$ with $\phi_{(a,b)}|A = \psi_{(a,b)}|A$ we have $\tilde{\phi}_{(a,b)}(\mu) = \tilde{\psi}_{(a,b)}(\mu)$.*

Proof. Let us consider any $\mu \in HX$ such that $\tilde{\phi}_{(a,b)}(\mu) = \tilde{\psi}_{(a,b)}(\mu)$ for each ϕ, $\psi \in C(X)$ and $(a,b) \subset [0,1)$ with $\phi_{(a,b)}|A = \psi_{(a,b)}|A$. Let us show that $\mu \in \mathrm{Cl}(HMA) \subset HX$.

It is enough to show that for each $\phi_1, \dots, \phi_n \in C(X)$, $(a_1,b_1), \dots, (a_n,b_n)$ with $(a_i,b_i) \subset [0,1)$ and each $\varepsilon > 0$ there exists $\beta \in HMA$ such that $|\tilde{\phi}_{i(a_i,b_i)}(\mu) - 1/(b_i - a_i)\int_{a_i}^{b_i} \phi_i \circ \beta dt| < \varepsilon$ for each $i \in \{1, \dots, n\}$. We will consider only the case $a_i = 0$ and $b_i = 1$. The proof of the general case is the same.

Let us consider the set $O_i = \{x \in X \mid \text{there exists } a \in A \text{ such that } |\phi_i(a) - \phi_i(x)| < \varepsilon/4\}$. Put $O = \bigcap_{i=1}^n O_i$. For each $i \in \{1, \dots, n\}$ choose the function $\psi_i \in C(X)$ such that $\psi_i|A = \phi_i$, $\psi_i|X \setminus O = \phi_i + 1$ and $\psi_i \geq \phi_i$. Put $k = \max\{1, \max_{x \in X, i \in \{1,\dots,n\}} 2|\phi_i(x)|\}$. By $\tilde{\phi}_i$ and $\tilde{\psi}_i$ we will denote the functions $\tilde{\phi}_{i(0,1)}$ and $\tilde{\psi}_{i(0,1)}$ Since HMX is dense in HX, there exists $\gamma \in HMX$ such that $|\tilde{\phi}_i(\mu) - \int_0^1 \phi_i \circ \gamma dt| < \varepsilon/4k$ and $|\tilde{\psi}_i(\mu) - \int_0^1 \psi_i \circ \gamma dt| < \varepsilon/4k$ for each $i \in \{1, \dots, n\}$.

Put $S = \{t \in [0,1) \mid \gamma(t) \in X \setminus O\}$. Then we have

$$m(S) = \int_S \psi_i \circ \gamma dt - \int_S \phi_i \circ \gamma dt \leq \int_0^1 \psi_i \circ \gamma dt - \int_0^1 \phi_i \circ \gamma dt \leq$$
$$\leq \left|\tilde{\phi}_i(\mu) - \int_0^1 \phi_i \circ \gamma dt\right| + \left|\tilde{\phi}_i(\mu) - \tilde{\psi}_i(\mu)\right| + \left|\tilde{\psi}_i(\mu) - \int_0^1 \psi_i \circ \gamma dt\right| < \frac{\varepsilon}{4k} + 0 + \frac{\varepsilon}{4k} = \frac{\varepsilon}{2k}.$$

Using the definition of the set O we can choose an element $\beta \in HMA$ such that $|\phi_i \circ \beta(t) - \phi_i \circ \gamma(t)| < \varepsilon/4$ for each $t \notin S$ and $i \in \{1, \dots, n\}$. Then we have

$$\left|\tilde{\phi}_i(\mu) - \int_0^1 \phi_i \circ \beta dt\right| \leq \left|\tilde{\phi}_i(\mu) - \int_0^1 \phi_i \circ \gamma dt\right| + \left|\int_0^1 \phi_i \circ \gamma dt - \int_0^1 \phi_i \circ \beta dt\right| \leq$$
$$\leq \left|\tilde{\phi}_i(\mu) - \int_0^1 \phi_i \circ \gamma dt\right| + \int_S \left|\phi_i \circ \gamma - \phi_i \circ \beta\right| dt + \int_{[0,1)\setminus S} \left|\phi_i \circ \gamma - \phi_i \circ \beta\right| dt <$$
$$< \frac{\varepsilon}{4} + m(S)k + \frac{\varepsilon}{4} < \varepsilon \text{ for each } i \in \{1, \dots, n\}.$$

The inverse implication is an easy exercise. □

Corollary 1. *Let A be a closed subset of X. Then $\mu \notin HA$ iff there exists $\phi \in C(X,I)$ such that $\tilde{\phi}_{(0,1)}(\mu) > 0$ and $\phi(a) = 0$ for each $a \in A$.*

Proposition 4. *The functor H preserves intersections.*

Proof. Since H is a continuous functor, it is sufficient to prove the proposition for the intersection of two closed subsets A_1 and A_2 of a compactum X.

It is evident that $H(A_1 \cap A_2) \subset H(A_1) \cap H(A_2)$. Let us show the inverse inclusion. Let $\mu \in H(A_1) \cap H(A_2)$. Choose any functions ψ_1, $\psi_2 \in C(X)$ such that $\psi_1|(A_1 \cap A_2) = \psi_2|(A_1 \cap A_2)$. By Lemma 4 it is sufficient to prove that $\mu(\psi_1) = \mu(\psi_2)$. Consider a function $\phi \in C(X)$ such that $\phi|A_1 = \psi_1$ and $\phi|A_2 = \psi_2$. Since $\mu \in H(A_1)$, we have $\mu(\phi) = \mu(\psi_2)$ and, since $\mu \in H(A_2)$, we have $\mu(\phi) = \mu(\psi_2)$. □

Lemma 5. *Let A be a closed subset of X. For each continuous map $f\colon X\to Y$, each $\phi\in C(X,I)$ such that $\phi|f^{-1}(A)=0$ there exists $\psi\in C(Y,I)$ such that $\psi\circ f\geq\phi$ and $\phi|A=0$.*

Proof. If $\phi(x)=0$ for each $x\in X$, we can define $\psi(y)=0$ for each $y\in Y$. Consider the case when $\phi(x)>0$ for some $x\in X$. We can suppose that $\phi(x)=1$ for some $x\in X$. For each $i\in\mathbb{N}$ define the set $A_i=\{y\in Y\mid \max_{x\in f^{-1}(y)}\phi(x)\geq 1/(i+1)\}$. We obtain the increasing sequence $(A_i)_{i=1}^\infty$ of closed subsets of Y with the property $A_i\cap A=\varnothing$ for each $i\in\mathbb{N}$. We can choose a sequence $(B_i)_{i=1}^\infty$ of closed subsets of Y such that $A_i\subset B_i$, $B_i\subset\operatorname{Int}B_{i+1}$ and $B_i\cap A=\varnothing$ for each $i\in\mathbb{N}$.

Let us construct by induction a sequence of function $\psi_i\colon Y\to[0,1]$ such that

1_i) $\psi_i|B_i\geq 1/i$,

2_i) $\psi_i|B_{i-1}=\psi_{i-1}|B_{i-1}$,

3_i) $\psi_i|Y\setminus B_l\leq 1/l$ for each $l\in\{1,\dots,i\}$.

Define ψ_1 by the rule $\psi_1(y)=1$ for each $y\in Y$. Let us assume that we have already defined the functions ψ_i for each $i\leq n$ which satisfy conditions 1_i)–3_i).

Let us define the function ψ_{n+1}. Choose a function $\gamma\colon Y\setminus\operatorname{Int}B_n\to[1/(i+1),1/i]$ such that $\gamma(\operatorname{Fr}B_n)\subset\{1/n\}$ and $\gamma(Y\setminus\operatorname{Int}B_{n+1})\subset\{1/(n+1)\}$. Define the function ψ_{n+1} by the conditions $\psi_{n+1}(x)=\psi_n(x)$ if $x\in\operatorname{Cl}B_n$ and $\psi_{n+1}(x)=\gamma(x)$ if $x\in Y\setminus B_n$.

Conditions 1_n) and 3_n) imply that ψ_{n+1} is well-defined and continuous. It is easy to check that ψ_{n+1} satisfies conditions 1_{n+1})-3_{n+1}).

The sequence ψ_i is fundamental in the complete space $C(X,I)$. Thus, there exists the limit function $\psi\colon Y\to I$. Condition 1_i) implies that $\psi\circ f\geq\phi$ and condition 3_i) implies that $\psi(a)=0$ for each $a\in A$. □

Proposition 5. *The functor H preserves preimages.*

Proof. Let $f\colon X\to Y$ be a continuous map and A a closed subset of Y. We should show that $H(f^{-1}(A))=(Hf)^{-1}(HA)$. Evidently, $H(f^{-1}(A))\subset(Hf)^{-1}(HA)$. Let us consider any $\alpha\notin H(f^{-1}(A))$. By Corollary 1 there exists a function $\phi\colon X\to I$ such that $\phi(z)=0$ for each $z\in f^{-1}(A)$ and $\tilde\phi_{(0,1)}(\alpha)>0$. By Lemma 5 we can choose a continuous function $\psi\colon Y\to I$ such that $\psi(a)=0$ for each $a\in A$ and $\psi\circ f\geq\phi$. Then we have $\widetilde{(\psi\circ f)}_{(0,1)}(\alpha)\geq\tilde\phi_{(0,1)}(\alpha)>0$. It follows from Lemma 2 that $\widetilde{(\psi\circ f)}_{(0,1)}(\alpha)=\tilde\psi(0,1)\circ Hf(\alpha)$. Hence $Hf(\alpha)\notin H(A)$ or $\alpha\notin(Hf)^{-1}(HA)$. Thus, we have proved $H(f^{-1}(A))=(Hf)^{-1}(HA)$. □

The following theorem is an immediate consequence of the results of this section and Lemma 1.

Theorem 2. *The functor H is normal.*

REFERENCES

1. Щепин Е. В. *Функторы и несчетные степени компактов,* Усп. мат. наук **36** (1981), 3–62.

2. Федорчук В. В., Филиппов В. В. Общая топология. Основные конструкции, Москва, 1988, 252 с.

3. Федорчук В. В., Заричный М. М. *Ковариантные функторы в категориях топологических пространств,* Итоги науки и техники. Алгебра. Топология. Геометрия. **28**, Москва, ВИНИТИ, 47–95.

4. Hartman S., Mycielski J. *On the embedding of topological groups into connected topological groups,* Colloq. Math **5** (1958), 167–169.

5. Telejko A., Zarichnyi M. Categorical topology of Compact Hausdorff Spaces, Lviv, VNTL, 1999, 263 pp.

6. Engelking R. General topology, Warszawa, 1978.

Faculty of Mechanics and Mathematics,
Lviv Ivan Franko National University

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