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FACTORIZATION OF MATRIX DIFFERENTIAL OPERATORS AND DARBOUX-LIKE TRANSFORMATIONS

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A factorization of a matrix differential operator of general form into elementary factors is constructed in explicit form. A notion of simplest elementary Darboux transformations is introduced. The theory of general Darboux type transformations for matrix evolutionary differential operators of arbitrary order is developed. Binary Darboux type transformations for such operators are also constructively represented.

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В явном виде построено разложение матричного дифференциального оператора общего вида на елементарные множители (факторы). Введено понятие простейших элементарных преобразований Дарбу. Конструктивно развита теория общих преобразований типа Дарбу для матричных эволюционных дифференциальных операторов произвольного порядка. Бинарные преобразования типа Дарбу для таких операторов представлены также в конструктивном виде.

Introduction

There are many mathematical and physical problems associated with the integrable dynamical systems. However, one of the most important results of the soliton theory – the construction of a large variety of exact solutions (e.g. multisoliton solutions) – can be obtained by using rather elementary tools applying the so-called Baecklund transformations. These transformations were first discovered for the famous sine-Gordon equation at the end of the 19th century [1]. Usually they are treated as nonlinear superpositions which allow one to create new solutions of a nonlinear evolution equation from a finite number of known solutions. In practice, however, Baecklund transformations are not very straightforward to apply in the construction of multisoliton solutions, despite the numerous statements of various authors.

Another, but closely related, elementary approach which also dates back to the 19th century. The idea of this approach was originated by Darboux (1882)(see [2]) in his study of the linear Sturm-Liouville problem. Darboux's idea can be applied to construct the solutions of linear and nonlinear partial differential equations including the nonstationary Schroedinger

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equation, Korteweg-de Vries and Kadomtsev-Petviashvili equations, $1+1$ and $2+1$ Toda lattice equations, sine-Gordon, nonlinear Schroedinger and Davey-Stewartson equations and many others.

It is necessary to mention that the presented technique does not allow to recover all the solutions or to give a solution of the Cauchy problem with arbitrary initial data. But, at the same time, it allows one to obtain rather nontrivial classes of solutions and to study their properties in many cases where the complete analysis based on the Inverse Scattering Transform method (IST), local or nonlocal Riemann-Hilbert problem approach is considerably more complicated.

The article is organized as follows: in the first section, we will briefly review the basic background information for the classical Darboux-Crum-Matveev transformations. In Section 2 we present our results on factorization matrix linear differential operator of the first order (operator of the elementary matrix Darboux-Matveev transformation) using Darboux-type factors. These factors are called the simplest matrix Darboux-like transformations (Theorem 1). A general form for dressing operators W (operators of transformations) is also shown. Using results of the second section, we construct in Section 3 the theory of general matrix binary Darboux-like transformations for the linear evolution partial differential equations.

1. Darboux, Crum and Matveev Theorems

Consider the Sturm-Liouville equation (the one-dimensional Schroedinger equation)

$$L\{f\} = \lambda f, \tag{1}$$

where $L := -\mathcal{D}^2 + u(x), \mathcal{D} := \frac{\partial}{\partial x}, \mathcal{D}\{f\} = \frac{\partial f}{\partial x} := f_x, \mathcal{D}f := f\mathcal{D} + f_x$, i.e. (1) has the form

$$-f_{xx} + uf = \lambda f.$$

Let $u(x) := u[1], f = f(x,\lambda) := f[1], L := L[1]$, then (1) can be rewritten as $L[1]\{f[1]\} = \lambda f[1]$ or $-f_{xx}[1] + u[1]f[1] = \lambda f[1]$.

We denote the fixed solution of (1) taken at the point $\lambda = \lambda_1$ by $\varphi_1[1]$, $\varphi_1[1] = \varphi_1[1](x,\lambda_1)$ and define an operator $W_e[\varphi_1[1]]$ as

$$W_e[\varphi_1[1]] := \varphi_1[1]\mathcal{D}\varphi_1^{-1}[1] = \mathcal{D} - \varphi_{1x}[1]\varphi_1^{-1}[1]. \tag{2}$$

Now the Darboux transformation (DT) $f := f[1] \to f[2]$ of an arbitrary solution of (1) is defined by

$$f[2] = W_e[\varphi_1[1]]\{f[1]\} = f_x[1] - \frac{\varphi_{1x}[1]}{\varphi_1[1]}f[1] = \frac{\mathcal{W}(\varphi_1[1], f[1])}{\varphi_1[1]}, \tag{3}$$

where $\mathcal{W}(\varphi_1[1], f[1]) = \varphi_1[1]f_x[1] - \varphi_{1x}[1]f[1]$ is the usual Wronskian.

Theorem (Darboux, 1882). *The function $f[2]$ satisfies the differential equation*

$$L[2]\{f[2]\} = \lambda f[2], \tag{4}$$

where

$$L[2] := -\mathcal{D}^2 + u[2], u[2] = u[1] - 2(\ln\varphi_1[1])_{xx}. \tag{5}$$

In other words, Darboux's theorem declares that the Sturm-Liouville equation (1) is covariant with respect to the action of the Darboux transformation $f[1] \to f[2], u[1] \to u[2]$.

It is evident that the Darboux transformation may be applied to (4) once more producing some new solvable Sturm-Liouville equation and that this operation can be repeated an arbitrary number of times. For the second step of this procedure we have

$$f[3] = (\mathcal{D} - \frac{\varphi_{2x}[2]}{\varphi_2[2]})(\mathcal{D} - \frac{\varphi_{1x}[1]}{\varphi_1[1]})\{f[1]\} = W_e[\varphi_2[2]]W_e[\varphi_1[1]]\{f[1]\}, \tag{6}$$

where $\varphi_2[2]$ is a fixed solution of (4) with $\lambda = \lambda_2$, generated by some fixed solution $\varphi_2[1](x, \lambda_2)$ of (1):

$$\varphi_2[2] := W_e[\varphi_1[1]]\{\varphi_2[1]\} = \varphi_1[1]\mathcal{D}\varphi_1^{-1}[1]\varphi_2[1] =$$
$$= \varphi_{2x}[1] - \frac{\varphi_{1x}[1]}{\varphi_1[1]}\varphi_2[1] = \frac{\mathcal{W}(\varphi_1[1], \varphi_2[1])}{\mathcal{W}(\varphi_1[1])} \tag{7}.$$

The potential of the linear Sturm-Liouville equation corresponding to $f[3]$ is given by

$$u[3] = u[2] - 2(\ln \varphi_2[2])_{xx} = u[1] - 2(\ln \mathcal{W}(\varphi_1[1], \varphi_2[1]))_{xx}. \tag{8}$$

Formulae (6), (8) can be generalized to include the case of N-times repeated Darboux transformation, expressed completely in terms of the solutions of the initial equation (1) without any use of the solutions related to the intermediate iterations of the process.

Let $\varphi_1[1], \dots, \varphi_N[1]$ be solutions of (1) fixed arbitrarily at $\lambda = \lambda_1, \dots, \lambda_N$ respectively. The following generalization of the Darboux theorem was discovered by Crum:

Theorem [3] (Crum, 1955). *The function*

$$f[N+1] = \Big(\mathcal{D} - \frac{\varphi_{Nx}[N]}{\varphi_N[N]}\Big) \dots \Big(\mathcal{D} - \frac{\varphi_{2x}[2]}{\varphi_2[2]}\Big)\Big(\mathcal{D} - \frac{\varphi_{1x}[1]}{\varphi_1[1]}\Big)\{f[1]\} =$$
$$= W_e[\varphi_N[N]] \dots W_e[\varphi_2[2]]W_e[\varphi_1[1]]\{f[1]\} = \frac{\mathcal{W}(\varphi_1[1], \dots, \varphi_N[1], f[1])}{\mathcal{W}(\varphi_1[1], \dots, \varphi_N[1])} \tag{9}$$

satisfies the differential equation

$$-f_{xx}[N+1] + u[N+1]f[N+1] = \lambda f[N+1] \tag{10}$$

with potential

$$u[N+1] = u[1] - 2(\ln \mathcal{W}(\varphi_1[1], \dots, \varphi_N[1]))_{xx}. \tag{11}$$

The Darboux theorem follows from the result of the Crum theorem in the case of $N = 1$.

The property of the Darboux covariance is even more general, it is not restricted by the second order scalar differential equations. The remarkable fact, first established by V. B. Matveev, is that the Darboux covariance holds also for the case of an arbitrary linear partial differential equation of the form

$$L\{f\} = 0, L = \alpha\partial_t - \sum_{i=0}^{n} u_i(x,t)\mathcal{D}^i, \mathcal{D}^i\{f\} := \frac{\partial^i f}{\partial x^i}, \partial_t\{f\} := \frac{\partial f}{\partial t}, \quad \alpha \in \mathbb{C} \tag{12}$$

even if the coefficients u_i are matrix-valued functions of x and t. In the case of $N \times N$ matrix coefficients $u_i(x,t)$ V. Matveev has defined the elementary Darboux transformation by the same formula analogous to the scalar case, taking for f a matrix solution of (12) and defining φ to be a fixed invertible $N \times N$ matrix solution of (1), i.e.,

$$f := f[1] \to f[2] := f_x[1] - \sigma f[1] := W[\varphi[1]]\{f[1]\},\ \sigma = \varphi_x[1]\varphi^{-1}[1], \tag{13}$$

$$W[\varphi[1]] := \varphi[1]\mathcal{D}\varphi^{-1}[1] = \mathcal{D} - \varphi_x[1]\varphi^{-1}[1].$$

Theorem (Matveev, 1979 [4], [5]). *The function $f[2]$ satisfies the linear partial evolution equation of the form*

$$L[2]\{f[2]\} = 0, \tag{14}$$

where

$$L[2] := \alpha\partial_t - \sum_{i=0}^{n} \tilde{u}_i(x,t)\mathcal{D}^i. \tag{15}$$

2. Factorization of matrix differential operator $W[\varphi] = \varphi\mathcal{D}\varphi^{-1} = \mathcal{D} - \varphi_x\varphi^{-1}$

We claim that Darboux-Matveev transformation (13) is not “elementary”, therefore the following theorem holds:

Theorem 1. *1. Every column-solution $\varphi_{.i}[1] := (\varphi_{1i}, \varphi_{2i}, \ldots, \varphi_{Ni})^\top[1], i \in \{1, \ldots, N\}$ of the matrix solution $\varphi[1]$ generates the simplest elementary Darboux-like transformation $W_{ii}[\varphi_{.i}[1]]$, $i \in \{1, \ldots, N\}$;*

$$f := f[1] \to W_{ii}[\varphi_{.i}[1]]\{f[1]\},$$

$$W_{ii}[\varphi_{.i}[1]]\{\varphi_{.i}[1]]\} = 0, \tag{16}$$

where, for example,

$$W_{11}[\varphi_{.1}[1]] = \begin{pmatrix} \varphi_{11}[1]\mathcal{D}\varphi_{11}^{-1}[1] & 0 \\ -\varphi_{21}[1]\varphi_{11}^{-1}[1] & \\ \vdots & I_{N-1} \\ -\varphi_{N1}[1]\varphi_{11}^{-1}[1] & \end{pmatrix}, \tag{17}$$

$I_{N-1} = \mathrm{diag}_{(N-1)\times(N-1)}(1 \ldots 1)$.
The function $f[i+1]$,

$$f[i+1] := W_{ii}[\varphi_{.i}[1]]\{f[1]\}, \quad i \in \{1, \ldots, N\},$$

satisfies the linear partial evolution equation of the form

$$L[i+1, \varphi_{.i}[1]]\{f[i+1]\} = 0,$$

where

$$L[i+1, \varphi_{.i}[1]] := \alpha\partial_t - \sum_{k=0}^{n} u_k[i+1](x,t)\mathcal{D}^k = W_{ii}[\varphi_{.i}[1]]L[1]W_{ii}^{-1}[\varphi_{.i}[1]].$$

2. *The differential operator $W[\varphi[1]]$ (13) admits a factorization to the form*

$$W[\varphi[1]] := \varphi[1]\mathcal{D}\varphi^{-1}[1] = \prod_{i=N}^{1} W_{ii}[\varphi_{.i}[i]], \tag{18}$$

where $\varphi_{.k}[k] := \prod_{i=k-1}^{1} W_{ii}[\varphi_{.i}[i]]\{\varphi_{.k}[1]\}$,

$$W_{ii}[\varphi_{.i}[i]] = E_i(\varphi_{ii}[i]\mathcal{D}\varphi_{ii}^{-1}[i] - 1) + I_N + A[i,i], \tag{19}$$

$$A[i,i] = \begin{pmatrix} & a_{1i} & \\ 0 & \vdots & 0 \\ a_{i1}\dots & a_{ii} & \\ 0 & \vdots & 0 \\ & a_{Ni} & \end{pmatrix}, \quad E_i = \mathrm{diag}(\delta_i^1, \dots, \delta_i^j, \dots, \delta_i^N), \tag{20}$$

where

$$\delta_i^j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Therefore, the Darboux-Matveev transformation is the composition of N simplest Darboux-type transformations like (18)–(20).

The proofs of this and the following theorems of our paper are contained in a constructive method of the construction of all transformations. We omit these proofs that we present in explicit form for operators of transformations (dressing operators W_{ii}, W_i and W – see below). An explicit form of the matrix $A[i,i]$ (20) are presented below in the second subitem of Theorem 2 (see formulae (25)). It allows us to check all our propositions by direct calculations.

Define the matrix differential operators

$$W_1 := W_{11}[\varphi_{.1}[1]], W_2 := W_{22}[\varphi_{.2}[2]]W_{11}[\varphi_{.1}[1]], \dots,$$

$$W_{k+1} := W_{k+1k+1}[\varphi_{.k+1}[k+1]]\dots W_{11}[\varphi_{.1}[1]], \dots, W_N = W[\varphi[1]], \tag{21}$$

$$\varphi[1] := \varphi := (\varphi_{.1}, \varphi_{.2}, \dots, \varphi_{.N}) = (\varphi_{ij}) =$$

$$\left(\begin{array}{ccc|ccc} \varphi_{11} & \dots & \varphi_{1k} & \varphi_{1k+1} & \dots & \varphi_{1N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \varphi_{k1} & \dots & \varphi_{kk} & \varphi_{kk+1} & \dots & \varphi_{kN} \\ \hline \varphi_{k+11} & \dots & \varphi_{k+1k} & \varphi_{k+1k+1} & \dots & \varphi_{k+1N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \varphi_{N1} & \dots & \varphi_{Nk} & \varphi_{Nk+1} & \dots & \varphi_{NN} \end{array}\right) :=$$

$$:= \left(\begin{array}{c|c} \hat{M}_k & \breve{D}_{k,N-k} \\ \hline \breve{M}_{N-k,k} & \hat{D}_{N-k} \end{array}\right), \quad i,j \in \{1, \dots, N\};$$

where $\hat{M}_k$ is the matrix of the principal minor $M_k : M_k = \det\hat{M}_k := |\hat{M}_k|$; $\hat{D}_{N-k}$ is the matrix of the inferior diagonal minor D_{N-k} of $(N-k)\times(N-k)$ dimension; $\hat{M}_1 = \varphi_{11}, \hat{D}_1 = \varphi_{NN}, \dots, \hat{M}_N = \hat{D}_N = \varphi$.

The matrices $\breve{M}_{N-k,k}$ and $\breve{D}_{k,N-k}$ are of $(N-k)\times k$ and $k\times(N-k)$ dimensions complement $\hat{M}_k$ and $\hat{D}_{N-k}$ to the matrices $(\varphi_{.1},\varphi_{.2},\ldots,\varphi_{.K})$ and $(\varphi_{.K},\ldots,\varphi_{.N})$, respectively.

Remark. It is evident that the following formulae hold:

1. $\varphi_{.k+1}[k+1]=W_k\{\varphi_{.k+1}[1]\},\ k\in\{1,\ldots,N-1\}$;
2. $W_k\{\varphi_{.j}[1]\}=0,\quad \text{for } j\le k$;
3. $W_{kk}[\varphi_{.k}[k]]=W_kW_{k-1}^{-1},\ k\in\{1,\ldots,N\}$.

Corollary. *The function* $\tilde{f}[i+1]$,

$$\tilde{f}[i+1]:=W_i\{f[1]\},\quad i\in\{1,\ldots,N\}, \tag{22}$$

satisfies the linear partial evolution equation of the form

$$\tilde{L}[i+1,\varphi[1]]\{\tilde{f}[i+1]\}=0,$$

where

$$\tilde{L}[i+1,\varphi[1]]:=W_iL[1]W_i^{-1}=\alpha\partial_t-\sum_{k=0}^{n}\tilde{u}_k[i+1](x,t)\mathcal{D}^k.$$

Formulae (19)–(22) can be expressed completely in terms of the solutions $\varphi=\varphi[1]$ of the initial equation (12) without any use of the solutions related to the intermediate iterations of the process.

Theorem 2. *1.*

$$W_{k+1}=W_{k+1}[\varphi[1]]=\begin{pmatrix}\hat{M}_{k+1}\mathcal{D}\hat{M}_{k+1}^{-1} & | & 0\\ - & - & -\\ -\breve{M}_{k+1}\hat{M}_{k+1}^{-1} & | & I_{N-k-1}\end{pmatrix}, \tag{23}$$

$$W_{k+1}^{-1}=W_{k+1}^{-1}[\varphi[1]]=\begin{pmatrix}\hat{M}_{k+1}\mathcal{D}^{-1}\hat{M}_{k+1}^{-1} & | & 0\\ - & - & -\\ \breve{M}_{k+1}\mathcal{D}^{-1}\hat{M}_{k+1}^{-1} & | & I_{N-k-1}\end{pmatrix}. \tag{24}$$

2.

$$W_{k+1k+1}[\varphi_{.k+1}[k+1]]=E_{k+1}(\varphi_{k+1k+1}[k+1]\mathcal{D}\varphi_{k+1k+1}^{-1}[k+1]-1)+I_N+A[k+1,k+1],$$

$$A[k+1,k+1]=\begin{pmatrix} & a_{1k+1} & \\ 0 & \vdots & 0\\ a_{k+11}\ldots & a_{k+1k+1} & \\ 0 & \vdots & 0\\ & a_{Nk+1} & \end{pmatrix},\ E_{k+1}=\operatorname{diag}(\delta_{k+1}^1,\ldots,\delta_{k+1}^j,\ldots,\delta_{k+1}^N),$$

where

$$a_{k+1.}=(a_{k+11},\ldots,a_{k+1k})=(\varphi_{k+11},\ldots,\varphi_{k+1k})\hat{M}_k^{-1},$$
$$a_{.k+1}:=(a_{1k+1},\ldots,a_{Nk+1})^\top=\begin{pmatrix}\hat{M}_{k+1x}\\ \breve{M}_{k+1}\end{pmatrix}\hat{M}_{k+1}^{-1}e_{k+1}, \tag{25}$$
$$e_{k+1}=(\delta_{k+1}^1,\ldots,\delta_{k+1}^j,\ldots,\delta_{k+1}^N)^\top.$$

3. Binary Darboux Transformation

Let $g = g[1](x,t)$ be an arbitrary solution and $\psi = \psi[1]$ be a fixed invertible $(N \times N)$ matrix of solutions of the transposed linear evolution equation

$$L^{\tau}[1]\{g[1]\} = 0, \tag{26}$$

where

$$L^{\tau}[1] := -\alpha\partial_t - \sum_{i=0}^{n}(-1)^i \mathcal{D}^i u_i[1](x,t). \tag{27}$$

The following theorem holds:

Theorem 3. *The functions*

$$g[i+1] := W_{ii}[\psi_{.i}[1]]\{g[1]\}, \tag{28}$$

and

$$\tilde{g}[i+1] := W_i[\psi[1]]\{g[1]\}, \tag{29}$$

satisfy the linear evolution equations of the form

$$L^{\tau}[i+1, \psi_{.i}[1]]\{g[i+1]\} = 0, \tag{30}$$

$$\tilde{L}^{\tau}[i+1, \psi[1]]\{\tilde{g}[i+1]\} = 0, \quad i \in \{1, \dots, N\}, \tag{31}$$

respectively.

Consider the transformations of the linear operators

$$\varphi_{.i}[1] : L[1] \to L[i+1] := L[i+1, \varphi_{.i}[1]] \to (L[i+1, \varphi_{.i}[1]])^{\tau} := L^{\tau}[i+1, \varphi_{.i}[1]],$$

$$\psi_{.i}[1] : L^{\tau}[1] \to L^{\tau}[i+1] := L^{\tau}[i+1, \psi_{.i}[1]] \to (L^{\tau}[i+1, \psi_{.i}[1]])^{\tau} := L[i+1, \psi_{.i}[1]]$$

and

$$L[1] \to L^{\tau}[i+1, \varphi_{.i}[1]] \to L^{\tau}[i+1, \varphi_{.i}[1], \psi[1]] \to L[i+1, \varphi_{.i}[1], \psi_{.i}[1]], \tag{32}$$

where

$$L[i+1, \varphi_{.i}[1], \psi_{.i}[1]] := W_{ii}^{-1,\tau}[\psi_{.i}[1]] W_{ii}[\varphi_{.i}[1]] L[1] W_{ii}^{-1}[\varphi_{.i}[1]] W_{ii}^{\tau}[\psi_{.i}[1]].$$

The following theorem holds:

Theorem 4. *The function* $F[i+1]$,

$$\begin{aligned} F[i+1] &:= W_{ii}^{-1,\tau}[\psi_{.i}[1]] W_{ii}[\varphi_{.i}[1]]\{f[1]\} = \\ &= f[1] - \varphi_{.i}\Omega^{-1}[\psi_{.i}[1], \varphi_{.i}[1]]\Omega[\psi_{.i}[1], f[1]] := \\ &:= \left(I - \varphi_{.i}\left[\mathcal{D}^{-1}\{\psi_{.i}^{T}[1]\varphi_i[1]\}\right]^{-1} \mathcal{D}^{-1}\psi_{.i}^{T}[1]\right)\{f[1]\} \end{aligned} \tag{33}$$

satisfies the linear partial differential equations

$$L[i+1, \varphi_{.i}[1], \psi_{.i}[1]]\{F[i+1]\} = 0, \tag{34}$$

where the scalar potential function Ω *is defined by*

$$\Omega[\psi_{.i}[1],\varphi_{.i}[1]] := \mathcal{D}^{-1}\{\psi_{.i}^T[1]\varphi_{.i}[1]\} = c + \int\limits_{(x_0,t_0)}^{(x,t)} Pdx + Qdt,\ c = \text{const}, \tag{35}$$

$$P_t = (\alpha\psi_{.i}^T[1]\varphi_{.i}[1])_t = Q_x = \left(\sum_{m=1}^{n}\sum_{k=0}^{m-1}(-1)^k(\psi_{.i}^T u_m)^{(k)}\varphi_{.i}^{(m-k-1)}\right)_x . \tag{36}$$

Theorem 5. *The function* $\hat{F}[i+1]$,

$$\hat{F}[i+1] := W_i^{-1,\tau}[\psi[1]]W_i[\varphi[1]]\{f[1]\} =$$
$$= \left(I - (\varphi_{.1},\varphi_{.2},\dots,\varphi_{.i})\Omega^{-1}\mathcal{D}^{-1}(\psi_{.1},\psi_{.2},\dots,\psi_{.i})^T\right)\{f[1]\} \tag{37}$$

satisfies the equation

$$\hat{L}[i+1]\left\{\hat{F}[i+1]\right\} = 0, \tag{38}$$

where

$$\Omega = C + \int\limits_{(x_0,t_0)}^{(x,t)} \alpha(\psi_{.1}.\psi_2.\dots,\psi_{.i})^T(\varphi_{.1},\varphi_{.2},\dots,\varphi_{.i})dx + Qdt \tag{39}$$

is an $(i\times i)$*-matrix potential function.*

Theorem 6. *The binary Darboux transformation*

$$f[1] \to W[\varphi[1],\psi[1]]\{f[1]\} := f([2],N),$$
$$L[1] \to W[\varphi[1],\psi[1]]L[1]W^{-1}[\varphi[1],\psi[1]] := L([2],N), \tag{40}$$

where

$$W[\varphi[1],\psi[1] := I - \varphi[1]\left(\mathcal{D}^{-1}\{\psi^T[1]\varphi[1]\}\right)\mathcal{D}^{-1}\psi^T[1]$$

admits a factorization by the simplest elementary binary Darboux-like transformations to the form

$$W[\varphi[1],\psi[1]] = \prod_{i=N}^{1} W_{ii}[\varphi_{.i}[i],\psi_{.i}[i]] :=$$

$$:= \prod_{i=N}^{1}\left(I - \tilde{\varphi}_{.i}[i]\left(\mathcal{D}^{-1}\{\tilde{\psi}_{.i}^T[i]\tilde{\varphi}_{.i}[i]\}\right)^{-1}\mathcal{D}^{-1}\tilde{\psi}_{.i}^T[i]\right), \tag{41}$$

$$\tilde{\varphi}_{.k}[1] := \varphi_{.k}[1],\ \tilde{\psi}_{.k}[1] := \psi_{.k}[1], \quad k\in\{1,\dots,N\}, \tag{42}$$

$$\tilde{\varphi}_{.k}[k] := \prod_{i=k-1}^{1} W_{ii}[\tilde{\varphi}_{.i}[i],\tilde{\psi}_{.i}[i]]\{\varphi_{.k}[1]\}, \tag{43}$$

$$\tilde{\psi}_{.k}[k] := \prod_{i=k-1}^{1} W_{ii}^{-1,\tau}[\tilde{\varphi}_{.i}[i],\tilde{\psi}_{.i}[i]]\{\psi_{.k}[1]\}, \quad k\in\{2,\dots,N\}; \tag{44}$$

and

$$W_{ii}^{-1,\tau}[\tilde{\varphi}_{.i}[i],\tilde{\psi}_{.i}[i]] = I - \tilde{\psi}_{.i}[i]\left(\mathcal{D}^{-1}\{\tilde{\varphi}_{.i}^T[i]\tilde{\psi}_{.i}[i]\}\right)^{-1}\mathcal{D}^{-1}\tilde{\varphi}_{.i}^T[i]. \tag{45}$$

Theorem 7. *Let f be an arbitrary solution of the evolution equation (12); φ, ψ be a fixed $(N \times K)$-matrix solutions of (12) and (26) respectively, $\mathcal{D}^{-1}\{\psi^\top\varphi\} := \Omega[\psi,\varphi]$ be a $(K \times K)$-matrix potential function (39). Then the function*

$$\hat{f} := W\{f\} := \left[I - \varphi\left(\mathcal{D}^{-1}\{\psi^T\varphi\}\right)^{-1}\mathcal{D}^{-1}\psi^T\right]\{f\} \tag{46}$$

satisfies the linear evolution differential equation

$$\hat{L}\{\hat{f}\} = 0,\ \hat{L} := WLW^{-1}, \tag{47}$$

and

$$W = \prod_{i=K}^{1}\left[I - \tilde{\varphi}_{.i}[i]\left(\mathcal{D}^{-1}\{\tilde{\psi}^T_{.i}[i]\tilde{\varphi}_{.i}[i]\}\right)^{-1}\mathcal{D}^{-1}\tilde{\psi}^T_{.i}\right], \tag{48}$$

where $\tilde{\varphi}_{.i}[i], \tilde{\psi}_{.i}[i]$ are defined by formulae (42)–(44) for $k \in \{1,\ldots,K\}$.

Definition 1. Transformation (46)–(47) is called a *general binary matrix Darboux-like transformation.*

Definition 2. A composition of simplest elementary binary Darboux-like transformations (48) is called a *canonical factorization of the general dressing operator* W (46).

Theorem 8. *1) Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in$ $\mathrm{Mat}_{N\times nN}$ be an $(N \times nN)$-matrix-valued function of $x \in \mathbf{R}$, $\varphi_k = (\varphi_{ijk})$; $i,j \in \{1,\ldots,N\}$; $k \in \{1,\ldots,n\}$; and $\mathcal{W}(\varphi) \neq 0$, where $\mathcal{W}(\varphi)$ is the usual block Wronskian determinant.*

*2) Let $W[\varphi]$ be the n-*order *$(N \times N)$-matrix differential operator of the form*

$$W[\varphi] := \mathcal{D}^n + \sum_{i=0}^{n-1} w_i(x)\mathcal{D}^i, \tag{49}$$

and $W[\varphi]\{\varphi\} = 0$.

Then

$$W[\varphi] = \prod_{k=n}^{1}\varphi_k[k]\mathcal{D}\varphi_k^{-1}[k] = \prod_{k=n}^{1}\prod_{i=N}^{1} W_{ii}[\varphi_{.ik}[i]], \tag{50}$$

where $W_{ii}[\varphi_{.ik}[i]]$ are defined by formulae (19), (20), (25).

This theorem is a corollary from the statement of Theorem 1.

Definition 3. The composition of simplest elementary Darboux-like transformations (50) will be called a *canonical factorization of the n-order $(N \times N)$-matrix differential operator (49).*

Concluding remarks

1. In papers [6-9] we used a particular case of the general dressing operator W (46) for integration of the Lax-Zakharov-Shabat equations in a matrix algebra of differential operators. This particular case corresponds to a $(K \times K)$-matrix potential function Ω of the form

$$\Omega[\psi, \varphi] := \mathcal{D}^{-1}\{\psi^\top \varphi\} := C + \int_{\pm\infty}^{x} \psi^\top(s,t)\varphi(s,t)\, ds.$$

Such a case of the potential Ω permits to construct the classes of exact solutions for corresponding nonlinear dynamical systems of the soliton theory which may be obtained by traditional method of solvation of the Gelfand-Levitan-Marchenko equation or by dressing method of Zakharov-Shabat.

Using general transformation (46) (that corresponds to simple substitution of the potential Ω in the formulae for solutions) as it is used to dressing by Darboux transformations [5], allows us to construct much more wide classes of solutions for nonlinear integrable systems.

2. It is possible to apply transformation (46)–(47) not only to the evolution differential operators. We used this transformation for investigation of nonlinear integrable systems from so-called constrained Kadomtsev–Petviashvili hierarchy (cKP) [10–16] in our papers [17,18] (see also [19-22]).

3. A modified transformation like (46)–(47) allows us to construct wide classes of exact solutions for the modified (non-standard) constrained KP hierarchy (m-cKP) (see e.g. [23, 24]).

4. The so-called inverses and elementary binary Darboux-Baeklund transformations for the Kadomtsev–Petviashvili equations has been noted in [25] and for the Davey-Stewartson equation in [5,26] (see also [27,28]).

5. The spectral properties of binary Darboux transformation (a composite transformation formed from an application of the basic elementary Darboux transformation with the second inverse transformation) are described for the "time"-dependent Schroedinger operator in [29]. The analytical and spectral theory of generalized Baecklund-Darboux transformations has been developed in [30–33] .

6. The Darboux transformations can be applied to constructing the solutions of linear and nonlinear partial differential equations of different types from being investigated in this paper (see, [34,35]). The author hopes that the explicit formulae for the composition of matrix binary Darboux transformations obtained in Section 3 will allow us to construct more wide classes of exact solutions for the self-dual Yang-Mills equations than those in [34]. (The solutions from [34] in $SU(N)$ are parameterized by matrices φ, ψ of dimension $N \times K$ only for the case $K = nN$.)

We also hope that the results of this paper will be useful for generalizations onto the matrix case of the theory of finite-gap integration of non-linear system from the soliton theory, a new view on it is proposed in papers [36–39].

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