УДК 517.95

K. Urbańska

PARABOLIC VARIATIONAL INEQUALITY IN UNBOUNDED DOMAINS

K. Urbańska. *Parabolic variational inequality in unbounded domains*, Matematychni Studii, **19** (2003) 165–180.

A non-linear parabolic variational inequality in unbounded domain with respect to the space variables is considered. Sufficient conditions for the existence and the uniqueness of the solution of the inequality were obtained, which does not depend on the behaviour of the solution and the coefficients, the free term, the initial condition as $|x| \to +\infty$.

К. Урбанская. *Параболическое вариационное неравенство в неограниченных областях* // Математичнi Студiї. – 2003. – Т.19, №2. – С.165–180.

Рассматривается нелинейное параболическое вариационное неравенство в неограниченной относительно пространственных переменных области. Получены достаточные условия существования и единственности решения этого неравенства, не зависящие от поведения решения, коэффициентов, свободного члена и начального условия $|x| \to +\infty$.

It is generally known that the uniqueness of the solution of Cauchy problem or the Dirichlet problem in unbounded domains with respect to the space variables for linear parabolic equations takes place in the class of the functions satisfying the inequality $|u(x,t)| \le ce^{a|x|^2}$, where the constants c, a depend on the coefficients of the equation [1, 2]. It turned out [3–7] that for some non-linear parabolic equations the uniqueness of the solutions of the problems in unbounded domain does not depend on the behaviour of the solution as $|x| \to +\infty$, also the existence of the solution does not depend on the behaviour of the coefficients, free term, and the initial condition as $|x| \to +\infty$.

In this paper a non-linear parabolic variational inequality in an unbounded domain with respect to the space variables is considered. Sufficient conditions of the existence and the uniqueness of the solution of the inequality were obtained, which did not depend on the behaviour of the solution and the coefficients, the free term, the initial condition as $|x| \to +\infty$. Let us note that the linear parabolic variational inequalities in unbounded domains were studied in the papers [8, 9].

Let Ω be any unbounded domain in $\mathbb{R}^n$, $\partial\Omega \subset C^1$ and let $\Omega^R = \Omega \cap B_R$ be a domain for every $R > 0$, where $B_R = \{x \in R^n : |x| < R\}$. Write $\Gamma_1^R = \partial\Omega \cap B_R$.
Let

$$H^1_{\Gamma_1^R}(\Omega^R) = \Big\{u \in H^1(\Omega^R) : u|_{\Gamma_1^R} = 0\Big\},$$

$$H^1_{\mathrm{loc}}(\bar{\Omega}) = \Big\{u : (\forall R > 0)\, u \in H^1(\Omega^R)\Big\},$$

2000 *Mathematics Subject Classification:* 35K85.

$$H^1_{0,\mathrm{loc}}(\bar\Omega) = \Big\{u : (\forall R > 0)\, u \in H^1_{\Gamma_1^R}(\Omega^R)\Big\},$$

$$L^r_{\mathrm{loc}}(\bar\Omega) = \Big\{u : (\forall R > 0)\, u \in L^r(\Omega^R)\ \Big\}, r \in [1,\infty],$$

$$V_{\mathrm{loc}}(\bar\Omega) = \Big\{u : (\forall R > 0)\, u \in V(\Omega^R)\Big\},$$

where $H^1_{\Gamma_1^R}(\Omega^R) \subset V(\Omega^R) \subset H^1(\Omega^R)$ and if $u \in V(\Omega^{R_2})$ then $u \in V(\Omega^{R_1})$ for $R_2 > R_1$.

Set $Q_\tau = \Omega \times (0,\tau)$, $Q^R_\tau = \Omega^R \times (0,\tau)$ for $\tau \in (0,T]$ and $T \in (0,+\infty)$, $R > 0$, $\Omega_\tau = Q_T \cap \{t = \tau\}$, $\Omega^R_\tau = Q^R_T \cap \{t = \tau\}$. Let

$$L^r_{\mathrm{loc}}(\bar{Q_T}) = \Big\{u : (\forall R > 0)\, u \in L^r(Q^R_T)\Big\},\quad r \in [1,+\infty],$$

$$H^1_{\mathrm{loc}}(\bar{Q_T}) = \Big\{u : (\forall R > 0)\, u \in H^1(Q^R_T)\Big\}.$$

From now on we make the assumption:

(K) : K is a closed subset of $L^2((0,T);V_{\mathrm{loc}}(\bar\Omega))$ such that the span of any finite number of elements from K with the non-negative coefficients belongs to K and $0 \in K$.

Definition 1. A function $u \in C\Big([0,T]; L^2_{\mathrm{loc}}(\bar\Omega)\Big) \cap L^2\Big((0,T); V_{\mathrm{loc}}(\bar\Omega)\Big) \cap L^p_{\mathrm{loc}}(\overline{Q}_T) \cap K$ is said to be a *solution* of the inequality

$$\int_{Q_\tau} \Big(v_t(v-u)\psi(x) + \sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}(v_{x_j} - u_{x_j})\psi(x) + \sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}(v-u)\psi_{x_j}(x) +$$

$$+ \sum_{i=1}^n b_i(x,t)u_{x_i}(v-u)\psi(x) - \frac{1}{2}\alpha(u-v)^2\psi(x) + c(x,t)u(u-v)\psi(x) + g(x,t,u)(v-u)\psi(x) -$$

$$-f(x,t)(v-u)\psi(x)\Big) e^{-\alpha t}dxdt \geq \frac{1}{2}\int_{\Omega_\tau} (u-v)^2 e^{-\alpha\tau}\psi(x)dx - \frac{1}{2}\int_{\Omega_0}(u_0 - v)^2\psi(x)dx \qquad (1)$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega \qquad (2)$$

if u satisfies (1) for some $\alpha \geq 0$ and for every $\tau \in (0,T]$, $v \in H^1_{\mathrm{loc}}(\bar{Q_T}) \cap L^p_{\mathrm{loc}}(\overline{Q}_T)$ and $\psi \in C_0(\mathbb{R}^n)$, $\psi_{x_i} \in L^\infty(\mathbb{R}^n)$ for $i \in \{1,...,n\}$, $\psi(x) \geq 0$ in $\mathbb{R}^n$.

Lemma. *Let*

$$w \in C\Big([0,T]; L^2_{\mathrm{loc}}(\bar\Omega)\Big) \cap L^2\Big((0,T); V_{\mathrm{loc}}(\bar\Omega)\Big) \cap K \cap L^p_{\mathrm{loc}}(\overline{Q}_T), \text{where } p \in (2,+\infty)$$

and $u_0 \in K \cap L^p_{\mathrm{loc}}(\bar\Omega)$. *Then a solution* w_η *of the problem*

$$\eta w_{\eta t} + w_\eta = w, \quad \eta > 0, \quad t \in (0,T],$$

$$w_\eta(x,0) = u_0(x) \qquad (3)$$

is weakly convergent to w *in the topology of* $L^2\Big((0,T); V_{\mathrm{loc}}(\bar\Omega)\Big) \cap L^p_{\mathrm{loc}}(\overline{Q}_T)$.

To prove the lemma we use the fact that problem (3) is the Cauchy problem for linear equation in a Banach space.

We will make the following assumptions concerning the coefficients of inequality (1):

(A): $a_{ij}(x,t) = a_{ji}(x,t)$ almost everywhere in Q_T, $a_{ij} \in L^\infty(\overline{Q}_T)$, $i,j \in \{1,...,n\}$,

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0 \sum_{i=1}^{n} \xi_i^2,\ a_0 = \text{const} > 0$$

for every $\xi \in \mathbb{R}^n$ and almost every $(x,t) \in Q_T$;

(B): $b_i \in L^\infty(Q_T)$, $i \in \{1,...,n\}$;

(C): $c \in L^\infty_{\text{loc}}(\overline{Q}_T)$, $c(x,t) \geq c_0$ almost everywhere in Q_T, $c_0 = \text{const}$;

(G): $g(\cdot,\cdot,\tau)$ is a measurable function in Q_T for every $\tau \in \mathbb{R}$, a function $g(x,t,\cdot)$ is continuous in $\mathbb{R}$ for almost every $(x,t) \in Q_T$; the following inequalities are satisfied: $\Big(g(x,t,\zeta) - g(x,t,\tau)\Big)(\zeta - \tau) \geq g_0|\zeta-\tau|^p$, $g_0 = \text{const} > 0$, $p \in (2,+\infty)$ for every $\zeta,\tau \in \mathbb{R}$ and almost every $(x,t) \in Q_T$, $|g(x,t,\tau)| \leq g_1|\tau|^{p-1}$, $g_1 = \text{const}$ for every $\tau \in \mathbb{R}$ and almost every $(x,t) \in Q_T$.

Set

$$b_0 = \sup_{Q_T} \sum_{i=1}^{n} b_i^2(x,t),\ a_1 = \sup_{Q_T} \sum_{i,j=1}^{n} a_{ij}^2(x,t),\ \alpha_0 = \max\Big\{0, -2c_0 + \frac{a_1+b_0}{2a_0}\Big\},$$

$q = \dfrac{p}{p-1}$, $p \in (2,+\infty)$.

Theorem 1. *Let assumptions (A),(B),(C),(G),(K) hold and $p < \frac{2n}{n-2}$ for $n > 2$. Inequality (1) has at most one solution for $\alpha = \alpha_0$.*

Proof. Let u_1 and u_2 be two solutions of inequality (1). Then

$$\int_{Q_T} \Bigg(v_t(v-u_k)\psi(x) + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{kx_i}(v_{x_j} - u_{kx_j})\psi(x) + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{kx_i}(v-u_k)\psi_{x_j} + \\ + \sum_{i=1}^{n} b_i(x,t)u_{kx_i}(v-u_k)\psi(x) - \frac{1}{2}\alpha_0(v-u_k)^2\psi(x) + c(x,t)u_k(v-u_k)\psi(x) + \\ + g(x,t,u_k)(v-u_k)\psi(x) - f(x,t)(v-u_k)\psi(x)\Bigg)e^{-\alpha_0 t}dxdt \geq \\ \geq \frac{1}{2}\int_{\Omega_T}(u_k - v)^2 e^{-\alpha_0 T}\psi(x)dx - \frac{1}{2}\int_{\Omega_0}(u_0 - v)^2\psi(x)dx, \quad k \in \{1,2\}. \tag{4}$$

In equation (3) we put $w = \frac{1}{2}(u_1 + u_2)$. Letting $v = w_\eta$ in inequality (4), where w_η is the solution of problem (3), and adding the corresponding inequalities we obtain

$$\int_{Q_T} \Bigg(2w_{\eta t}(w_\eta - w)\psi(x) + \sum_{i,j=1}^{n} a_{ij}(x,t)\Big(u_{1x_i}(w_{\eta x_j} - u_{1x_j}) + u_{2x_i}(w_{\eta x_j} - u_{2x_j})\Big)\psi(x) +$$

$$+\sum_{i,j=1}^{n} a_{ij}(x,t)\Big(u_{1x_i}(w_\eta-u_1)+u_{2x_i}(w_\eta-u_2)\Big)\psi_{x_j}(x)+$$

$$+\sum_{i=1}^{n} b_i(x,t)\Big(u_{1x_i}(w_\eta-u_1)+u_{2x_i}(w_\eta-u_2)\Big)\psi(x)-$$

$$-\frac{1}{2}\alpha_0\Big((u_1-w_\eta)^2+(u_2-w_\eta)^2\Big)\psi(x)+c(x,t)\Big(u_1(w_\eta-u_1)+u_2(w_\eta-u_2)\Big)\psi(x)+$$

$$+\Big(g(x,t,u_1)(w_\eta-u_1)+g(x,t,u_2)(w_\eta-u_2)\Big)\psi(x)-2f(x,t)(w_\eta-w)\psi(x)\Bigg)e^{-\alpha_0 t}dxdt\geq$$

$$\geq\frac{1}{2}\int_{\Omega_T}\Big((u_1-w_\eta)^2+(u_2-w_\eta)^2\Big)\psi(x)e^{-\alpha_0 T}dx. \tag{5}$$

From (3) we have $w_\eta - w = -\eta w_{\eta t}$, so

$$2\int_{Q_T} w_{\eta t}(w_\eta-w)\psi(x)e^{-\alpha_0 t}dxdt=-2\eta\int_{Q_T}(w_{\eta t})^2\psi(x)e^{-\alpha_0 t}dxdt\leq 0.$$

Moreover,

$$\int_{\Omega_T}\Big((u_1-w_\eta)^2+(u_2-w_\eta)^2\Big)\psi(x)e^{-\alpha_0 T}dx\geq 0.$$

We conclude from (5) that

$$\int_{Q_T}\Bigg(\sum_{i,j=1}^{n} a_{ij}(x,t)\Big(u_{1x_i}(w_{\eta x_j}-u_{1x_j}+u_{2x_i}(w_{\eta x_j}-u_{2x_j})\Big)\psi(x)+$$

$$+\sum_{i,j=1}^{n} a_{ij}(x,t)\Big(u_{1x_i}(w_\eta-u_1)+u_{2x_i}(w_\eta-u_2)\Big)\psi_{x_j}(x)+$$

$$+\sum_{i=1}^{n} b_i(x,t)\Big(u_{1x_i}(w_eta-u_1)+u_{2x_i}(w_\eta-u_2)\Big)\psi_{x_j}(x)+$$

$$+c(x,t)\Big(u_1(w_\eta-u_1)+u_2(w_\eta-u_2)\Big)\psi(x)+\Big(g(x,t,u_1)(w_\eta-u_1)+$$

$$+g(x,t,u_2)(w_\eta-u_2)\Big)\psi(x)-2f(x,t)(w_\eta-w)\psi(x)\Bigg)e^{-\alpha_0 t}dxdt\geq$$

$$\geq\frac{1}{2}\alpha_0\int_{Q_T}\Big((u_1-w_\eta)^2+(u_2-w_\eta t)^2\Big)\psi(x)e^{-\alpha_0 t}dxdt. \tag{6}$$

By the lemma, $w_\eta \to w$ in the weak topology of $L^2\Big((0,T);V_{\rm loc}(\bar\Omega)\Big)\cap L^p_{\rm loc}(\overline{Q}_T)$, thus

$$\liminf_{\eta\to+0}\int_{Q_T}\Big((u_1-w_\eta)^2+(u_2-w_\eta)^2\Big)\psi(x)e^{-\alpha_0 t}dxdt\geq\frac{1}{2}\int_{Q_T}u^2\psi(x)e^{-\alpha_0 t}dxdt,$$

where $u = u_1 - u_2$.

Comparing this with (6), we get

$$\int_{Q_T}\Bigg(\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}u_{x_j}\psi(x)+\sum_{i,j=1}^{n} a_{ij}(x,t)\psi_{x_j}u_{x_i}u+\sum_{i=1}^{n} b_i(x,t)u_{x_i}u\psi(x)+\frac{1}{2}\alpha_0 u^2\psi(x)+$$

$$+c(x,t)u^2\psi(x)+\Big(g(x,t,u_1)-g(x,t,u_2)\Big)u\psi(x)\Big)e^{-\alpha_0 t}dxdt\le 0. \qquad (7)$$

By assumptions (A), (B), (C), (G), we have

$$I_1=\int_{Q_T}\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}u_{x_j}\psi(x)e^{-\alpha_0 t}dxdt\ge a_0\int_{Q_T}\sum_{i=1}^n u_{x_i}^2\psi(x)e^{-\alpha_0 t}dxdt,$$

$$I_2=\int_{Q_T}\sum_{i,j=1}^n a_{ij}(x,t)\psi_{x_j}u_{x_i}ue^{-\alpha_0 t}dxdt\le\frac{1}{2}a_1\delta_0\int_{Q_T}\sum_{i=1}^n u_{x_i}^2\psi(x)e^{-\alpha_0 t}dxdt+$$

$$+\frac{\delta_1}{p\delta_0}\int_{Q_T}|u|^p\psi(x)e^{-\alpha_0 t}dxdt+I_2^1,\quad \delta_1>0,$$

where

$$I_2^1=\frac{\delta_1^{\frac{2}{2-p}}(p-2)}{2\delta_0 p}\int_{Q_T}(\psi(x))^{\frac{2}{2-p}}\Big(\sum_{j=1}^n\frac{\psi_{x_j}^2(x)}{\psi(x)}\Big)^{\frac{p}{p-2}}e^{-\alpha_0 t}dxdt,$$

$$I_3=\int_{Q_T}\sum_{i=1}^n b_i(x,t)u_{x_i}u\psi(x)e^{-\alpha_0 t}dxdt\le\frac{1}{2}\delta_0 b_0\int_{Q_T}\sum_{i=1}^n u_{x_i}^2\psi(x)e^{-\alpha_0 t}dxdt+$$

$$+\frac{1}{2\delta_0}\int_{Q_T}u^2\psi(x)e^{-\alpha_0 t}dxdt,$$

$$I_4=\int_{Q_T}c(x,t)u^2\psi(x)e^{-\alpha_0 t}dxdt\ge c_0\int_{Q_T}u^2\psi(x)e^{-\alpha_0 t}dxdt,$$

$$I_5=\int_{Q_T}\Big(g(x,t,u_1)-g(x,t,u_2)\Big)u\psi(x)e^{-\alpha_0 t}dxdt\ge g_0\int_{Q_T}|u|^p\psi(x)e^{-\alpha_0 t}dxdt.$$

Putting these estimates into (7), we obtain

$$\int_{Q_T}\Big(\Big(a_0-\frac{1}{2}\delta_0 a_1-\frac{1}{2}b_0\delta_0\Big)\sum_{i=1}^n u_{x_i}^2+\Big(\frac{1}{2}\alpha_0+c_0-\frac{1}{2\delta_0}\Big)u^2+\Big(g_0-\frac{\delta_1}{p\delta_0}\Big)|u|^p\Big)\times$$

$$\times\psi(x)e^{-\alpha_0 t}dxdt\le I_2^1. \qquad (8)$$

Let $\delta_0=\dfrac{a_0}{a_1+b_0}$, $\delta_1=\dfrac{(g_0p\delta_0)}{2}$,

$$\psi(x)=\begin{cases}\Big(\frac{1}{R}(R^2-|x|^2\Big), & |x|\le R,\\ 0, & |x|>R,\end{cases}\ ,R>0.$$

It is easy to check that $\sum_{j=1}^n\psi_{x_j}^2(x)\le 4$. From (8) we get the inequality

$$\int_{Q_T}|u|^p\psi(x)e^{-\alpha_0 t}dxdt\le\mu_5\int_{Q_T}\Big(\psi(x)\Big)^{1-\frac{2p}{p-2}}e^{-\alpha_0 t}dxdt, \qquad (9)$$

where μ_5 is a constant independent of R.

Next

$$\int_{Q_T}\Big(\psi(x)\Big)^{1-\frac{2p}{p-2}}e^{-\alpha_0 t}dxdt\le T\int_{B_R}\Big(\frac{1}{R}(R^2-|x|^2)\Big)^{1-\frac{2p}{p-2}}dx\le$$
$$\le T2^{1-\frac{2p}{p-2}}\int_{B_R}(R-|x|)^{1-\frac{2p}{p-2}}dx\le \mu_6 R^{1+n-\frac{2p}{p-2}},$$

μ_6 is a constant independent of R.
Let us now take $R_0>0, R_0<R$. Then

$$\int_{Q_T}|u|^p\psi(x)e^{-\alpha_0 t}dxdt\ge\int_{Q_T^{R_0}}|u|^p(R-R_0)e^{-\alpha_0 T}dxdt,$$

so from (9) we find that

$$\int_{Q_T^{R_0}}|u|^p dxdt\le\mu_7\Big(\frac{R}{R-R_0}\Big)R^{n-\frac{2p}{p-2}}, \tag{10}$$

where μ_7 is a constant independent of R.
By the assumption $n<\frac{2p}{p-2}$, we may choose R (for fixed R_0) large enough such that the right-hand side of (10) be smaller than any small number. Thus $u=0$ in $Q_T^{R_0}$. As R_0 is an arbitrary number, the proof is complete. □

For every $R>0$ we define $\mathcal{B}$ to be a bounded, monotonic, semi-continuous operator defined in $L^2\Big((0,T);V(\Omega^R)\Big)$ with values in $L^2\Big((0,T);V^*(\Omega^R)\Big)$ and $\mathcal{B}(0)=0$.

Set

$$K=\{v\in V_{\rm loc}(\bar\Omega):(\forall R>0)\ v\in K_R\},\quad K_R=\{v\in V(\Omega^R):\mathcal{B}(v)=0\}.$$

Theorem 2. *Let assumptions (A), (B), (C), (G), (K) hold, we are given* $f\in L^q_{\rm loc}(\overline{Q}_T)$, $u_0\in K\cap L^p_{\rm loc}(\bar\Omega)$ *and* $p<\frac{2n}{n-2}$ *for* $n>2$. *Then there exists a solution of inequality (1).*

Proof. Let $R>0$ be an arbitrary fixed number,

$$W_m(\Omega^R)=H^m(\Omega^R)\cap V(\Omega^R),$$

where $m=[\frac{n}{2}]+2$.

The space $W_m(\Omega^R)$ is dense in $W(\Omega^R)=V(\Omega^R)\cap L^p(\Omega^R)$. We choose a base $\{\varphi_k\}$ of $W_m(\Omega^R)$ as a system of the eigenfunctions of the problem

$$(\varphi_k,v)_{W_m(\Omega^R)}=\lambda_k\Big(\varphi_k,v\Big)_{L^2(\Omega^R)},\quad v\in W_m(\Omega^R), \tag{11}$$

where we denote by $\big(\cdot,\cdot\big)_H$ the inner product in a Hilbert space H. Without loss of generality we can assume that $\{\varphi_k\}$ is orthonormal in $L^2(\Omega^R)$.

Consider the functions

$$u^N(x,t)=\sum_{k=1}^N c_k^N(t)\varphi_k(x),\quad N\in\{1,2,...\}$$

where $c_1^N, ..., c_N^N$ are the solutions of the system

$$\int_{\Omega^R}\Big(u_t^N\varphi_k+\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^N\varphi_{kx_j}+\sum_{i=1}^n b_i(x,t)u_{x_i}^N\varphi_k+c(x,t)u^N\varphi_k+g(x,t,u^N)\varphi_k-$$

$$-f(x,t)\varphi_k\Big)dx+\frac{1}{\varepsilon}\Big\langle\mathcal{B}(u^N),\varphi_k\Big\rangle_{A_0(R)}=0,\quad k\in\{1,...,N\},\ \varepsilon>0, \tag{12}$$

with the initial conditions

$$c_k^N(0)=u_{0,k}^N,\quad k\in\{1,...,N\}, \tag{13}$$

where $u_0^N(x)=\sum_{k=1}^N u_{0,k}^N\varphi_k(x)$ and $\lim_{N\to+\infty}\|u_0^N-u_0\|_{W(\Omega^R)}=0$. We will denote by $\langle\cdot,\cdot\rangle>_{A_0(R)}$ the duality between $V^*(\Omega^R)$ and $V(\Omega^R)$. Multiplying (12) by $c_k^N(t)e^{-\alpha_0 t}$, summing on k and integrating from 0 to $\tau\in(0,T]$ gives

$$\int_{Q_\tau^R}\Big(u_t^Nu^N+\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^Nu_{x_j}^N+\sum_{i=1}^n b_i(x,t)u_{x_i}^Nu^N+c(x,t)(u^N)^2+g(x,t,u^N)u^N-$$

$$-f(x,t)u^N\Big)e^{-\alpha_0 t}dxdt+\frac{1}{\varepsilon}\int_0^\tau\Big\langle\mathcal{B}(u^N),u^N\Big\rangle_{A_0(R)}e^{-\alpha_0 t}dt=0. \tag{14}$$

Proceeding as in Theorem 1, we obtain

$$I_6=\int_{Q_\tau^R}\Big(u_t^Nu^N+\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^Nu_{x_j}^N+\sum_{i=1}^n b_i(x,t)u_{x_i}^Nu^N+c(x,t)(u^N)^2\Big)e^{-\alpha_0 t}dxdt\geq$$

$$\geq\int_{Q_\tau^R}\Big(\Big(a_0-\frac{1}{2}\delta_2a_1-\frac{1}{2}b_0\delta_2\Big)\sum_{i=1}^n(u_{x_i}^N)^2+\Big(\frac{1}{2}\alpha_0+c_0-\frac{1}{2\delta_2}\Big)(u^N)^2\Big)e^{-\alpha_0 t}dxdt+$$

$$+\frac{1}{2}\int_{\Omega_\tau^R}(u^N)^2e^{-\alpha_0\tau}dx-\frac{1}{2}\int_{\Omega_0^R}(u_0^N)^2dx,\quad \delta_2>0.$$

By assumption (G) we find

$$I_7=\int_{Q_\tau^R}g(x,t,u^N)u^Ne^{-\alpha_0 t}dxdt\geq g_0\int_{Q_\tau^R}|u^N|^pe^{-\alpha_0 t}dxdt.$$

The operator $\mathcal{B}$ is monotonic, this implies that

$$I_8=\frac{1}{\varepsilon}\int_0^\tau\Big\langle\mathcal{B}(u^N),u^N\Big\rangle_{A_0(R)}e^{-\alpha_0 t}dt\geq 0.$$

Furthermore

$$I_9=-\int_{Q_\tau^R}f(x,t)u^Ne^{-\alpha_0 t}dxdt\geq-\frac{\delta_3}{p}\int_{Q_\tau^R}|u^N|^pe^{-\alpha_0 t}dxdt,\quad \delta_3>0.$$

Putting the estimates of $I_6, ..., I_9$ into (14), we get

$$\int_{\Omega_\tau^R} (u^N)^2 e^{-\alpha\tau} dx + \int_{Q_\tau^R} \Big(\Big(2a_0 - \delta_2(a_1 + b_0)\Big) \sum_{i=1}^n (u_{x_i}^N)^2 + \Big(\alpha_0 + 2c_0 - \frac{1}{\delta_2}\Big)(u^N)^2 +$$
$$+ \Big(2g_0 - \frac{2\delta_3}{p}\Big)|u^N|^p \Big) e^{-\alpha_0 t} dx dt + \frac{1}{\varepsilon} \int_0^\tau \Big\langle \mathcal{B}(u^N), u^N \Big\rangle_{A_0(R)} e^{-\alpha_0 t} dt \le$$
$$\le \int_{\Omega_0^R} (u_0^N)^2 dx + \frac{2}{q\delta_3^{\frac{2}{p}}} \int_{Q_\tau^R} |f(x,t)|^q e^{-\alpha_0 t} dx dt, \quad \tau \in (0,T]. \tag{15}$$

Now choose in (15) $\delta_2 = \frac{a_0}{a_1 + b_0}, \delta_3 = \frac{pg_0}{2}$. It follows that

$$\int_{\Omega_\tau^R} |u^N|^2 dx \le \mu_8, \ \ \tau \in [0,T], \tag{16}$$

$$\int_{Q_T^R} \Big((u^N)^2 + |u^N|^p + \sum_{i=1}^n (u_{x_i}^N)^2 \Big) dx dt \le \mu_8, \tag{17}$$

$$\frac{1}{\varepsilon} \int_0^T \Big\langle \mathcal{B}(u^N), u^N \Big\rangle_{A_0(R)} dt \le \mu_8, \tag{18}$$

where μ_8 is a constant independent on N.

Making use of (G) and (17) we conclude that

$$\int_{Q_T^R} |g(x,t,u^N)|^q dx dt \le \mu_9, \tag{19}$$

where μ_9 is a constant independent of N.

We will denote by S_N the linear span of the set $\{\varphi_1, ..., \varphi_N\}$. Then the projection P_N in $L^2(\Omega^R)$ onto S_N is bounded by one in $\mathcal{L}\Big(L^2(\Omega^R), L^2(\Omega^R)\Big)$, $\mathcal{L}\Big(W_m(\Omega^R), W_m(\Omega^R)\Big)$, $\mathcal{L}\Big(W_m^*(\Omega^R), W_m^*(\Omega^R)\Big)$.

From (12) we obtain

$$u_t^N + P_N A u^N + P_N g^N + P_N \Big(\frac{1}{\varepsilon}\mathcal{B}(u^N)\Big) = P_N f, \tag{20}$$

where $g^N = g(x,t,u^N)$ and an operator A is defined by the formula

$$(\forall v \in V(\Omega^R)) \ \Big\langle Au^N, v \Big\rangle = \int_{\Omega^R} \Big(\sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}^N v_{x_j} + \sum_{i=1}^n b_i(x,t) u_{x_i}^N v + c(x,t) uv \Big) dx.$$

It follows from (17) that the set $\{Au^N\}$ is bounded in $L^2\Big((0,T); V^*(\Omega^R)\Big)$, so $\{Au^N\}$ is bounded in $L^2\Big((0,T); W_m^*(\Omega^R)\Big)$. Then $\{P_N A u^N\}$ is bounded in $L^2\Big((0,T); W_m^*(\Omega^R)\Big)$. Analogously we get that $\{P_N g_N\}$ is bounded in $L^q\Big((0,T); W_m^*(\Omega^R)\Big)$, $\{P_N(\frac{1}{\varepsilon}\mathcal{B}(u^N))\}$ is bounded in $L^2\Big((0,T); W_m^*(\Omega^R)\Big)$, $\{P_N f\}$ is bounded in $L^q\Big((0,T); W_m^*\Big)$.

We have from (20)

$$\|u_t^N\|_{L^q\left((0,T);W_m^*\right)} \le \mu_{10}, \tag{21}$$

where μ_{10} is a constant independent on N.

Consider the space $S(Q_T^R) = \Big\{u : u \in L^2\Big((0,T);W(\Omega^R)\Big), u_t \in L^q\Big((0,T);W_m^*(\Omega^R)\Big)\Big\}$.

Obviously $W(\Omega^R) \subset L^2(\Omega^R) \subset W_m^*(\Omega^R)$ at the same time the inclusion $W(\Omega^R) \subset L^2(\Omega^R)$ is compact.

Estimates (16), (17), (19), (21) imply that there exists a subsequence $\{u^{N_k}\}$ of $\{u^N\}$ such that $u^{N_k} \to u^{\varepsilon,R}$ in the weak-star topology of $L^\infty\Big((0,T);L^2(\Omega^R)\Big)$, $u^{N_k} \to u^{\varepsilon,R}$ in the weak topology of $L^2\Big((0,T);V(\Omega^R)\Big) \cap L^p(Q_T^R)$, $u_t^{N_k} \to u_t^{\varepsilon,R}$ in the weak topology of $L^q\Big((0,T);W_m^*(\Omega^R)\Big)$, $u^{N_k} \to u^{\varepsilon,R}$ in $L^q(Q_T^R)$ and almost everywhere in Q_T^R and $g(\cdot,\cdot,u^{N_k}) \to z^{\varepsilon,R}$ in the weak topology of $L^q(Q_T^R)$ as $k \to +\infty$.

Then $\{g(x,t,u^{N_k})\}$ is convergent to $g(x,t,u^{\varepsilon,R})$ almost everywhere in Q_T^R and $g(x,t,u^{\varepsilon,R}) = z^{\varepsilon,R}(x,t)$. Since the operator $\mathcal{B}$ is bounded, we can write

$$\|\mathcal{B}(u^N)\|_{L^2\left((0,T);V^*(\Omega^R)\right)} \le \mu_{12},$$

where μ_{12} is a constant independent of N.

Without loss of generality we can assume that $\mathcal{B}(u^{N_k}) \to \theta^{\varepsilon,R}$ weakly in $L^2\Big((0,T);V^*(\Omega^R)\Big)$ as $k \to +\infty$. Using (12) it is easy to prove an equality

$$\int_0^\tau \Big\langle u_t^{\varepsilon,R}, v\Big\rangle_{(W_m^*(\Omega^R),W_m(\Omega^R))} dt + \int_{Q_T^R}\Bigg(\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^{\varepsilon,R}v_{x_j} + \sum_{i=1}^n b_i(x,t)u_{x_i}^{\varepsilon,R}v +$$

$$+c(x,t)u^{\varepsilon,R}v + g(x,t,u^{\varepsilon,R})v - f(x,t)v\Bigg)dxdt + \frac{1}{\varepsilon}\int_0^\tau \Big\langle \theta^{\varepsilon,R}, v\Big\rangle_{A_0(R)} dt = 0, \tag{22}$$

$$\forall \tau \in (0,T], \forall v \in H^1(Q_T^R) \cap Y(Q_T^R), \text{where } Y(Q_T^R) = L^2\Big((0,T);V(\Omega^R)\Big) \cap L^p(Q_T^R).$$

We deduce from (22) that $u_t^{\varepsilon,R} \in Y^*(Q_T^R)$. Therefore by Theorem 1.17 [10], we get $u^{\varepsilon,R} \in C\Big([0,T];L^2(\Omega^R)\Big)$ and

$$\Big\langle u_t^{\varepsilon,R}, v\Big\rangle_{(Y^*(Q_T^R),Y(Q_T^R))} = \int_{\Omega_\tau^R} u^{\varepsilon,R}vdx - \int_{\Omega_0^R} u^{\varepsilon,R}vdx - \int_{Q_\tau^R} u^{\varepsilon,R}v_tdxdt. \tag{23}$$

We have

$$u_k^N(\cdot,0) \to u^{\varepsilon,R}(\cdot,0) \text{ weakly in } L^2(\Omega^R) \text{ and } u_k^N(\cdot,0) \to u_0 \text{ in } L^2(\Omega^R).$$

Thus

$$u^{\varepsilon,R}(x,0) = u_0(x).$$

By using (22), (23) becomes

$$\int_{\Omega_\tau^R} u^{\varepsilon,R} w dx + \int_{Q_T^R} \Big(- u^{\varepsilon,R} w_t + \sum_{i,j=1}^n a_{ij}(x,t) u^{\varepsilon,R}_{x_i} w_{x_j} + \sum_{i=1}^n b_i(x,t) u^{\varepsilon,R}_{x_i} w + c(x,t) u^{\varepsilon,R} w +$$

$$+ g(x,t,u^{\varepsilon,R}) w - f(x,t) w \Big) dx dt + \frac{1}{\varepsilon} \int_0^\tau \Big\langle \theta^{\varepsilon,R} w \Big\rangle_{A_0(R)} dt = \int_{\Omega_0^R} u_0 w dx \tag{24}$$

$\forall w \in H^1(Q_T^R) \cap Y(Q_T^R), \tau \in (0,T]$. Applying (23) we have

$$\frac{1}{2} \int_{\Omega_\tau^R} (u^{\varepsilon,R})^2 e^{-\alpha_0 \tau} dx + \int_{Q_\tau^R} \Big(\sum_{i,j=1}^n a_{ij}(x,t) u^{\varepsilon,R}_{x_i} u^{\varepsilon,R}_{x_j} + \frac{1}{2} \alpha_0 (u^{\varepsilon,R})^2 + \sum_{i=1}^n b_i(x,t) u^{\varepsilon,R}_{x_i} u^{\varepsilon,R} +$$

$$+ c(x,t)(u^{\varepsilon,R})^2 + g(x,t,u^{\varepsilon,R}) u^{\varepsilon,R} - f(x,t) u^{\varepsilon,R} \Big) e^{-\alpha_0 t} dx dt + \frac{1}{\varepsilon} \int_0^\tau \Big\langle \theta^\varepsilon, u^{\varepsilon,R} \Big\rangle_{A_0(R)} e^{-\alpha_0 t} dt =$$

$$= \int_{\Omega_0^R} u_0^2 dx, \qquad \tau \in (0,T]. \tag{25}$$

Proceeding as in [11, II,1.2] we prove that $\theta^{\varepsilon,R} = \mathcal{B}(u^{\varepsilon,R})$. Hence $\forall v \in H^1(Q_T^R) \cap Y(Q_T^R)$ and $\forall \tau \in (0,T]$ we obtain

$$\int_{\Omega_\tau^R} u^{\varepsilon,R} v dx + \int_{Q_\tau^R} \Big(- u^{\varepsilon,R} v_t + \sum_{i,j=1}^n a_{ij}(x,t) u^{\varepsilon,R}_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x,t) u^{\varepsilon,R}_{x_i} v + c(x,t)(u^{\varepsilon,R})^2 +$$

$$+ g(x,t,u^{\varepsilon,R}) v - f(x,t) v \Big) dx dt + \frac{1}{\varepsilon} \int_0^\tau \Big\langle \mathcal{B}(u^{\varepsilon,R}), v \Big\rangle_{A_0(R)} dt = \int_{\Omega_0^R} u_0 v dx. \tag{26}$$

Let R be any natural number. We are given a sequence $\{u^{\varepsilon,k}\}$, where $u^{\varepsilon,k}$ satisfies the equation

$$\int_{\Omega_\tau^k} u^{\varepsilon,k} v dx + \int_{Q_\tau^k} \Big(- u^{\varepsilon,k} v_t + \sum_{i,j=1}^n a_{ij}(x,t) u^{\varepsilon,k}_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x,t) u^{\varepsilon,k}_{x_i} v + c(x,t) u^{\varepsilon,k} v +$$

$$+ g(x,t,u^{\varepsilon,k}) v - f(x,t) v \Big) dx dt + \frac{1}{\varepsilon} \int_o^\tau \Big\langle \mathcal{B}(u^{\varepsilon,k}), v \Big\rangle_{A_0(k)} dt = \int_{\Omega_0^k} u_0 v dx, \tag{27}$$

$\forall v \in H^1_{\text{loc}}(\overline{Q}_T) \cap L^2\Big((0,T); V_{\text{loc}}(\bar{\Omega})\Big) \cap L^p_{\text{loc}}(\overline{Q}_T) = H^1_{\text{loc}}(\overline{Q}_T) \cap Y_{\text{loc}}(\overline{Q}_T), \forall \tau \in [0,T]$.

Obviously $u^{\varepsilon,k}(x,t) = u^{\varepsilon,l}(x,t)$, $l > k$ in Q_T^k. Write

$$u^\varepsilon(x,t) = u^{\varepsilon,k}(x,t), \ (x,t) \in Q_T^k, \ \ k \in \mathbb{N}.$$

Therefore (by (27)) the function u^ε satisfies the equation

$$- \int_{\Omega_\tau} u^\varepsilon w \psi(x) e^{-\alpha_0 \tau} dx - \int_{Q_\tau} \Big(- u^\varepsilon w_t \psi(x) + \alpha_0 u^\varepsilon w \psi(x) + \sum_{i,j=1}^n a_{ij}(x,t) u^\varepsilon_{x_i} w_{x_j} \psi(x) +$$

$$+\sum_{i,j=1}^{n} a_{ij}(x,t)u^{\varepsilon}_{x_i} w\psi_{x_j}(x) + \sum_{i=1}^{n} b_i(x,t)u^{\varepsilon}_{x_i} w\psi(x) + c(x,t)u^{\varepsilon} w\psi(x) + g(x,t,u^{\varepsilon})w\psi(x)-$$

$$-f(x,t)w\psi(x)\Big)e^{-\alpha_0 t}dxdt - \frac{1}{\varepsilon}\int_0^{\tau}\Big\langle \mathcal{B}(u^{\varepsilon}), w\psi(x)\Big\rangle_{A_0(R)} e^{-\alpha_0 t}dt = -\int_{\Omega_0} u_0 w\psi(x)dx, \quad (28)$$

$\forall \tau \in (0,T], \forall w \in H^1_{\mathrm{loc}}(\overline{Q}_T) \cap Y_{\mathrm{loc}}(\overline{Q}_T) \cap K$, and

$$\frac{1}{2}\int_{\Omega_\tau} (u^{\varepsilon})^2 \psi(x) e^{-\alpha_0 T}dx + \int_{Q_T}\Big(\sum_{i,j=1}^{n} a_{ij}(x,t)u^{\varepsilon}_{x_i}u^{\varepsilon}_{x_j}\psi(x) + \sum_{i,j=1}^{n} a_{ij}(x,t)u^{\varepsilon}_{x_i}u^{\varepsilon}\psi_{x_j}(x)+$$

$$+\sum_{i=1}^{n} b_i(x,t)u^{\varepsilon}_{x_i}u^{\varepsilon}\psi(x) + \Big(\frac{1}{2}\alpha_0 + c(x,t)\Big)(u^{\varepsilon}(x,t))^2\psi(x) + g(x,t,u^{\varepsilon})u^{\varepsilon}\psi(x)-$$

$$-f(x,t)u^{\varepsilon}\psi(x)\Big)e^{-\alpha_0 t}dxdt + \frac{1}{\varepsilon}\int_0^{\tau}\Big\langle \mathcal{B}(u^{\varepsilon}), u^{\varepsilon}\psi\Big\rangle_{A_0(R)} e^{-\alpha_0 t}dt = \int_{\Omega_0} u_0^2\psi(x)dx, \quad (29)$$

$\forall \tau \in (0,T]$, $\psi \in C^1(\mathbb{R}^n)$, $\psi_{x_i} \in L^{\infty}(\mathbb{R}^n)$, $i \in \{1,\dots,n\}$, $\operatorname{supp}\psi \subset B_R, R > 0$, $\psi(x) \geq 0$ in $\mathbb{R}^n$. Moreover,

$$-\int_{Q_\tau} ww_t\psi(x)e^{-\alpha_0 t}dxdt = -\frac{1}{2}\int_{\Omega_\tau} w^2\psi(x)e^{-\alpha_0\tau}dx + \frac{1}{2}\int_{\Omega_0} w^2\psi(x)dx-$$

$$-\frac{1}{2}\alpha_0\int_{Q_\tau} w^2\psi(x)e^{-\alpha_0 t}dxdt. \quad (30)$$

Adding (28), (29) and (30) we obtain

$$\frac{1}{2}\int_{\Omega_\tau}\Big(u^{\varepsilon} - w\Big)^2\psi(x)e^{-\alpha_0\tau}dx + \int_{Q_\tau}\Big(\sum_{i,j=1}^{n} a_{ij}(x,t)u^{\varepsilon}_{x_i}\Big(u^{\varepsilon}_{x_j} - w_{x_j}\Big)\psi(x)+$$

$$+\sum_{i,j=1}^{n} a_{ij}(x,t)u^{\varepsilon}_{x_i}\Big(u^{\varepsilon} - w\Big)\psi_{x_j}(x) + \sum_{i=1}^{n} b_i(x,t)u^{\varepsilon}_{x_i}\Big(u^{\varepsilon} - w\Big)\psi(x)w_t\Big(u^{\varepsilon} - w\Big)\psi(x)+$$

$$+c(x,t)u^{\varepsilon}\Big(u^{\varepsilon} - w\Big)\psi(x) + \frac{1}{2}\alpha_0\Big(u^{\varepsilon} - w\Big)^2\psi(x) + g(x,t,u^{\varepsilon})\Big(u^{\varepsilon} - w\Big)\psi(x)-$$

$$-f(x,t)\Big(u^{\varepsilon} - w\Big)\psi(x)\Big)e^{-\alpha_0 t}dxdt + \frac{1}{\varepsilon}\int_0^{\tau}\Big\langle \mathcal{B}(u^{\varepsilon}), (u^{\varepsilon} - w)\psi\Big\rangle_{A_0(R)} e^{-\alpha_0 t}dt =$$

$$= \frac{1}{2}\int_{\Omega_0}\Big(u_0 - w\Big)^2\psi(x)dx. \quad (31)$$

But $w \in K$ and the operator $\mathcal{B}$ is monotonic, it follows that $\mathcal{B}(w) = 0$ and from (31) we get the inequality

$$\int_{\Omega_\tau}\Big(u^{\varepsilon} - w\Big)^2\psi(x)e^{-\alpha_0\tau}dx - \int_{\Omega_0}\Big(u_0 - w\Big)^2\psi(x)dx \leq 2\int_{Q_\tau}\Big(w_t\Big(w - u^{\varepsilon}\Big)\psi(x)-$$

$$-\frac{1}{2}\alpha_0\Big(w-u^\varepsilon\Big)^2\psi(x)+\sum_{i,j=1}^n a_{ij}(x,t)u^\varepsilon_{x_i}\Big(w_{x_j}-u^\varepsilon_{x_j}\Big)\psi(x)+\sum_{i,j=1}^n a_{ij}(x,t)u^\varepsilon_{x_i}\Big(w-u^\varepsilon\Big)\psi_{x_j}(x)+$$

$$+\sum_{i=1}^n b_i(x,t)u^\varepsilon_{x_i}\Big(w-u^\varepsilon\Big)\psi(x)+c(x,t)u^\varepsilon\Big(w-u^\varepsilon\Big)\psi(x)+g(x,t,u^\varepsilon)\Big(w-u^\varepsilon\Big)\psi(x)-$$

$$-f(x,t)\Big(w-u^\varepsilon\Big)\psi(x)\Big)e^{-\alpha_0 t}dxdt,\quad \tau\in(0,T]. \tag{32}$$

Let

$$\psi(x)=\begin{cases}\frac{R^2-|x|^2}{R}, & |x|\le R,\\ 0, & |x|>R.\end{cases}$$

We conclude from (16)–(19) (for $R=l$), that

$$\int_{\Omega^l_\tau}(u^\varepsilon)^2dx\le\mu_{12},\ \tau\in[0,T],\quad \int_{Q^l_T}\Big(|u^\varepsilon|^2+|u^\varepsilon|^p+\sum_{i=1}^n(u^\varepsilon_{x_i})^2\Big)dxdt\le\mu_{13},$$

$$\int_0^T\Big\langle\mathcal{B}(u^\varepsilon),u^\varepsilon\Big\rangle_{A_0(l)}dt\le\mu_{13}\varepsilon,\|g(\cdot,\cdot,u^\varepsilon)\|_{L^q(Q^l_T)}\le\mu_{13},$$

where μ_{13} is a constant independent on ε (but depends on l), $l\in\mathbb{N}$.

Therefore we may extract a subsequence $\{u^{\varepsilon(k),1}\}$ of $\{u^\varepsilon\}$ such that

$$u^{\varepsilon(k),1}\to u^1 \text{ weakly-star in } L^\infty\Big((0,T);L^2(\Omega^1)\Big),$$

$$u^{\varepsilon(k),1}\to u^1 \text{ weakly in } L^2\Big((0,T);V(\Omega^1)\Big)\cap L^p(Q^1_T),$$

$$g(\cdot,\cdot,u^{\varepsilon(k),1})\to z^1 \text{ weakly in } L^q(Q^1_T),$$

$\int_0^T\Big\langle\mathcal{B}(u^{\varepsilon(k),1}),u^{\varepsilon(k),1}\Big\rangle_{A_0(1)}dt\to0$, where $\lim_{k\to+\infty}\varepsilon(k)=0$.

There exists a subsequence $\{u^{\varepsilon(k),2}\}\subset\{u^{\varepsilon(k),1}\}$ such that

$$u^{\varepsilon(k),2}\to u^2 \text{ weakly-star in } L^\infty\Big((0,T);L^2(\Omega^2)\Big),$$

$$u^{\varepsilon(k),2}\to u^2 \text{ weakly in } L^2\Big((0,T);V(\Omega^2)\Big)\cap L^p(Q^2_T),$$

$$g(\cdot,\cdot,u^{\varepsilon(k),2})\to z^2 \text{ weakly in} L^q(Q^2_T),$$

$\int_0^T\Big\langle\mathcal{B}(u^{\varepsilon(k),2}),u^{\varepsilon(k),2}\Big\rangle_{A_0(2)}dt\to0$ as $\varepsilon(k)\to0$ and so on.

Obviously $u^l(x,t)=u^k(x,t)$ in Q^l_T, $z^l(x,t)=z^k(x,t)$ in Q^l_T, for $k>l$.

Write $u(x,t)=u^l(x,t)$, $(x,t)\in Q^l_T, z(x,t)=z^l(x,t)$, $(x,t)\in Q^l_T, l\in\mathbb{N}$. Thus $u^{\varepsilon(k),k}\to u$ weakly-star in $L^\infty\Big((0,T);L^2_{\mathrm{loc}}(\bar\Omega)\Big)$, $u^{\varepsilon(k),k}\to u$ weakly in $L^2\Big((0,T);V_{\mathrm{loc}}(\bar\Omega)\Big)\cap L^p(\overline{Q}_T)$, $g(\cdot,\cdot,u^{\varepsilon(k),k})\to z$ weakly in $L^q_{\mathrm{loc}}(\overline{Q}_T)$.

By (27), we have

$$\lim_{k\to+\infty}\Big\langle\mathcal{B}(u^{\varepsilon(k),k}),v\Big\rangle_{A_l}=0$$

$\forall v\in L^2\Big((0,T);V^*(\Omega^l)\Big)$, $l\in\mathbb{N}$, where by $\Big\langle\cdot,\cdot\Big\rangle_{A_l}$ we denote the action of an element from $L^2((0,T);V^*(\Omega^l))$ on $L^2((0,T);V(\Omega^l)\Big)$.

Consider a sequence $\{\sigma_k\}$ with non-negative members of the form

$$\sigma_k=\Big\langle\mathcal{B}(u^{\varepsilon(k),k})-\mathcal{B}(v),u^{\varepsilon(k),k}-v\Big\rangle_{A_l}=\Big\langle\mathcal{B}(u^{\varepsilon(k),k}),u^{\varepsilon(k),k}\Big\rangle_{A_l}-\Big\langle\mathcal{B}(v),u^{\varepsilon(k),k}-v\Big\rangle_{A_l}-$$
$$-\Big\langle\mathcal{B}(u^{\varepsilon(k),k}),v\Big\rangle_{A_l},\quad\forall v\in L^2\Big((0,T);V^*(\Omega^l)\Big),\quad k>l.$$

Therefore

$$(\forall l\in\mathbb{N})\ \Big\langle\mathcal{B}(v),u-v\Big\rangle_{A_l}\ge0\quad\text{and}\quad\mathcal{B}(u)=0\quad.$$

Thus $u\in K$.
We write (32) for $u^{\varepsilon(k),k}$ and $u^{\varepsilon(s),s}$, $s>k$, take $w=w_\eta$, where w_η is a solution of problem

$$\eta w_{\eta t}+w_\eta=v,\ w_\eta(0)=u_0,\quad v=\frac{1}{2}\Big(u^{\varepsilon(k),k}+u^{\varepsilon(s),s}\Big)$$

and add the corresponding inequalities. The result is

$$\int_{Q_\tau}\Bigg(2w_{\eta t}(w_\eta-v)\psi(x)+\sum_{i,j=1}^n a_{ij}(x,t)\Big(u_{x_i}^{\varepsilon(k),k}(w_{\eta x_j}-u_{x_j}^{\varepsilon(k),k})+u_{x_i}^{\varepsilon(s),s}(w_{\eta x_j}-u_{x_j}^{\varepsilon(s),s})\Big)\psi(x)+$$
$$+\sum_{i,j=1}^n a_{ij}\psi_{x_j}(x)\Big(u_{x_i}^{\varepsilon(k),k}(w_\eta-u^{\varepsilon(k),k})+u_{x_i}^{\varepsilon(s),s}(w_\eta-u^{\varepsilon(s),s})\Big)+$$
$$+\sum_{i=1}^n b_i(x,t)\Big(u_{x_i}^{\varepsilon(k),k}(w_\eta-u^{\varepsilon(k),k})+u_{x_i}^{\varepsilon(s),s}(w_\eta-u^{\varepsilon(s),s})\Big)\psi(x)+$$
$$+c(x,t)\Big(u^{\varepsilon(k),k}(w_\eta-u^{\varepsilon(k),k})+u^{\varepsilon(s),s}(w_\eta-u^{\varepsilon(s),s})\Big)\psi(x)-$$
$$-\frac{1}{2}\alpha_0\Big((u^{\varepsilon(k),k}-w_\eta)^2+(u^{\varepsilon(s),s}-w_\eta)^2\Big)\psi(x)+\Big(g(x,t,u^{\varepsilon(k),k})(w_\eta-u^{\varepsilon(k),k})+$$
$$+g(x,t,u^{\varepsilon(s),s})(w_\eta-u^{\varepsilon(s),s})\Big)\psi(x)-2f(x,t)(w_\eta-v)\psi(x)\Bigg)e^{-\alpha_0 t}dxdt\ge$$
$$\ge\frac{1}{2}\int_{\Omega_\tau}\Big((u^{\varepsilon(k),k}-w_\eta)^2+(u^{\varepsilon(s),s}-w_\eta)^2\Big)\psi(x)e^{-\alpha_0\tau}dx,\quad\tau\in(0,T].\tag{33}$$

We have

$$\int_{Q_\tau}w_{\eta t}(w_\eta-v)\psi(x)e^{-\alpha_0 t}dxdt\le0,\ w_\eta\to v\ \text{ weakly in }\ L^2\Big((0,T);V_{\rm loc}(\bar\Omega)\Big)\cap L^p_{\rm loc}(\overline{Q}_T)$$

as $\eta\to+0$ (by Lemma), so (33) implies the inequality below

$$\frac{1}{2}\int_{\Omega_\tau}(u^{k,s})^2\psi(x)e^{-\alpha_0\tau}dx+\int_{Q_\tau}\Bigg(\sum_{i,j=1}^{n}a_{ij}(x,t)u^{k,s}_{x_i}u^{k,s}_{x_j}\psi(x)+\sum_{i,j=1}^{n}a_{ij}(x,t)u^{k,s}_{x_i}u^{k,s}\psi_{x_j}(x)+$$

$$+\sum_{i=1}^{n}b_i(x,t)u^{k,s}_{x_i}u^{k,s}\psi(x)+\frac{1}{2}\alpha_0(u^{k,s})^2\psi(x)+c(x,t)(u^{k,s})^2\psi(x)+$$

$$+\Big(g(x,t,u^{\varepsilon(k),k})-g(x,t,u^{\varepsilon(s),s})\Big)u^{k,s}\psi(x)\Bigg)e^{-\alpha_0 t}dxdt\le 0, \tag{34}$$

where $u^{k,s}=u^{\varepsilon(k),k}-u^{\varepsilon(s),s}$.
Inequality (34) coincides with (7). Hence applying (34) gives that $\{u^{\varepsilon(k),k}\}$ is a Cauchy sequence in $C\Big((0,T];L^2_{\mathrm{loc}}(\bar\Omega)\Big)\cap L^2\Big((0,T);V_{\mathrm{loc}}(\bar\Omega)\Big)\cap L^p_{\mathrm{loc}}(\overline{Q}_T)$.

Writing (32) for $u^{\varepsilon(k),k}$ and passing to the limit as $k\to+\infty$, we obtain inequality (1) and initial condition (2), which completes the proof of Theorem 2. □

Example 1. Let $V_{\mathrm{loc}}(\bar\Omega)=H^1_{0,\mathrm{loc}}(\bar\Omega)$, $K=L^2\Big((0,T);H^1_{0,\mathrm{loc}}(\bar\Omega)\Big)$, $\mathcal{B}=0$, u be a solution of (1) with condition (2) and $u_t\in L^2\Big((0,T);(H^1_{0,\mathrm{loc}}(\bar\Omega))^*\Big)$.

Then in equation (1) we put $v=u+\lambda\phi$, where $\phi\in H^1_{\mathrm{loc}}(\overline{Q}_T)\cap L^2\Big((0,T);\ H^1_{0,\mathrm{loc}}(\bar\Omega)\Big)\cap L^p_{\mathrm{loc}}(\overline{Q}_T)$, $\phi(x,0)=0$, $\lambda\in\mathbb{R}$.

Thus we have from (1)

$$\int_{\Omega_\tau}u\phi e^{-\alpha\tau}\psi(x)dx+\int_{Q_\tau}\Bigg(-u\phi_t\psi(x)+\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}\Big(\phi\psi(x)\Big)_{x_j}+\sum_{i=1}^{n}b_i(x,t)u_{x_i}\phi\psi(x)+$$

$$+c(x,t)u\phi\psi(x)+\alpha u\phi\psi(x)+g(x,t,u)\phi\psi(x)-f(x,t)\phi\psi(x)\Bigg)e^{-\alpha t}dxdt=0$$

which implies that u is a weak solution of the following problem

$$u_t-\sum_{i,j=1}^{n}\Big(a_{ij}(x,t)u_{x_i}\Big)_{x_j}+\sum_{i=1}^{n}b_i(x,t)u_{x_i}+c(x,t)u+g(x,t,u)=f(x,t),$$

$$u|_{\partial\Omega\times(0,T)}=0,\ \ u(x,0)=u_0(x). \tag{35}$$

Example 2. Let

$$V_{\mathrm{loc}}(\bar\Omega)=H^1_{\mathrm{loc}}(\bar\Omega),\ K=\Big\{v:v\in L^2\Big((0,T);H^1_{\mathrm{loc}}(\bar\Omega)\Big),v\ge 0\text{ on }\partial\Omega\times(0,T)\Big\},$$

u be a solution of (1) with condition (2) and $u_t\in L^2\Big((0,T);(H^1_{\mathrm{loc}}(\bar\Omega))^*\Big)$.

In this case we can define

$$\mathcal{B}:L^2\Big((0,T);H^1_{\mathrm{loc}}(\bar\Omega)\Big)\to L^2\Big((0,T);(H^1_{\mathrm{loc}}(\bar\Omega))^*\Big)$$

by the formula

$$\Big\langle \mathcal{B}(u), v \Big\rangle_{\Big(L^2((0,T);(H^1(\Omega^R))^*),L^2((0,T);H^1(\Omega^R))\Big)} = -\int_0^T \int_{\Gamma_1^R} u^- v d\Gamma,$$

$u, v \in L^2\Big((0,T); H^1(\Omega^R)\Big), \forall R > 0$, where

$$u^-(x,t) = \begin{cases} 0, & \text{if } u(x,t) \geq 0 \\ -u(x,t), & \text{if } u(x,t) < 0. \end{cases}$$

The set K satisfies condition (K), the operator $\mathcal{B}$ is bounded, semilinear, monotonic and $K = \Big\{v : v \in L^2\Big((0,T); H^1_{\text{loc}}(\bar{\Omega})\Big), \mathcal{B}(v) = 0\Big\}$.

Proceeding as in [11, II, 9.5], we obtain that u is a weak solution of (35) with initial condition (2) and the boundary conditions

$$u \geq 0 \text{ on } \partial\Omega \times (0,T), \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i} \cos(\nu, x_i) \geq 0 \text{ on } \partial\Omega \times (0,T),$$

$$u \sum_{i=1}^n a_{ij}(x,t) u_{x_i} \cos(\nu, x_i) = 0 \text{ on } \partial\Omega \times (0,T), \text{ where } \nu \text{ is the normal vector}$$

to $\partial\Omega \times (0,T)$.

Example 3. Let $V_{\text{loc}}(\bar{\Omega}) = H^1_{0,\text{loc}}(\bar{\Omega})$, $K = \Big\{v : v \in L^2\Big((0,T); V_{\text{loc}}(\bar{\Omega})\Big), v \geq 0 \text{ in } Q_T\Big\}$, u be a solution of (1) with condition (2) and $u_t \in L^2\Big((0,T); (H^1_{0,\text{loc}}(\bar{\Omega}))^*\Big)$. We define in $L^2_{\text{loc}}(Q_T)$ an operator $\mathcal{B}$ by the formula

$$(\mathcal{B}(u), v)_{L^2(Q_T^R)} = -\int_{Q_T^R} u^- v dx dt, \quad \forall u, v \in L^2_{\text{loc}}(Q_T), \quad \forall R > 0.$$

The set K satisfies (K), the operator $\mathcal{B}$ is bounded, semilinear, monotonic in $L^2\Big((0,T); V_{\text{loc}}(\Omega)\Big)$ and $K = \Big\{v : v \in L^2\Big((0,T); H^1_{0,\text{loc}}(\bar{\Omega})\Big), \mathcal{B}(v) = 0\Big\}$ (because $L^2\Big((0,T); V_{\text{loc}}(\bar{\Omega})\Big)$ is dense in $L^2_{\text{loc}}(\overline{Q}_T)$). Analogously like in [11, I, 9.5] we prove that u is a weak solution of (35) in this part of Q_T^R, where $u(x,t) > 0, u(x,t) = 0$ in the remaining parts of $Q_T, u = 0$ on $\partial\Omega \times (0,T)$ and satisfies condition (2).

REFERENCES

1. Tychonoff A. *Théorèmes d'unicité pour l'equation de la chaleur* // Mat. Sbornik. – 1935. – V. 42. №2. – P.199–216.
2. Олейник О.А., Радкевич Е.В. *Метод введения параметра для исследования эволюционных уравнений* // Успехи мат. наук. – 1978. – Т.33, №5. – С.7–72.
3. Brezis H. *Semilinear equations in R^n without condition at infinity* // Appl. Math. Optim. – 1984. – V. 12. – P.271–282.

4. Бокало М. М. *Об однозначной разрешимости краевых задач для полулинейных параболических уравнений в неограниченной области без условий на бесконечности* // Сиб. мат. журн. – 1993. – Т.34, №4. – С.33–40.

5. Бокало М. М. *Краевые задачи для полулинейных параболических уравнений в неограниченной области без условий на бесконечности,* // Сиб. мат. журн. – 1996. – Т.37, №5. – С.977–985.

6. Бокало М. М. *Задача Фур'є для напівлінійних параболічних рівнянь довільного порядку в необмежених областях* // Нелінійні граничні задачі. – 2000. – №10. – С.9–15.

7. Herrero M. A., Pirre M. *The Cauchy problem for* $u_t - \Delta u^m = 0$, *when* $0 < m < 1$ // Trans. Amer. Math. Soc. – 1985. – V.291, no.1. – P.145–158.

8. Friedman A. *Regularity theorems for variational inequalities in unbounded domains and application to stopping time problems* // Arch. Rational Mech. Anal. – 1973. – V.52. – P.134–160.

9. Бугрій О. М. *Системи параболічних варіаційних нерівностей в необмеженій області* // Вісник Львівськ. ун-ту. Сер. мех.-мат. – 1999. – №53. – С.77–86.

10. Gajewski H., Gröger K., Zacharias K. Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akademie Verlag, Berlin, 1974.

11. Lions J.L. Quelques methodes de résolution des problémes aux limites non linéaires, Dunod Gautier-Villars, Paris, 1969.

Cracow University of Technology, Institute of Mathematics
24, Warszawska str., 31-155 Cracow, Poland

Received 5.09.2002