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FREE BOUNDARY PROBLEM FOR NONLINEAR DIFFUSION EQUATION

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In the paper we consider an initial boundary value problem for one-dimensional diffusion equation in the domain with moving unknown boundary. By transformation of variables the problem is reduced to a coefficient inverse problem for some parabolic equation. Existence and uniqueness of solution of this problem are established.

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В работе рассматривается начально-краевая задача для уравнения диффузии в области с неизвестной движущейся границей. Преобразованием переменных задача сводится к коэффициентной обратной задаче для некоторого параболического уравнения. Установлены условия существования и единственности решения данной задачи.

The problem that we study in the paper may be considered as an application of the theory of coefficient inverse problems with time-dependent unknown parameters.

In the domain $\Omega_T = \{(x,t) : 0 < x < h(t), 0 < t < T < \infty\}$ with unknown moving boundary we consider the diffusion equation

$$u_t = (a(u)u_x)_x + f(x,t) \tag{1}$$

subject to the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0, h(0)], \tag{2}$$

and the boundary conditions

$$u(0,t) = \mu_1(t), \quad u(h(t),t) = \mu_2(t), \quad t \in [0,T]. \tag{3}$$

In order to determine an unknown boundary we impose the overdetermination condition of integral type

$$\int_0^{h(t)} u(x,t)dx = \mu_3(t), \quad t \in [0,T]. \tag{4}$$

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In (1)–(4) the functions a, φ, μ_i, $i \in \{1,2,3\}$, f are given.

Note that in problem (1)–(4) we neglect why the boundary is moving and we shall determine the moving boundary only using the additional condition (4) which may be considered as control on summary concentration of a solid in a solution.

By the solution of problem (1)–(4) we shall mean a pair of functions $(h(t), u(x,t)) \in C^1[0,T] \times C^{2,1}(\overline{\Omega}_T), h(t) > 0, t \in [0,T]$, which satisfies conditions (1)–(4). Here $C^1[0,T]$ denotes the Banach space of continuous functions having continuous first derivative on $[0,T]$, and $C^{2,1}(\overline{\Omega}_T)$ is the Banach space of function continuously differentiable to the second order with respect to x and to first order with respect to t.

By the transformation

$$y = \frac{x}{h(t)}, \quad t = t \tag{5}$$

we reduce the free boundary problem (1)–(4) to the problem in a rectangle:

$$v_t = \frac{(a(v)v_y)_y}{h^2(t)} + \frac{yh'(t)}{h(t)}v_y + f(yh(t),t), \quad (y,t) \in Q_T \equiv (0,1) \times (0,T), \tag{6}$$

$$v(y,0) = \varphi(h(0)y), \quad y \in [0,1], \tag{7}$$

$$v(0,t) = \mu_1(t), \quad v(1,t) = \mu_2(t), \quad t \in [0,T], \tag{8}$$

$$h(t)\int_0^1 v(y,t)dy = \mu_3(t), \quad t \in [0,T], \tag{9}$$

where $v(y,t) = u(yh(t),t)$.

Theorem 1. *Suppose that the following assumptions are fulfilled:*

(A1) $\varphi \in C^2[0,h_0], \mu_i \in C^1[0,T]$, $i \in \{1,2,3\}$, $f \in C^{1,0}([0,H_1] \times [0,T]), a \in C^1[M_0,M_1]$;

(A2) $\varphi(x) > 0, x \in [0,h_0], \mu_i(t) > 0, t \in [0,T]$, $i \in \{1,2,3\}$, $f(x,t) \geq 0$, $(x,t) \in [0,H_1] \times [0,T]$, $a(s) \geq a_0 > 0, s \in [M_0,M_1]$, $a_0 = const$;

(A3) $\mu_1(0) = \varphi(0), \mu_2(0) = \varphi(h_0), \int_0^{h(0)} \varphi(x)dx = \mu_3(0)$,
$\mu_1'(0) = a(\mu_1(0))\varphi''(0) + a'(\mu_1(0))\varphi'^2(0) + f(0,0)$,
$\mu_2'(0) = a(\mu_2(0))\varphi''(h_0) + a'(\mu_2(0))\varphi'^2(h_0) + \varphi'(h_0)h'(0) + f(h_0,0)$.

Then there exists a number t_0, $0 < t_0 \leq T$, defined by given data such that problem (6)–(9) has at least one solution for $y \in [0,1], t \in [0,t_0]$.

Proof. We begin with explication of meaning of constants h_0, H_1, M_0, M_1 and the value $h'(0)$ introduced in assumptions (A1)–(A3).

Consider one of compatibility conditions

$$\int_0^{h(0)} \varphi(x)dx = \mu_3(0). \tag{10}$$

Taking into account the assumptions on φ and μ_3 we conclude the existence of unique value $h(0) = h_0 > 0$ satisfying (10). This value h_0 is used in assumptions (A1)–(A3).

In order to explain the origin of constant H_1, note that by the maximum principle [1] we have the following estimate for a solution of problem (1)–(3) for any continuous positive function $a(s)$:

$$0 < M_0 \leq u(x,t),\ (x,t) \in \overline{\Omega}_T \quad \text{or} \quad 0 < M_0 \leq v(y,t),\ (y,t) \in \overline{Q}_T, \tag{11}$$

where $M_0 = \min\{\min_{[0,h_0]} \varphi(x), \min_{[0,T]} \mu_1(t), \min_{[0,T]} \mu_2(t)\}$.

Estimate (11) allows us to reduce (9) to the form

$$h(t) = \frac{\mu_3(t)}{\int_0^1 v(y,t)dy}, \quad t \in [0,T], \tag{12}$$

and, hence, to establish the estimate

$$h(t) \le H_1 < \infty, \quad t \in [0,T], \tag{13}$$

where $H_1 = \frac{1}{M_0} \max\limits_{[0,T]} \mu_3(t)$. It follows from (13) that

$$0 \le f(yh(t),t) \le C_1 < \infty, \quad (y,t) \in \overline{Q}_T,$$

with a known constant $C_1 > 0$. Once more applying the maximum principle to the problem (6)–(8) we obtain the estimate from above

$$v(y,t) \le M_1 < \infty, \quad (y,t) \in \overline{Q}_T, \tag{14}$$

where

$$M_1 = \max\Big\{\max_{[0,h_0]} \varphi(x), \max_{[0,T]} \mu_1(t), \max_{[0,T]} \mu_2(t), \max_{[0,H_1]\times[0,T]} f(x,t)\Big\}.$$

Then we can also establish the estimate

$$h(t) \ge H_0 > 0, \quad t \in [0,T], \tag{15}$$

with an evidently defined constant H_0.

Finally, if we differentiate condition (4) with respect to t, use equation (1) and put $t=0$ then we get

$$h'(0)\mu_2(0) + a(\mu_2(0))\varphi'(h_0) - a(\mu_1(0))\varphi'(0) + \int_0^{h_0} f(x,0)dx = \mu_3(0).$$

From this equality we determine the constant $h'(0)$. So the meaning of the constants h_0, H_1, M_0, M_1, $h'(0)$ is clear.

Now we turn to the existence of solution of problem (6)–(9). For this aim we reduce problem (6)–(9) to a system of equations and then we shall apply to it the Schauder fixed-point theorem.

By differentiating condition (9) with respect to t and using equation (6) we get

$$h'(t)\int_0^1 v(y,t)dy + h(t)\int_0^1 \left(\frac{(a(v)v_y)_y}{h^2(t)} + \frac{yh'(t)}{h(t)}v_y + f(yh(t),t)\right)dy = \mu_3'(t).$$

After integrating by parts we obtain the equation

$$h'(t) = \frac{1}{\mu_2(t)}\bigg(\mu_3'(t) - h(t)\int_0^1 f(yh(t),t)dy +$$
$$+\frac{1}{h(t)}\Big(a(\mu_1(t))v_y(0,t) - a(\mu_2(t))v_y(1,t)\Big)\bigg), \quad t \in [0,T]. \tag{16}$$

Denote $p(t) = h'(t)$. Now we can rewrite equations (6) and (16) as follows:

$$v_t = \frac{(a(v)v_y)_y}{h^2(t)} + \frac{yp(t)}{h(t)}v_y + f(yh(t),t), \quad (y,t) \in Q_T, \tag{17}$$

$$p(t) = \frac{1}{\mu_2(t)}\Bigg(\mu_3'(t) - h(t)\int_0^1 f(yh(t),t)dy +$$
$$+\frac{1}{h(t)}\Big(a(\mu_1(t))v_y(0,t) - a(\mu_2(t))v_y(1,t)\Big)\Bigg), \quad t \in [0,T]. \tag{18}$$

Hence, problem (6)–(9) is reduced to equations (12) and (18) where $v = v(y,t)$ is a solution of problem (17), (7), (8) which corresponds to given functions $(h(t),p(t)) \in C[0,T] \times C[0,T]$. It is known [2] that under assumptions that we made, a solution of problem (17), (7), (8) exists and is unique in $C^{2,1}(\overline{Q}_T)$ for each pair of functions $(h(t),p(t)) \in (C[0,T])^2$.

To apply the Schauder fixed-point theorem to system (12), (18), we need estimates of its solution. We have already established estimates (11), (13)–(15). Now we are going to evaluate the first derivative $v_y(y,t)$. For this purpose we shall use a special representation of the solution of problem (17), (7), (8) and its derivatives.

Let $\xi \in [0,1]$ be an arbitrary fixed point. We represent equation (17) as follows:

$$v_t = \frac{a(v(\xi,t))}{h^2(t)}v_{yy} + \frac{((a(v(y,t) - a(v(\xi,t))v_y)_y}{h^2(t)} + \frac{yh'(t)}{h(t)}v_y + f(yh(t),t), \quad (y,t) \in Q_T. \tag{19}$$

If the solution $v = v(y,t)$ is known we may consider the coefficient $a(v(\xi,t))$ as given function of t (the point ξ is a parameter) and construct the Green function $G_1^\xi = G_1^\xi(y,t,\eta,\tau)$ of initial-boundary problem (7), (8) for the equation

$$v_t = a(v(\xi,t))v_{yy} + f(yh(t),t). \tag{20}$$

It is easy to check that G_1^ξ is determined by the formula

$$G_1^\xi(y,t,\eta,\tau) = \frac{1}{\left(\pi\int_\tau^t a(v(\xi,\sigma))h^{-2}(\sigma)\,d\sigma\right)^{1/2}}\sum_{n=-\infty}^{\infty}\left(\exp\left(-\frac{(y-\eta+2n)^2}{4\int_\tau^t a(v(\xi,\sigma))h^{-2}(\sigma)\,d\sigma}\right) - \right.$$
$$\left. - \exp\left(-\frac{(y+\eta+2n)^2}{4\int_\tau^t a(v(\xi,\sigma))h^{-2}(\sigma)\,d\sigma}\right)\right), \quad \forall\xi \in [0,1].$$

With the aid of the Green function we can reduce direct problem (19), (7), (8) to the equivalent integro-differential equation

$$v(y,t) = v_0(y,t) +$$
$$+\int_0^t\int_0^1 G_1^\xi(y,t,\eta,\tau)\left(\frac{((a(v(\eta,\tau) - a(v(\xi,\tau))v_\eta(\eta,\tau))_\eta}{h^2(\tau)} + \frac{\eta p(\tau)}{h(\tau)}v_\eta(\eta,\tau)\right)d\eta d\tau, \tag{21}$$

where $v_0(y,t)$ is a solution of problem (20), (7), (8).

Integrating by parts we transform equation (21) to the form

$$v(y,t) = v_0(y,t) + \int_0^t \int_0^1 \Big(G_{1\eta}^{\xi}(y,t,\eta,\tau) \frac{(a(v(\xi,\tau) - a(v(\eta,\tau))v_\eta(\eta,\tau)}{h^2(\tau)} + $$
$$+ G_1^{\xi}(y,t,\eta,\tau)\frac{\eta p(\tau)}{h(\tau)} v_\eta(\eta,\tau) \Big) d\eta d\tau. \qquad (22)$$

Differentiating (22) by y and putting $\xi = y$ we obtain

$$v_y(y,t) = v_{0y}(y,t) + \int_0^t \int_0^1 \Big(G_{1\eta y}^{\xi}(y,t,\eta,\tau)\Big|_{\xi=y} \frac{(a(v(\xi,\tau) - a(v(\eta,\tau))v_\eta(\eta,\tau)}{h^2(\tau)} + $$
$$+ G_{1y}^{\xi}(y,t,\eta,\tau)\Big|_{\xi=y} \frac{\eta p(\tau)}{h(\tau)} v_\eta(\eta,\tau) \Big) d\eta d\tau. \qquad (23)$$

Here v_{0y} is found by the formula

$$v_{0y}(y,t) = h_0 \int_0^1 G_2^y(y,t,\eta,0)\varphi'(h_0\eta)d\eta - \int_0^t G_2^y(y,t,0,\tau)\mu_1'(\tau)d\tau +$$
$$+ \int_0^t G_2^y(y,t,1,\tau)\mu_2'(\tau)d\tau + \int_0^t \int_0^1 G_{2y}^{\xi}(y,t,\eta,\tau)\big|_{\xi=y} f(\eta h(\tau),\tau)d\eta d\tau,$$

where $G_2^{\xi}(y,t,\eta,\tau)$ is the Green function for the second initial-boundary problem for equation (20). It is easy to verify that the following estimate holds

$$|v_{0y}(y,t)| \le C_1 < \infty,$$

where the constant $C_1 > 0$ is determined by the given data.

Denote $V(t) = \max_{y\in[0,1]} |v_y(y,t)|$. Using the estimates of the Green function [2, p.469] we have from (18) and (23)

$$|p(t)| \le C_2 + C_3 V(t), \qquad (24)$$

$$V(t) \le C_1 + C_4 \int_0^t \frac{V^2(\tau)d\tau}{\sqrt{t-\tau}} + C_5 \int_0^t \frac{|p(\tau)|V(\tau)d\tau}{\sqrt{t-\tau}}, \quad t \in [0,T]. \qquad (25)$$

After substituting (24) into (25) we obtain the following inequality with respect to $V(t)$:

$$V(t) \le C_1 + C_6 \int_0^t \frac{(V(\tau) + V^2(\tau))d\tau}{\sqrt{t-\tau}}.$$

If we denote $V_1(t) = V(t) + \frac{1}{2}$ then we have

$$V_1(t) \le C_7 + C_8 \int_0^t \frac{V_1^2(\tau)d\tau}{\sqrt{t-\tau}}. \qquad (26)$$

To resolve inequality (26) we square both parts of (26) and use the Cauchy inequality:

$$V_1^2(t) \le C_9 + C_{10} \left(\int\limits_0^t \frac{V_1^2(\tau)d\tau}{\sqrt{t-\tau}} \right)^2 .$$

Applying the Hölder inequality we transform the previous inequality to the form

$$V_1^2(t) \le C_9 + C_{11} \int\limits_0^t \frac{V_1^4(\tau)d\tau}{\sqrt{t-\tau}}. \tag{27}$$

Now we put $t = \sigma$ in (27) and multiply the result by $(t-\sigma)^{-\frac{1}{2}}$. Integrating the obtained inequality with respect to σ from 0 to t we arrive to the inequality

$$\int\limits_0^t \frac{V_1^2(\sigma)d\sigma}{\sqrt{t-\sigma}} \le C_{12} + C_{11} \int\limits_0^t \frac{d\sigma}{\sqrt{t-\sigma}} \int\limits_0^\sigma \frac{V_1^4(\tau)d\tau}{\sqrt{\sigma-\tau}}.$$

Changing the order of integration we get

$$\int\limits_0^t \frac{V_1^2(\sigma)d\sigma}{\sqrt{t-\sigma}} \le C_{12} + C_{11} \int\limits_0^t V_1^4(\tau)d\tau \int\limits_\tau^t \frac{d\sigma}{\sqrt{(t-\sigma)(\sigma-\tau)}}. \tag{28}$$

Taking into account the equality

$$\int\limits_\tau^t \frac{d\sigma}{\sqrt{(t-\sigma)(\sigma-\tau)}} = \pi$$

and putting (28) into (26) we obtain the following inequality

$$V_1(t) \le C_{12} + C_{13} \int\limits_0^t V_1^4(\tau)d\tau, \quad t \in [0,T]. \tag{29}$$

To resolve inequality (29) we apply Gronwall's method. Denote $W(t) = C_{12} + C_{13} \int_0^t V_1^4(\tau)d\tau$. From (29) we find

$$W'(t) = C_{13}V_1^4(t) \le C_{13}W^4(t).$$

After separating the variables and integrating from 0 to t we obtain

$$\frac{1}{3C_{12}^3} - \frac{1}{3W^3(t)} \le C_{13}t. \tag{30}$$

Let the number $t_0, 0 < t_0 \le T$, be such that the following inequality holds:

$$1 - 3C_{12}^3 C_{13} t_0 > 0.$$

Then we have from (30)

$$W(t) \le \frac{C_{12}}{\sqrt[3]{1 - 3C_{12}^3 C_{13} t}}, \quad t \in [0, t_0].$$

This means that the estimates

$$|v_y(y,t)| \le M_2 < \infty, \quad (y,t) \in [0,1]\times[0,t_0],$$
$$|p(t)| \le H_2 < \infty, \quad t\in[0,t_0], \tag{31}$$

are established.

The system (12), (18) can be treated as an operator equation

$$\omega = P\omega,$$

where $\omega = (h,p)$ and the operator P is defined by the right hand parts of equations (12) and (18). Denote $\mathcal{N} = \{(h(t),p(t)) \in (C[0,t_0])^2 : H_0 \le h(t) \le H_1, |p(t)| \le H_2\}$. From (13), (15), (31) it is clear that the operator P maps $\mathcal{N}$ onto $\mathcal{N}$. Using the method of [3] it is easy to establish that the operator P is compact on $\mathcal{N}$. This means that the conditions of Schauder fixed-point theorem are fulfilled and, hence, there exists a solution $(h(t),p(t)) \in (C[0,t_0])^2$ of system (12), (18). Substituting the known functions $(h(t),p(t))$ into equation (6) we find $v(y,t) \in C^{2,1}(\overline{Q}_{t_0})$. □

Theorem 2. *Suppose that (A2) and the following assumption holds:*

(A4) $a(s) \in C^1[M_0,M_1], f(x,t) \in C^{1,0}([0,H_1]\times[0,T])$ *(for notations see Theorem 1).*

Then the solution of problem (6)–(9) is unique.

Proof. Suppose that $(h_i(t), v_i(y,t)) \in C^1[0,T]\times C^{2,1}(\overline{Q}_T)$, $i \in \{1,2\}$, are two different solutions of problem (6)–(9). Denote $p(t) = h_1(t) - h_2(t)$, $q(t) = h_1'(t) - h_2'(t)$, $w(y,t) = v_1(y,t) - v_2(y,t)$. Since the functions $(h_i(t), v_i(y,t))$, $i\in\{1,2\}$, satisfy conditions (6)–(9), we have

$$w_t = \frac{1}{h_1^2(t)}(a(v_1)w_y)_y + \frac{yh_1'(t)}{h_1(t)}w_y + \left(\left(\frac{a(v_1)}{h_1^2(t)} - \frac{a(v_2)}{h_2^2(t)}\right)v_{2y}(y,t)\right)_y +$$
$$+\left(\frac{h_1'(t)}{h_1(t)} - \frac{h_2'(t)}{h_2(t)}\right)yv_{2y}(y,t) + f(yh_1(t),t) - f(yh_2(t),t), \quad (y,t)\in Q_T, \tag{32}$$

$$w(y,0) = 0, \quad y\in[0,1], \tag{33}$$

$$w(0,t) = w(1,t) = 0, \quad t\in[0,T], \tag{34}$$

$$p(t) = -\frac{\mu_3(t)}{\int_0^1 v_1(y,t)dy \int_0^1 v_2(y,t)dy}\int_0^1 w(y,t)dy, \quad t\in[0,T]. \tag{35}$$

Note that conditions (A2) provide $v_i(y,t) > 0$, $i\in\{1,2\}$, in $\overline{Q}_T$. Using the Green function $G(y,t,\eta,\tau)$ for the linear equation

$$w_t = \left(\frac{a(v_1(y,t))}{h_1^2(t)}w_y\right)_y + \frac{yh_1'(t)}{h_1(t)}w_y$$

with conditions (33), (34) we find the solution for problem (32)–(34)

$$w(y,t) = \int_0^t\int_0^1 G(y,t,\eta,\tau)\Bigg(\left(\left(\frac{a(v_1(\eta,\tau))}{h_1^2(\tau)} - \frac{a(v_2(\eta,\tau))}{h_2^2(\tau)}\right)v_{2\eta}(\eta,\tau)\right)_\eta +$$
$$+\left(\frac{h_1'(\tau)}{h_1(\tau)} - \frac{h_2'(\tau)}{h_2(\tau)}\right)\eta v_{2\eta}(\eta,\tau) + f(\eta h_1(\tau),\tau) - f(\eta h_2(\tau),\tau)\Bigg)d\eta d\tau. \tag{36}$$

Using the introduced notations we can transform (36) as follows:

$$w(y,t)=\int_0^t\int_0^1 G(y,t,\eta,\tau)\Big(\frac{a(v_1(\eta,\tau))-a(v_2(\eta,\tau))}{h_1^2(\tau)}-\frac{h_1(\tau)+h_2(\tau)}{h_1^2(\tau)h_2^2(\tau)}a(v_2(\eta,\tau))p(\tau)+$$
$$+\left(\frac{q(\tau)}{h_1(\tau)}-\frac{h_2'(\tau)}{h_1(\tau)h_2(\tau)}p(\tau)\right)\eta v_{2\eta}(\eta,\tau)+f(\eta h_1(\tau),\tau)-f(\eta h_2(\tau),\tau)\Big)d\eta d\tau. \tag{37}$$

Applying to (37) the well-known formula for difference of values of a function at two points

$$\omega(t)-\omega(\tau)=\int_0^1\frac{d}{d\sigma}\omega(\tau+\sigma(t-\tau))d\sigma=(t-\tau)\int_0^1\omega'(z)\big|_{z=\tau+\sigma(t-\tau)}d\sigma \tag{38}$$

we get

$$w(y,t)=\int_0^t\int_0^1 G(y,t,\eta,\tau)\Bigg(\Bigg(-\frac{\eta h_2'(\tau)}{h_1(\tau)h_2(\tau)}v_{2\eta}(\eta,\tau)-\frac{h_1(\tau)+h_2(\tau)}{h_1^2(\tau)h_2^2(\tau)}a(v_2(\eta,\tau))+$$
$$+\int_0^1\eta f_z(\eta z,\tau)\big|_{z=h_2(\tau)+z(h_1(\tau)-h_2(\tau))}dz\Bigg)p(\tau)+\frac{q(\tau)}{h_1(\tau)}+$$
$$+w(\eta,\tau)\int_0^1 a'(z)\big|_{z=v_2(\eta,\tau)+z(v_1(\eta,\tau)-v_2(\eta,\tau))}dz\Bigg)d\eta d\tau,\quad (y,t)\in\overline{Q}_T. \tag{39}$$

Another equation is obtained if for each $i\in\{1,2\}$ we find the expressions for $h_i'(t)$, analogous to (18) and then consider their difference:

$$q(t)=\frac{1}{\mu_2(t)}\Bigg(-\int_0^1(h_1(t)f(yh_1(t),t)-h_2(t)f(yh_2(t),t))dy+$$
$$+a(\mu_1(t))\left(\frac{v_{1y}(0,t)}{h_1(t)}-\frac{v_{2y}(0,t)}{h_2(t)}\right)-a(\mu_2(t))\left(\frac{v_{1y}(1,t)}{h_1(t)}-\frac{v_{2y}(1,t)}{h_2(t)}\right)\Bigg),\quad t\in[0,T]. \tag{40}$$

We use (38) to reduce (40) to the form

$$q(t)=\frac{1}{\mu_2(t)}\Bigg(\Bigg(-\int_0^1 f(yh_1(t),t)dy-\int_0^1 dy\int_0^1\eta f_z(\eta z,t)\big|_{z=h_2(\tau)+z(h_1(\tau)-h_2(\tau))}dz+$$
$$+\frac{v_{2y}(1,t)a(\mu_2(t))-v_{2y}(0,t)a(\mu_1(t))}{h_1(t)h_2(t)}\Bigg)p(t)+$$
$$+\frac{a(\mu_1(t))}{h_1(t)}w_y(0,t)-\frac{a(\mu_2(t))}{h_1(t)}w_y(1,t)\Bigg),\quad t\in[0,T]. \tag{41}$$

Finally, we differentiate (39) with respect to y :

$$w_y(y,t)=\int\limits_0^t\int\limits_0^1 G_y(y,t,\eta,\tau)\Bigg(\Bigg(-\frac{\eta h_2'(\tau)}{h_1(\tau)h_2(\tau)}v_{2\eta}(\eta,\tau)-\frac{h_1(\tau)+h_2(\tau)}{h_1^2(\tau)h_2^2(\tau)}a(v_2(\eta,\tau))+$$
$$+\int\limits_0^1 \eta f_z(\eta z,\tau)\big|_{z=h_2(\tau)+z(h_1(\tau)-h_2(\tau))}dz\Bigg)p(\tau)+\frac{q(\tau)}{h_1(\tau)}+$$
$$+w(\eta,\tau)\int\limits_0^1 a'(z)\big|_{z=v_2(\eta,\tau)+z(v_1(\eta,\tau)-v_2(\eta,\tau))}dz\Bigg)d\eta d\tau,\quad (y,t)\in\overline{Q}_T. \qquad (42)$$

Taking into account (42) it is easy to see that system (35), (39), (41) is a homogeneous system of Volterra integral equations of second kind with respect to unknown (w,p,q). Since the kernels of these equations are integrable, the system possesses the unique solution which is trivial: $p(t)\equiv 0, q(t)\equiv 0, t\in[0,T], w(y,t)\equiv 0, (y,t)\in\overline{Q}_T$. □

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