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ON SOLUTIONS OF HOMOGENEOUS CONVOLUTION EQUATION GENERATED BY SINGULARITY

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Explicit solutions f in the Hardy-Smirnov space in a semistrip of the convolution equation $\int_{\partial D_\sigma} f(w+\tau)g(w)dw = 0, \tau \leqslant 0$, are found in the case when a singular boundary function of the characteristic function of the equation has discontinuities.

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В пространстве Харди-Смирнова в полуполосе найдены явные решения f уравнения свертки $\int_{\partial D_\sigma} f(w+\tau)g(w)dw = 0, \tau \leqslant 0$, в случае, когда сингулярная граничная функция характеристической функции уравнения имеет разрыв.

It is well known [9] that solutions $f \in L^2(-\infty;0)$ of the convolution equation

$$\int_{-\infty}^{0} f(w+\tau)g(w)dw = 0, g \in L^2(-\infty;0), \quad \tau \leqslant 0, \tag{1}$$

exists if and only if the function

$$\widetilde{G}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} g(u)e^{zu}du$$

is not outer for the Hardy space $H^2(\mathbb{C}_+)$. This space consists of the analytic functions f in the right half-plane $\mathbb{C}_+ = \{z : \operatorname{Re} z > 0\}$, for which

$$\sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int_{0}^{+\infty} |f(re^{i\varphi})|^p dr \right\} < +\infty.$$

It is known [7, p. 59] that

$$\|f\| = \left\{ \int_{-\infty}^{+\infty} |f(iy)|^p dy \right\}^{1/p}.$$

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Lemma 1. *A function f belongs to $H^p(\mathbb{C}_+)$, if and only if*

$$\|f\|_* := \sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p dr \right\}^{1/p} < +\infty$$

and

$$2^{-1/p}\|f\| \leqslant \|f\|_* \leqslant \|f\|.$$

This lemma is contained in [8].

A function G is outer (that is equation (1) has a nonzero solution) if and only if at least one of the following conditions holds:

a) $\widetilde{G}$ has at least one zero in $\mathbb{C}_+$;

b) $\overline{\lim\limits_{x\to+\infty}} \dfrac{\ln|\widetilde{G}(x)|}{x} = a_1 < 0$;

c) a singular boundary function of the function $\widetilde{G}$ is not an identical constant.

Let $D_\sigma = \{z : |\mathrm{Im} z| < \sigma, \mathrm{Re} z < 0\}, 0 \leqslant \sigma < +\infty$ $D^*_\sigma = \mathbb{C}\backslash\overline{D}_\sigma$. Let $E^p[D_\sigma]$ (respectively $E^p_*[D_\sigma]$), $1 \leqslant p < +\infty$, is the space of analytic functions in D_σ (in $D^*_\sigma = \mathbb{C}\backslash\overline{D}_\sigma$) for which

$$\sup \left\{ \int_\gamma |f(z)|^p |dz| \right\} < +\infty,$$

where the supremum is taken over all segments γ that lie in D_σ (D^*_σ) and are parallel to one of sides of ∂D_σ. The functions from these spaces have [3] almost everywhere (a. e.) on ∂D_σ the angle boundary values, which will be denoted by $f(z)$, and $f \in L^p[\partial D_\sigma]$. Further, let $H^p_\sigma(\mathbb{C}_+)$, $1 \leqslant p < \infty$, be the class of analytic functions f in $\mathbb{C}_+$, for which

$$\sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr \right\} < +\infty.$$

The functions from this space also have a. e. on $\partial\mathbb{C}_+$ the angle boundary values $f(iy)$ and $f(iy)e^{-\sigma|y|} \in L^p(-\infty;\infty)$.

If $g \in E^2_*[D_\sigma]$, then [1] the function

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{\partial D_\sigma} g(z)e^{zw} dw \tag{2}$$

belongs to the class $H^2_\sigma(\mathbb{C}_+)$ and equality (2) determines a bijection between $H^2_\sigma(\mathbb{C}_+)$ and $E^2_*[D_\sigma]$.

The equation

$$\int_{\partial D_\sigma} f(w+\tau)g(w)dw = 0, \quad \tau \leqslant 0, \ g \in E^2_*[D_\sigma], \tag{3}$$

is studied in [1]–[3], [10]. In these papers it is proved that equation (3) has a nonzero solution f in the class $E^2[D_\sigma]$ provided that at least one of conditions a), c) or

$$b') \qquad \underline{\lim_{r\to\infty}} \int_{1<|t|\leqslant r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln\left|G(it)e^{-\sigma|t|}\right| dt > -\infty$$

holds, and condition b) is not sufficient for the existence of a solution for (3). In [2] a solution for case a) was found. In the case c) the problem of a form for a solution of (3) was open. In the following result we find a solution of (3) when singular boundary function has discontinuities.

Theorem. *If* $G(z) = G_1(z)\exp\{-\frac{b}{z-ia}\}$, *where* $G_1 \in H^2_\sigma(\mathbb{C}_+), b > 0, a \in \mathbb{R}$, *then the function* $f(w) = d(w)e^{iaw}$ *is a solution of equation (3) in the class* $E^2[D_\sigma]$, *where* d *is an arbitrary entire function of order* $1/2$ *and type* $\leqslant 2\sqrt{b}$ *such that* $d \in E^2[D_\sigma]$.

In order to prove this theorem we need some auxiliary results.

Lemma 2. *If* $f \in E^2[D_\sigma], g \in E^2_*[D_\sigma]$, *then*

$$(\forall \tau \leqslant 0): \int\limits_{\partial D_\sigma} f(w+\tau)g(w)dw = i\int\limits_0^{+\infty} \Phi_1(iy)e^{i\tau y}dy - i\int\limits_{-\infty}^{0} \Phi_3(iy)e^{i\tau y}dy + \int\limits_0^{+\infty} \Phi_2(x)e^{\tau x}dx$$

and $\Phi_1(z) + \Phi_2(z) + \Phi_3(z) = 0$ *a. e. on* $\partial\mathbb{C}_+$, *where* $\Phi_j = F_jG, j \in \{1,2,3\}$. *Here* G *is determined by equality (2), and*

$$F_j(z) = \frac{1}{\sqrt{2\pi}}\int\limits_{l_j} f(w)e^{-zw}dw, \qquad f \in E^2[D_\sigma],$$

l_1 *and* l_3 *are the halflines of* ∂D_σ *lying above and under the real axis, respectively, and* l_2 *is the segment* $[-i\sigma; i\sigma]$ *oriented according to the positive orientation of* ∂D_σ.

Let $E^p[\mathbb{C}(\alpha,\beta)], 0 < \beta - \alpha < 2\pi, 1 \leqslant p < \infty$, be the space of analytic functions f in $\mathbb{C}(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}$ for which

$$\sup_{\alpha<\varphi<\beta}\left\{\int\limits_0^{+\infty} |f(re^{i\varphi})|^p dr\right\} < +\infty.$$

The functions f from the class $E^p[\mathbb{C}(\alpha,\beta)]$ have [4] a. e. on $\partial\mathbb{C}(\alpha,\beta)$ the angle boundary values and $f \in L^p[\partial\mathbb{C}(\alpha,\beta)]$.

Lemma 3. *If* $f \in E^1[\mathbb{C}(\alpha,\beta)]$, *then*

$$\int\limits_{\partial\mathbb{C}(\alpha,\beta)} f(z)dz = 0.$$

Lemma 4. *Let a function* ψ *be analytic in* $\mathbb{C}_+$,

$$\sup_{|\varphi|<\frac{\pi}{2}} \varlimsup_{r\to\infty} \frac{\ln|\psi(re^{i\varphi})|}{r} \leqslant 2\sqrt{b},$$

continuous in $\overline{\mathbb{C}}_+$ *and* $\psi \in L^2(\partial\mathbb{C}_+)$. *Then* $\psi(w)e^{-2\sqrt{b}w} \in H^2(\mathbb{C}_+)$.

Lemma 2 is from [2], Lemma 3 from [4], and Lemma 4 from [5].

Lemma 5. *Let a function d satisfy the conditions of Theorem. Then*

$$d_{\pm}(w) \stackrel{\text{def}}{=} d(w \pm i\sigma) e^{-2\sqrt{b}\sqrt{w}} \in E^2[\mathbb{C}(-\pi, \pi)].$$

Proof. The functions $\psi_{\pm}(w) = \sqrt{2w} d(w^2 \pm i\sigma)$ satisfy the conditions of Lemma 4 and $\psi_{\pm}(w) e^{-2\sqrt{b}w} \in H^2(\mathbb{C}_+)$. Hence

$$\frac{\psi_{\pm}(\sqrt{w})}{\sqrt[4]{4w}} e^{-2\sqrt{b}\sqrt{w}} \in E^2[\mathbb{C}(-\pi; \pi)].$$

Thus, $d(w \pm i\sigma) e^{-2\sqrt{b}\sqrt{w}} \in E^2[\mathbb{C}(-\pi; \pi)]$, and Lemma 5 is proved. □

Lemma 6. *If $f \in E^p[\mathbb{C}(\alpha, \beta)]$, then*

$$\sup_{\alpha < \varphi < \beta} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p dr \right\}^{1/p} \leqslant \left\{ \int_{\partial \mathbb{C}(\alpha, \beta)} |f(z)|^p |dz| \right\}^{1/p}.$$

This lemma is a consequence of Lemma 1.

Lemma 7. *If a function f is analytic in $\mathbb{C}[\alpha, \beta]$, has the angle boundary values a.e. on $\partial \mathbb{C}(\alpha, \beta)$, $f \in L^p(\partial \mathbb{C}(\alpha, \beta))$, and for some $\gamma \in (0; \pi/(\beta - \alpha))$*

$$(\forall \varepsilon > 0) : \sup_{\alpha < \varphi < \beta} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-\varepsilon\left(r^\gamma + \frac{1}{r^\gamma}\right)} dr \right\}^{1/p} < +\infty, \tag{4}$$

then $f \in E^p[\mathbb{C}(\alpha, \beta)]$.

Proof. Let

$$f_\varepsilon(z) = f(z) e^{-\varepsilon(z^\gamma + 1/z^\gamma)}.$$

From (4) we have $f_\varepsilon \in \mathbb{C}(\alpha; \beta)$, and

$$\sup_{\alpha < \varphi < \beta} \left\{ \int_0^{+\infty} |f_\varepsilon(z)|^p |dz| \right\}^{1/p} \leqslant \left\{ \int_{\partial \mathbb{C}(\alpha, \beta)} |f(z)|^p |dz| \right\}^{1/p}.$$

Hence, by the Fatou lemma, Lemma 7 is proved. □

Lemma 8. *If a function f is analytic in $\mathbb{C}_+$, has the angle boundary values a.e. on $\partial \mathbb{C}_+$, $f \in L^p(\partial \mathbb{C}_+)$, for some $\delta > 0$ $f(x) e^{-\delta\left(x + \frac{1}{x}\right)} \in L^p(0; +\infty)$ and for all $\gamma \in (0; 2)$*

$$(\forall \varepsilon > 0) : \sup_{|\varphi| < \pi/2} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-\varepsilon\left(r^\gamma + \frac{1}{r^\gamma}\right)} dr \right\}^{1/p} < +\infty, \tag{5}$$

then $f \in H^p(\mathbb{C}_+)$.

Proof. Let
$$F_\varepsilon(z) = f(z)e^{-\delta(z+1/z)}.$$
From (5) we have $|F_\delta(iy)| = |f(iy)|$, $F_\delta \in L^2(0;+\infty)$. Then by Lemma 7 $F_\delta \in E^p[\mathbb{C}(-\frac{\pi}{2};0)]$, $F_\delta \in E^p[\mathbb{C}(0;\frac{\pi}{2})]$. Hence $F_\delta \in H^p(\mathbb{C}_+)$ and
$$\sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int\limits_0^{+\infty} |F_\delta(re^{i\varphi})|^p dr \right\}^{1/p} \leqslant \left\{ \int\limits_{-\infty}^{+\infty} |f(iy)|^p dy \right\}^{1/p}.$$
By the Fatou lemma, we obtain $f \in H^p(\mathbb{C}_+)$. □

Lemma 9. *If a function d satisfies the conditions of Theorem, then the functions*
$$Q_\pm(z) = -\frac{e^{i\alpha}}{\sqrt{2\pi}} \int\limits_0^{+\infty} d(te^{i\alpha} \pm i\sigma)e^{-(t\exp(i\alpha)\pm i\sigma)z}dt, \quad 0 < \cos(\alpha + \arg z), \tag{6}$$
do not depend on $\alpha \in [-\pi;\pi)$ and $Q_+(z)e^{-\frac{b}{z}}e^{i\sigma z} \in H^2(\mathbb{C}_+)$, $Q_-(z)e^{-\frac{b}{z}}e^{-i\sigma z} \in H^2(\mathbb{C}_+)$.

Proof. The functions $d(w \pm i\sigma)$ are entire of order $1/2$, because the functions $Q_\pm$ are analytic in $\mathbb{C}_+\backslash\{0\}$ and do not depend on α (see [6]) as Borel's transforms of the functions $d(z \pm i\sigma)$. If $\alpha = -\pi$ then from Paley-Wiener's theorem [12, p.20] the functions $Q_\pm(z)e^{\pm i\sigma z}$ belong to the Hardy space H^2 in the halfplane $\{z : \text{Re }\, z < 0\}$, hence $Q_\pm(z)e^{\pm i\sigma z}e^{-\frac{b}{z}} \in L^2(\partial\mathbb{C}_+)$.

From Schwartz' inequality
$$\int\limits_0^{+\infty} \left|e^{\pm i\sigma z}Q_\pm(z)e^{-\frac{b}{z}}\right|^2 e^{-\varepsilon\left(r^\gamma + \frac{1}{r^\gamma}\right)}dr \leqslant$$
$$\leqslant c_1 \int\limits_0^{+\infty} e^{-\varepsilon\left(r^\gamma + \frac{1}{r^\gamma}\right)} \exp\left\{-\frac{2b\cos\varphi}{r}\right\} I(r)dr,$$
where by Lemma 5
$$c_1 = \sup_{\alpha\in(-\pi;\pi)} \int\limits_0^{+\infty} |d(te^{i\alpha} \pm i\sigma)|^2 e^{4\sqrt{t}\cos\frac{\alpha}{2}}dt < +\infty,$$
$$I(r) = \int\limits_0^{+\infty} e^{4\sqrt{t}\cos\frac{\alpha}{2} - 2tr\cos(\alpha+\varphi)}dt =$$
$$= \frac{\cos^2\frac{\alpha}{2}}{r^2\cos^2(\alpha+\varphi)} \int\limits_0^{+\infty} \exp\left\{\frac{2b}{r}\frac{\cos^2\frac{\alpha}{2}}{\cos(\alpha+\varphi)}(2\sqrt{u} - u)\right\} du.$$
Splitting the last integral on the sum of the integrals on $[0;1]$ and $[1;+\infty)$ and substituting $\sqrt{u} = 1 \pm \sqrt{\tau}$, we obtain
$$I(r) = \frac{\cos^2\frac{\alpha}{2}}{r^2\cos^2(\alpha+\varphi)} \left(\frac{e^{b_1}}{\sqrt{b_1}} \int\limits_0^{+\infty} \frac{e^{-\sqrt{\nu}}}{\sqrt{\nu}}d\nu - \frac{e^{b_1}-1}{b_1} + \frac{1}{b_1} \right),$$

where $b_1 = (2b\cos^2\alpha/2)\,/\,(r\cos(\alpha+\varphi))$. Taking $\alpha = -\varphi$, we find that the functions

$$e^{\pm i\sigma z}Q_\pm(z)e^{-\frac{b}{z}}$$

satisfy the conditions of Lemma 8. Hence they belong to $H^2(\mathbb{C}_+)$. □

Proof of Theorem. The functions

$$D_\pm(z) := -\frac{e^{i\alpha}}{\sqrt{2\pi}}\int_0^{+\infty} f(te^{i\alpha}\pm i\sigma)e^{-(te^{i\alpha}\pm i\sigma)z}dt =$$

$$= -\frac{e^{i\alpha}}{\sqrt{2\pi}}\int_0^{+\infty} d(te^{i\alpha}\pm i\sigma)e^{-(te^{i\alpha}\pm i\sigma)(z-ia)}dt = Q_\pm(z-ia)$$

are analytic continuations in $\mathbb{C}/\{ia\}$ of F_1 and $-F_3$, respectively (see Lemma 2). As in Lemma 9 $Q_\pm(z)e^{\pm i\sigma z}e^{-\frac{b}{z}} \in H^2(\mathbb{C}_+)$, so $D_\pm(z)e^{-\frac{b}{z-ia}}e^{\pm i\sigma(z-ia)} \in H^2(\mathbb{C}_+)$, and hence $D_\pm(z)e^{-\frac{b}{z-ia}}e^{\pm i\sigma z} \in H^2(\mathbb{C}_+)$. By (2) we get that $G_1(z)e^{i\sigma z} \in E^2[\mathbb{C}(0;\pi/2)]$ and $G_1(z)e^{-i\sigma z} \in E^2[\mathbb{C}(-\pi/2;0)]$. So $\Phi_1 \in E^1[\mathbb{C}(0;\pi/2)]$ and $\Phi_3 \in E^1[\mathbb{C}(-\pi/2;0)]$. Since $\Phi_1(z)e^{\tau z} \in E^1[\mathbb{C}(0;\pi/2)]$ and $\Phi_3(z)e^{\tau z} \in E^1[\mathbb{C}(-\pi/2;0)], \tau \leqslant 0$, by Lemma 3

$$-\int\limits_{\partial\mathbb{C}(0;\pi/2)} \Phi_1(z)e^{\tau z}dz + \int\limits_{\partial\mathbb{C}(-\pi/2;0)} \Phi_1(z)e^{\tau z}dz = 0, \quad \tau \leqslant 0.$$

Therefore, the right side of (3) is equal to zero and hence the theorem is proved. □

The following proposition is a consequence of Lemma 8 and is an extension of Martirosian's results (see [11]).

Proposition. *If function f is analytic in $\mathbb{C}_+$, has a.e on $\partial\mathbb{C}_+$ the angle boundary values, $f \in L^p(\partial\mathbb{C}_+)$, and satisfies condition (5) then $f \in H^p(\mathbb{C}_+)$.*

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