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ASYMPTOTIC RELATIONS BETWEEN THE MEANS OF DIRICHLET SERIES AND THEIR APPLICATION

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For a Dirichlet series $F(s)=\sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s=\sigma+it$, $0\le\lambda_n\uparrow+\infty$, such that $G_p(\sigma,F):=(\sum_{n=0}^{\infty}|a_n|^p e^{\sigma\lambda_n p})^{1/p}<+\infty$ for every $\sigma\in\mathbb{R}$, a necessary and sufficient condition on (a_n) is established in order that $G_p(\sigma,F)\le(1+o(1))G_q(\sigma,F)h(\ln G_q(\sigma,F))$ as $\sigma\to+\infty$, where $q>p>0$ and h is a positive continuous function on $\mathbb{R}$. This result is applied for establishing estimates of exceptional sets in some relations between characteristics of Dirichlet series and for obtaining relations between the maximum modulus and the Nevanlinna characteristic for power series with gaps.

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Для ряда Дирихле $F(s)=\sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s=\sigma+it$, $0\le\lambda_n\uparrow+\infty$, такого, что $G_p(\sigma,F):=(\sum_{n=0}^{\infty}|a_n|^p e^{\sigma\lambda_n p})^{1/p}<+\infty$ для любого $\sigma\in\mathbb{R}$, получены необходимые и достаточные условия на (a_n), при которых $G_p(\sigma,F)\le(1+o(1))G_q(\sigma,F)h(\ln G_q(\sigma,F))$, $\sigma\to+\infty$, где $q>p>0$, а h — положительная, непрерывная на $\mathbb{R}$ функция. Этот результат применен для установления оценок исключительных множеств в некоторых соотношениях между характеристиками ряда Дирихле, а также для получения соотношений между максимумом модуля и характеристикой Неванлинны для лакунарных степенных рядов.

1. Introduction. Let Λ be the class of nonnegative increasing to $+\infty$ sequences $\lambda=(\lambda_n)_{n=0}^{\infty}$. For every number $p>0$ and a sequence $\lambda\in\Lambda$ by $S_p(\lambda)$ we denote the class of Dirichlet series

$$F(s)=\sum_{n=0}^{\infty} a_n e^{s\lambda_n} \qquad (s=\sigma+it) \tag{1.1}$$

such that $|\{n\ge 0: a_n\ne 0\}|=+\infty$ and

$$\forall\sigma\in\mathbb{R}: \qquad G_p(\sigma,F)=\left(\sum_{n=0}^{\infty}|a_n|^p e^{\sigma\lambda_n p}\right)^{1/p}<+\infty. \tag{1.2}$$

Let $p>0$. It follows from (1.2) that $|a_n|e^{\sigma\lambda_n}\to 0$ as $n\to\infty$. Hence, for each Dirichlet series $F\in S_p(\lambda)$ and every $\sigma\in\mathbb{R}$ we can define $\mu(\sigma,F)=\max\{|a_n|e^{\sigma\lambda_n}: n\ge 0\}$. It is clear

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that $\mu(\sigma, F) \le G_p(\sigma, F)$. If $p \ge 1$, then $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\} < +\infty$ and $\mu(\sigma, F) \le M(\sigma, F)$ for each $\sigma \in \mathbb{R}$.

As it is easily seen,

$$G_q^q(\sigma, F) \le \mu^{q-p}(\sigma, F) G_p^p(\sigma, F) \qquad (q > p > 0). \tag{1.3}$$

Therefore, $G_q(\sigma, F) \le G_p(\sigma, F)$ and $S_q(\lambda) \subset S_p(\lambda)$.

By C_+ we denote the class of continuous positive functions on $\mathbb{R}$, and let L be the subclass of nondecreasing functions $l(\sigma)$ from the class C_+ with the property $l(\sigma) \to +\infty$ $(\sigma \to +\infty)$.

For a sequence $\lambda \in \Lambda$ and a function $\psi \in L$ by $S_p(\lambda, \psi)$ we denote the class of Dirichlet series (1.1) for which condition (1.2) holds and

$$|a_n| \le e^{-\lambda_n \psi(\lambda_n)} \qquad (n \ge n_0(F)). \tag{1.4}$$

Put

$$\varepsilon(\lambda, \psi) = \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n \psi(\lambda_n)}, \qquad \tau(\lambda) = \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n}.$$

If $\varepsilon(\lambda, \psi) < p$ and condition (1.4) holds, then also condition (1.2) holds (see Lemma 1 below). Hence, if $\varepsilon(\lambda, \psi) = 0$, then $S_p(\lambda, \psi) = S_q(\lambda, \psi)$ for every $p > 0$ and $q > 0$. In the case $\tau(\lambda) < +\infty$ we have $S_p(\lambda, \psi) = S_q(\lambda, \psi)$ and $S_p(\lambda) = S_q(\lambda)$ for every $\psi \in L$, $p > 0$ and $q > 0$.

In [1,2] there was considered a problem of the establishing conditions on the coefficients a_n of Dirichlet series (1.1) under which some asymptotic relations between $M(\sigma, F)$ and $\mu(\sigma, F)$ are fulfilled. In particular, in [1] the following assertion is proved.

Theorem A [2]. *Let $\lambda \in \Lambda$, $\psi \in L$, $\varepsilon(\lambda, \psi) < 1$ and $h \in C_+$. In order that*

$$(\forall F \in S_1(\lambda, \psi))(\forall \sigma \ge \sigma_0(F)):\ M(\sigma, F) < \mu(\sigma, F) h(\ln \mu(\sigma, F))$$

it is necessary and sufficient that

$$(\forall \gamma_1, \gamma_2 \in L)(\forall n \ge n_0(\gamma_1, \gamma_2)):\ n < \gamma_1(n) + h(\gamma_2(n)\psi(\lambda_n)).$$

Note that in Theorem A one can replace $M(\sigma, F)$ by $G_1(\sigma, F)$. By using this fact, we obtain the following assertion.

Theorem B. *Let $\lambda \in \Lambda$, $\psi \in L$, $\varepsilon(\lambda, \psi) < p$ and $h \in C_+$. In order that*

$$(\forall F \in S_p(\lambda, \psi))(\forall \sigma \ge \sigma_0(F)):\ G_p(\sigma, F) < \mu(\sigma, F) h(\ln \mu(\sigma, F))$$

it is necessary and sufficient that

$$(\forall \gamma_1, \gamma_2 \in L)(\forall n \ge n_0(\gamma_1, \gamma_2)):\ n < \gamma_1(n) + h^p(\gamma_2(n)\psi(\lambda_n)).$$

In order to prove Theorem B, in addition to series (1.1) we consider the series $F_p(s) = \sum_{n=0}^{\infty} |a_n|^p e^{s\lambda_n p}$, note that $F_p \in S_1(\lambda p, \psi) \Leftrightarrow F \in S_p(\lambda, \psi)$, $G_1(\sigma, F_p) = G_p^p(\sigma, F)$, $\mu(\sigma, F_p) = \mu^p(\sigma, F)$, $\varepsilon(\lambda, \psi) = \varepsilon(\lambda p, \psi)$, and apply Theorem A with λp, F_p, h^p instead of λ, F, h, respectively.

In this paper we establish conditions on the coefficients a_n of Dirichlet series (1.1) under which some asymptotic relations between the two means $G_p(\sigma, F)$ and $G_q(\sigma, F)$ are fulfilled, where $q > p > 0$.

Theorem 1. *Let $q > p$, $\lambda \in \Lambda$, $\psi \in L$, $\varepsilon(\lambda, \psi) < p$ and $h \in C_+$. In order that*

$$\forall F \in S_p(\lambda, \psi): \ G_p(\sigma, F) \le (1 + o(1)) G_q(\sigma, F) h(\ln G_q(\sigma, F)) \quad (\sigma \to +\infty) \tag{1.5}$$

it is necessary and sufficient that

$$\forall \gamma \in L: \ n^{\frac{q-p}{qp}} \le (1 + o(1)) h(\gamma(n)\psi(\lambda_n)) \quad (n \to \infty). \tag{1.6}$$

We derive two simple corollaries of Theorem 1, which we use below.

Corollary 1. *Let $q > p$. For every sequence $\lambda \in \Lambda$ and each function $l \in L$ there exists a Dirichlet series $F \in S_p(\lambda)$ such that*

$$\varlimsup_{\sigma \to +\infty} \frac{G_p(\sigma, F)}{l(G_q(\sigma, F))} = +\infty. \tag{1.7}$$

Proof. Consider a function $h \in C_+$ such that $l(x) = o(xh(\ln x))$ as $x \to +\infty$, and assume that $\tau(\lambda) < +\infty$ (in the opposite case we consider a subsequence λ^* of the sequence λ such that $\tau(\lambda^*) < +\infty$, and notice that $S_p(\lambda^*) \subset S_p(\lambda)$). Then $\varepsilon(\lambda, \psi) = 0 < p$. Choose a function $\psi \in L$ such that condition (1.6) is not fulfilled. By Theorem 1, condition (1.5) does not hold, and this will mean that there exists a Dirichlet series $F \in S_p(\lambda, \psi) \subset S_p(\lambda)$ such that (1.7) is valid. □

Corollary 2. *Let $q > p$. For every sequence $\lambda \in \Lambda$ and each function $\psi \in L$ there exists a Dirichlet series $F \in S_p(\lambda, \psi)$ such that*

$$\varlimsup_{\sigma \to +\infty} \frac{G_p(\sigma, F)}{G_q(\sigma, F)} = +\infty. \tag{1.8}$$

Proof. We may assume that $\tau(\lambda) < +\infty$. Then $\varepsilon(\lambda, \psi) = 0 < p$. Choose a function $h \in L \subset C_+$ such that condition (1.6) is not valid. Then, by Theorem 1, condition (1.5) is not fulfilled, i.e. there exists a Dirichlet series $F \in S_p(\lambda, \psi)$ such that (1.8) holds. □

Note that Corollary 2 follows also from (1.3) and Theorem B (or Theorem A).

Theorem 2. *Let $q > p$, $\lambda \in \Lambda$, $\psi, \varphi \in L$, $\varepsilon(\lambda, \psi) < p$ and*

$$\varphi(x + 1) \sim \varphi(x) \quad (x \to +\infty). \tag{1.9}$$

In order that

$$\forall F \in S_p(\lambda, \psi): \ \varphi(\ln G_p(\sigma, F)) \sim \varphi(\ln G_q(\sigma, F)) \quad (\sigma \to +\infty) \tag{1.10}$$

it is necessary and sufficient that

$$\forall \gamma \in L: \qquad \varphi\biggl(\gamma(n)\psi(\lambda_n) + \frac{q-p}{qp} \ln n\biggr) \sim \varphi(\gamma(n)\psi(\lambda_n)) \quad (n \to \infty). \tag{1.11}$$

Condition (1.9) is essential in Theorem 2. This fact follows from the following assertion.

Theorem 3. *Let $q > p$, $\lambda \in \Lambda$, $\varphi \in L$. In order that there exists a function $\psi \in L$ such that (1.10) is fulfilled it is necessary and sufficient that condition (1.9) holds.*

Theorems 2 and 3 will be obtained from Theorem 1. Furthermore, Theorem 1 will be applied for establishing estimates of exceptional sets in some relations between characteristics of Dirichlet series (in particular, we prove a conjecture of O. B. Skaskiv, which is formulated in [3]), and also for sharpening result a of A. A. Gol'dberg concerning the Paley effect for power series with gaps.

2. Proof of Theorem 1. We begin with the following lemma (see also [1, p. 144]).

Lemma 1. *Let $\lambda \in \Lambda$, $\psi \in L$, $\varepsilon = \varepsilon(\lambda, \psi) < p$. If $\delta \in (\varepsilon, p)$, $c = \frac{p}{p-\delta}$, $n(\sigma) = \min\{n \geq 1 : \psi(\lambda_n) \geq c\sigma\}$, then*

$$\sum_{n \geq n(\sigma)} e^{-\lambda_n \psi(\lambda_n) p} e^{\sigma \lambda_n p} \leq K \qquad (\sigma \geq \sigma_1), \tag{2.1}$$

where K is some constant.

Proof. Let $\varepsilon_1 \in (\varepsilon, \delta)$. Then, from the condition $\varepsilon(\lambda, \psi) < \varepsilon_1$, we have

$$\ln n \leq \varepsilon_1 \lambda_n \psi(\lambda_n) \qquad (n \geq n_1). \tag{2.2}$$

Since the function $n(\sigma)$ is nondecreasing to $+\infty$ on $\mathbb{R}$, $n(\sigma) \geq n_1$ for $\sigma \geq \sigma_1$, and hence, by the definition of $n(\sigma)$ and by (2.2),

$$\sum_{n \geq n(\sigma)} e^{-\lambda_n \psi(\lambda_n) p} e^{\sigma \lambda_n p} \leq \sum_{n \geq n(\sigma)} e^{-\lambda_n \psi(\lambda_n) p} e^{p \lambda_n \psi(\lambda_n)/c} =$$
$$= \sum_{n \geq n(\sigma)} e^{-\delta \lambda_n \psi(\lambda_n)} \leq \sum_{n \geq n(\sigma)} e^{-\delta \ln n / \varepsilon_1} \leq \sum_{n \geq 1} n^{-d/\varepsilon_1} = K$$

for every $\sigma \geq \sigma_1$. □

Now, we will prove Theorem 1.

Sufficiency. We assume that condition (1.6) is satisfied. Suppose that (1.5) is not fulfilled, i.e., there exists a Dirichlet series $F \in S_p(\lambda, \psi)$ of form (1.1), a number $\eta > 0$ and an increasing to $+\infty$ sequence (σ_k) such that

$$G_p(\sigma_k, F) \geq (1 + \eta) G_q(\sigma_k, F) h(\ln G_q(\sigma_k, F)) \qquad (k \geq 0). \tag{2.3}$$

Since F is not an exponential polynomial, $\sigma = o(\ln G_q(\sigma, F))$ as $\sigma \to +\infty$. But $\psi(\lambda_{n(\sigma)-1}) < c\sigma$ for $\sigma \geq \sigma_2$, where $n(\sigma)$ is the function from Lemma 1. Therefore, $\psi(\lambda_{n(\sigma)-1}) = o(\ln G_q(\sigma, F))$ as $\sigma \to +\infty$, and hence there exist a function $\gamma \in L$ and a sequence (σ_{k_m}) such that $l(n(\sigma_{k_m}) - 1)\psi(\lambda_{n(\sigma_{k_m})-1}) = \ln G_q(\sigma_{k_m}, F)$ for each $m \geq 0$. Then, by (1.6),

$$n(\sigma_{k_m})^{\frac{q-p}{qp}} \leq (1 + o(1)) h(\ln G_q(\sigma_{k_m}, F)) \qquad (m \to \infty). \tag{2.4}$$

Put $\sigma_2 = \min\{\sigma \geq \sigma_1 : n(\sigma) \geq n_0(F)\}$, where σ_1 and $n_0(F)$ are the numbers from (2.1) and (1.4), respectively. Then, by (2.1), (1.4) and the Hölder inequality, we have

$$G_p^p(\sigma, F) = \sum_{n < n(\sigma)} |a_n|^p e^{\sigma \lambda_n p} + \sum_{n \geq n(\sigma)} |a_n|^p e^{\sigma \lambda_n p} \leq$$
$$\leq n^{\frac{q-p}{q}}(\sigma) \Bigg(\sum_{n < n(\sigma)} |a_n|^q e^{\sigma \lambda_n q} \Bigg)^{\frac{p}{q}} + K \leq n^{\frac{q-p}{q}}(\sigma) G_q^p(\sigma, F) + K.$$

Therefore, by (2.4),

$$\varlimsup_{m\to\infty} \frac{G_p(\sigma_{k_m},F)}{G_q(\sigma_{k_m},F)h(\ln G_q(\sigma_{k_m},F))} \le \varlimsup_{m\to\infty} \frac{n^{\frac{q-p}{qp}}(\sigma_{k_m})}{h(\ln G_q(\sigma_{k_m},F))} \le 1$$

that contradicts to (2.3). The sufficiency of condition (1.6) is proved.
Necessity. Suppose that condition (1.6) is not fulfilled, i.e., there exist a number $\delta>0$ and a function $\gamma\in L$ such that the set

$$E_1=\bigl\{n\ge 1: n^{\frac{q-p}{qp}} \ge (1+\delta)h(\gamma(n)\psi(\lambda_n))\bigr\}$$

is not bounded. We will prove that in this case (1.5) is not fulfilled.

Consider the function $h_1(x)=e^x h(x)$ and show that there exists a Dirichlet series $F\in S_p(\lambda,\psi)$ such that

$$\varlimsup_{\sigma\to+\infty} \frac{G_p(\sigma,F)}{h_1(\ln G_q(\sigma,F))} > 1. \tag{2.5}$$

If $\varliminf_{x\to+\infty} h_1(x)<+\infty$, there is nothing to prove (in this case inequality (2.5) holds for each Dirichlet series $F\in S_p(\lambda,\psi)$).

Assume that $h_1(x)\to+\infty$ as $x\to+\infty$. Put $l(x)=\min\{h_1(t): t\ge x\}$. It is clear that $l\in L$. Since $l(x)\le h_1(x)=e^x h(x)$, the set

$$E_2=\bigl\{n\ge 1: n^{\frac{q-p}{qp}} \ge (1+\delta)l(\gamma(n)\psi(\lambda_n))e^{-\gamma(n)\psi(\lambda_n)}\bigr\}$$

contains the set E_1. Hence, E_2 is not bounded too.

We fix some $n_0\in E_2$ and put $m_0=n_0$, $\eta_0=\varkappa_{-1}=\varepsilon_0=1$. Suppose that for certain $k\ge 0$ integer numbers $n_k\in E_2$ and m_k and real numbers η_k, $\varkappa_{k-1}$ and ε_k are defined. Then, we choose integer numbers $n_{k+1}\in E_2$ and m_{k+1} so that for them and for

$$\varkappa_k=\frac{\gamma(n_{k+1})\psi(\lambda_{n_{k+1}})}{\lambda_{n_{k+1}}}+\varepsilon_k\psi(\lambda_{n_k}) \tag{2.6}$$

the inequalities

$$\lambda_{m_{k+1}}\ge 2\lambda_{n_k},\quad \varkappa_k\ge 2\psi(\lambda_{n_{k+1}}),\quad n_{k+1}\ge\sqrt{m_{k+1}},\quad m_{k+1}\ge n(\varkappa_{k-1}), \tag{2.7}$$

$$\varkappa_k\ge 2\varkappa_{k-1},\quad \frac{n_k}{e^{(\varkappa_k-\varkappa_{k-1})(\lambda_{n_k}-\lambda_{n_k-1})q}}<\frac{1}{k+1} \tag{2.8}$$

are fulfilled, where $n(\sigma)$ is the function from Lemma 1. Furthermore, we put

$$\varepsilon_{k+1}=\frac{\gamma(n_{k+1})(\lambda_{n_{k+1}}-\lambda_{n_k})}{\lambda_{n_{k+1}}\lambda_{n_k}}+\varepsilon_k\frac{\psi(\lambda_{n_k})}{\psi(\lambda_{n_{k+1}})}. \tag{2.9}$$

Define $b_{n_k}=e^{-\varepsilon_k\lambda_{n_k}\psi(\lambda_{n_k})}$ for every $k\ge 0$, and let

$$b_n=b_{n_k}e^{(\lambda_{n_k}-\lambda_n)\varkappa_k},\quad n\in[m_{k+1},n_{k+1}),\quad k\ge 0. \tag{2.10}$$

Let $b_n=0$ in the other cases.

Consider the auxiliary Dirichlet series

$$F^*(s)=\sum_{n=0}^{\infty} b_n e^{s\lambda_n}=\sum_{k=0}^{\infty}\sum_{n=m_k}^{n_k} b_n e^{s\lambda_n} \tag{2.11}$$

and show that $F^*\in S_p(\lambda,\psi)$. By (2.10) and (2.7), for each $n\in[m_{k+1},n_{k+1}]$ and $k\ge 0$ we have

$$-\ln b_n=-\ln b_{n_k}+(\lambda_n-\lambda_{n_k})\varkappa_k\ge$$
$$\ge(\lambda_n-\lambda_{n_k})\varkappa_k\ge\left(\frac{\lambda_n}{2}+\frac{\lambda_{m_{k+1}}}{2}-\lambda_{n_k}\right)\cdot 2\psi(\lambda_{n_{k+1}})\ge\lambda_n\psi(\lambda_n).$$

Consequently, $|b_n|\le e^{-\lambda_n\psi(\lambda_n)}$ for every $n\ge 0$. Then $F^*\in S_p(\lambda,\psi)$, by Lemma 1.

Now, we prove that

$$\frac{\ln b_{n_k}-\ln b_{n_{k+1}}}{\lambda_{n_{k+1}}-\lambda_{n_k}}=\varkappa_k \qquad (k\ge 0). \tag{2.12}$$

We multiply the two sides of equality (2.9) by the expression $\lambda_{n_{k+1}}\psi(\lambda_{n_{k+1}})$. Then, by (2.6), we obtain

$$-\ln b_{n_{k+1}}=\varepsilon_{k+1}\lambda_{n_{k+1}}\psi(\lambda_{n_{k+1}})=\frac{\gamma(n_{k+1})\psi(\lambda_{n_{k+1}})}{\lambda_{n_k}}(\lambda_{n_{k+1}}-\lambda_{n_k})+\varepsilon_k\lambda_{n_{k+1}}\psi(\lambda_{n_k})=$$
$$=(\varkappa_k-\varepsilon_k\psi(\lambda_{n_k}))(\lambda_{n_{k+1}}-\lambda_{n_k})+\varepsilon_k\lambda_{n_{k+1}}\psi(\lambda_{n_k})=$$
$$=(\lambda_{n_{k+1}}-\lambda_{n_k})\varkappa_k+\varepsilon_k\lambda_{n_k}\psi(\lambda_{n_k})=(\lambda_{n_{k+1}}-\lambda_{n_k})\varkappa_k-\ln b_{n_k}.$$

This implies (2.12).

By (2.8), $\varkappa_k\to+\infty$ as $k\to\infty$. Moreover, $b_n\le b_{n_k}e^{(\lambda_{n_k}-\lambda_n)\varkappa_k}$ for every $n\in[n_k,n_{k+1})$ and $k\ge 0$. From this and from (2.12), as is well known, it follows that for Dirichlet series (2.11) the equalities $\mu_k:=\mu(\varkappa_k,F^*)=b_{n_{k+1}}e^{\varkappa_k\lambda_{n_{k+1}}}$ hold for every $k\ge 0$. Hence, by (2.6), we have

$$\mu_k=e^{-\varepsilon_k\lambda_{n_k}\psi(\lambda_{n_k})}e^{\varkappa_k\lambda_{n_k}}=e^{\gamma(n_{k+1})\psi(\lambda_{n_{k+1}})} \qquad (k\ge 0). \tag{2.13}$$

Next, we note that for a fixed number $r>0$, by (2.10),

$$\sum_{n=m_{k+1}}^{n_{k+1}-1} b_n^r e^{\varkappa_k\lambda_n r}=\sum_{n=m_{k+1}}^{n_{k+1}-1} b_{n_k}^r e^{\varkappa_k\lambda_{n_k} r}=(n_{k+1}-m_{k+1})\mu_k^r \qquad (k\ge 0). \tag{2.14}$$

Furthermore, it can be easily verified that

$$\max_{0\le n\le n_k-1} b_n e^{\varkappa_k\lambda_n}=b_{n_k-1}e^{\varkappa_k\lambda_{n_k-1}} \qquad (k\ge 0),$$

by (2.10) and (2.12). Therefore, by (2.8), we obtain

$$\sum_{n=0}^{n_k-1} b_n^q e^{\varkappa_k\lambda_n q}\le n_k b_{n_k-1}^q e^{\varkappa_k\lambda_{n_k-1}q}=\frac{n_k b_{n_k}^q e^{\varkappa_k\lambda_{n_k}q}}{e^{(\varkappa_k-\varkappa_{k-1})(\lambda_{n_k}-\lambda_{n_k-1})q}}<\frac{\mu_k}{k+1} \qquad (k\ge 0). \tag{2.15}$$

Put $k_0=\min\left\{k\ge 0:\frac{1}{k+1}+\frac{K}{\mu_k^q}<1\right\}$, where K is the number from Lemma 1. Let $k\ge 0$. We put $N_{k+1}=1$ for $0\le k\le k_0$ and $N_{k+1}=\min\left\{n\ge 1:\frac{1}{k+1}+\frac{n_{k+1}-m_{k+1}}{n}+\frac{K}{\mu_k^q}\le 1\right\}$ for $k\ge k_0$. Then, by (2.7), $N_{k+1}\sim n_{k+1}-m_{k+1}\sim n_{k+1}$ as $k\to\infty$.

Define $a_n = b_n/N_{k+1}^{1/q}$ for every $n \in [m_{k+1}, n_{k+1})$ and $k \geq 0$, and let $a_n = 0$ in the other cases. Consider the Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n} = \sum_{k=0}^{\infty} \sum_{n=m_{k+1}}^{n_{k+1}-1} a_n e^{s\lambda_n}.$$

By (2.15), (2.14) with $r = q$, (2.7) and Lemma 1, we obtain

$$\begin{aligned} G_q^q(\varkappa_k, F) &\leq \sum_{n=0}^{n_k-1} b_n^q e^{\varkappa_k\lambda_n q} + \frac{1}{N_{k+1}} \sum_{n=m_{k+1}}^{n_{k+1}-1} b_n^q e^{\varkappa_k\lambda_n q} + \sum_{n\geq m_{k+2}} b_n^q e^{\varkappa_k\lambda_n q} \leq \\ &\leq \frac{\mu_k^q}{k+1} + \frac{n_{k+1}-m_{k+1}}{N_{k+1}}\mu_k + \sum_{n\geq n(\varkappa_k)} b_n^q e^{\varkappa_k\lambda_n q} \leq \\ &\leq \mu_k^q \left(\frac{1}{k+1} + \frac{n_{k+1}-m_{k+1}}{N_{k+1}} + \frac{K}{\mu_k^q}\right) \leq \mu_k^q \qquad (k \geq k_0). \end{aligned} \tag{2.16}$$

On the other hand, by (2.14) with $r = p$, we have

$$G_p^p(\varkappa_k, F) \geq \frac{1}{N_{k+1}^{p/q}} \sum_{n=m_{k+1}}^{n_{k+1}-1} b_n^p e^{\varkappa_k\lambda_n p} = \frac{n_{k+1}-m_{k+1}}{N_{k+1}^{p/q}}\mu_k^p \qquad (k \geq 0). \tag{2.17}$$

Since $n_{k+1} \in E_2$, from (2.16), (2.17) and (2.13) we obtain

$$\varlimsup_{k\to\infty} \frac{G_p(\varkappa_k, F)}{l(\ln G_q(\varkappa_k, F))} \geq \varlimsup_{k\to\infty} \frac{(n_{k+1}-m_{k+1})^{\frac{1}{p}}\mu_k}{N_{k+1}^{1/q} l(\ln\mu_k)} = \varlimsup_{k\to\infty} \frac{n_{k+1}^{\frac{q-p}{qp}} e^{\gamma(n_{k+1})\psi(\lambda_{n_{k+1}})}}{l(\gamma(n_{k+1})\psi(\lambda_{n_{k+1}}))} \geq 1+\delta.$$

This implies (2.5). Indeed, from the definition of the function l and from the continuity and the monotonicity of the function $\ln G_q(\sigma, F)$ it follows that for every $k \geq 0$ there exists a point $\sigma_k \geq \varkappa_k$ such that $l(\ln G_q(\varkappa_k, F)) = h_1(\ln G_q(\varkappa_k, F))$. Then,

$$\varlimsup_{k\to\infty} \frac{G_p(\sigma_k, F)}{h_1(\ln G_q(\sigma_k, F))} \geq \varlimsup_{k\to\infty} \frac{G_p(\varkappa_k, F)}{l(\ln G_q(\varkappa_k, F))} \geq 1+\delta,$$

i.e. (2.5) holds. This completes the proof of Theorem 1.

3. Proof of Theorem 2. We can assume, without loss of generality, that the function $\varphi \in L$ is increasing on $\mathbb{R}$.

Sufficiency. Suppose that conditions (1.9) and (1.11) are fulfilled, and prove that (1.10) holds.

We fix an arbitrary number $\delta > 0$. From (1.11) we obtain

$$(\forall\gamma \in L)(\forall n \geq n_0): \ \varphi\left(\gamma(n)\psi(\lambda_n) + \frac{q-p}{qp}\ln n\right) \leq (1+\delta)\varphi(\gamma(n)\psi(\lambda_n)). \tag{3.1}$$

Introduce the function

$$h_\delta(x) = \exp\{\varphi^{-1}((1+\delta)\varphi(x)) - x\}. \tag{3.2}$$

Then condition (3.1) can be written in the form

$$(\forall\gamma \in L)(\forall n \geq n_0): \ n^{\frac{q-p}{qp}} \leq h_\delta(\gamma(n)\psi(\lambda_n)).$$

As is easily seen, for the function $h(x)=h_\delta(x)$ condition (1.6) is fulfilled. By Theorem 1

$$(\forall F\in S_p(\lambda,\psi))(\forall\sigma\ge\sigma_0(F)):\ G_p(\sigma,F)\le eG_q(\sigma,F)h(\ln G_q(\sigma,F)). \tag{3.3}$$

Using (3.2), we rewrite (3.3) in the form

$$(\forall F\in S_p(\lambda,\psi))(\forall\sigma\ge\sigma_0(F)):\ \varphi(\ln G_p(\sigma,F)-1)\le(1+\delta)\varphi(\ln G_q(\sigma,F)).$$

Thus, by (1.9), we have

$$\forall F\in S_p(\lambda,\psi):\ 1\le\varliminf_{\sigma\to+\infty}\frac{\varphi(\ln G_p(\sigma,F))}{\varphi(\ln G_q(\sigma,F))}\le\varlimsup_{\sigma\to+\infty}\frac{\varphi(\ln G_p(\sigma,F))}{\varphi(\ln G_q(\sigma,F))}\le 1+\delta. \tag{3.4}$$

Since $\delta>0$ is arbitrary, from (3.4) we obtain (1.10). The sufficiency of condition (1.11) is proved.

Necessity. We assume that conditions (1.9) and (1.10) are fulfilled. Suppose that (1.11) is not fulfilled, i.e. there exist a function $\gamma\in L$, a number $\delta>0$ and an increasing to $+\infty$ sequence (n_k) such that

$$\varphi\left(\gamma(n_k)\psi(\lambda_{n_k})+\frac{q-p}{qp}\ln n_k\right)\ge(1+\delta)\varphi(\gamma(n_k)\psi(\lambda_{n_k}))\qquad(k\ge 0). \tag{3.5}$$

Using (3.2), we rewrite (3.5) in the form

$$n_k^{\frac{q-p}{qp}}\ge h_\delta(\gamma(n_k)\psi(\lambda_{n_k}))\qquad(k\ge 0).$$

Hence, for the function $h(x)=e^{-1}h_\delta(x)$ condition (1.6) is not fulfilled. By Theorem 1, (1.5) is not fulfilled. Then there exist a Dirichlet series $F\in S_p(\lambda,\psi)$ and an increasing to $+\infty$ sequence (σ_k) such that

$$G_p(\sigma_k,F)>G_q(\sigma_k,F)h(\ln G_q(\sigma_k,F))=e^{-1}G_q(\sigma_k,F)h_\delta(\ln G_q(\sigma_k,F))\qquad(k\ge 0).$$

This inequality can be written in the form

$$\varphi(\ln G_p(\sigma_k,F)-1)>(1+\delta)\varphi(\ln G_q(\sigma_k,F))\qquad(k\ge 0).$$

Then, from this and (1.9), we see that (1.10) is not fulfilled, but this is impossible. Theorem 2 is proved.

4. Proof of Theorem 3. We can assume, without restricting generality, that the function $\varphi\in L$ is increasing on $\mathbb{R}$.

Necessity. Suppose that for the function φ there exists a function $\psi\in L$ such that (1.10) is fulfilled. Condition (1.10) holds true if we replace λ by some subsequence of λ. Hence, we can assume, without loss of generality, that $\tau(\lambda)<+\infty$ and therefore $\varepsilon(\lambda,\psi)=0<p$.

Suppose that condition (1.9) is not fulfilled. Then there exist a number $\delta>0$ and a positive increasing to $+\infty$ sequence (x_n) such that

$$\varphi(x_n+1)\ge(1+\delta)\varphi(x_n)\qquad(n\ge 0) \tag{4.1}$$

and $x_n/\psi(\lambda_n)\uparrow+\infty$ as $n\to\infty$. Therefore, there exists a function $\gamma\in L$ such that $\gamma(n)=x_n/\psi(\lambda_n)$ for every $n\ge 0$. Using (3.2), we rewrite (4.1) in the form $1\ge h_\delta(\gamma(n)\psi(\lambda_n))$,

$n \geq 0$. From this it follows that for the function $h(x) = 2h_\delta(x)$ condition (1.6) is not fulfilled. By Theorem 1, there exist a Dirichlet series $F \in S_p(\lambda, \psi)$ and an increasing to $+\infty$ sequence (σ_k) such that $G_p(\sigma_k, F) \geq G_q(\sigma_k, F) h_\delta(\ln G_q(\sigma_k, F))$ for $k \geq 0$, i.e., using (3.2), we obtain $\varphi(\ln G_p(\sigma_k, F)) \geq (1+\delta)\varphi(\ln G_q(\sigma_k, F))$ for $k \geq 0$, that contradicts to (1.10). The necessity of condition (1.9) is proved.

Sufficiency. We assume that for the function $\varphi \in L$ condition (1.9) holds. Then there exists a function $\alpha \in L$ such that $\varphi(x + \alpha(x)) \sim \varphi(x)$ as $x \to +\infty$. Consider functions $\psi_1, \psi_2 \in L$ such that $\psi_1(\lambda_n) = \alpha^{-1}\left(\frac{q-p}{qp} \ln n\right)$ and $\psi_2(\lambda_n) = \ln n$ for every $n \geq n_0$. Put $\psi(x) = \max\{\psi_1(x), \psi_2(x)\}$. Then $\psi \in L$, $\varepsilon(\lambda, \psi) = 0 < p$ and for each function $\gamma \in L$ we have

$$1 \leq \varlimsup_{n\to\infty} \frac{\varphi\left(\gamma(n)\psi(\lambda_n) + \frac{q-p}{qp}\ln n\right)}{\varphi(\gamma(n)\psi(\lambda_n))} = \varlimsup_{n\to\infty} \frac{\varphi(\gamma(n)\psi(\lambda_n) + \alpha(\psi_1(\lambda_n)))}{\varphi(\gamma(n)\psi(\lambda_n))} \leq$$
$$\leq \varlimsup_{n\to\infty} \frac{\varphi(\gamma(n)\psi(\lambda_n) + \alpha(\gamma(n)\psi(\lambda_n)))}{\varphi(\gamma(n)\psi(\lambda_n))} \leq \varlimsup_{x\to+\infty} \frac{\varphi(x+\alpha(x))}{\varphi(x)} = 1,$$

i.e. for ψ condition (1.11) is fulfilled. By Theorem 2, for ψ condition (1.10) holds. Hence, Theorem 3 is proved.

5. On exceptional sets in some relations between characteristics of Dirichlet series. Let $F \in S_1(\lambda)$ and $m(\sigma, F) = \inf\{|F(\sigma + it)| : t \in \mathbb{R}\}$. Then

$$m(\sigma, F) \leq \left(\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2 dt\right)^{1/2} = G_2(\sigma, F) \leq M(\sigma, F).$$

O. B. Skaskiv [5,6] proved the following assertions.

Theorem C [5]. *Let $\lambda \in \Lambda$. In order that for every Dirichlet series $F \in S_1(\lambda)$ there exists a set $E(F)$ of finite measure such that $M(\sigma, F) \sim \mu(\sigma, F) \sim m(\sigma, F)$ as $\sigma \to +\infty$ and $\sigma \notin E(F)$ it is necessary and sufficient that*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty. \tag{5.1}$$

Theorem D [6]. *Let $\lambda \in \Lambda$. In order that for every Dirichlet series $F \in S_1(\lambda)$ there exists a set $E(F)$ of finite measure such that $\ln M(\sigma, F) \sim \ln \mu(\sigma, F)$ as $\sigma \to +\infty$ and $\sigma \notin E(F)$ it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \frac{1}{n\lambda_n} < +\infty. \tag{5.2}$$

More precisely, if condition (5.1) (or (5.2)) is not fulfilled, then [5] ([6]) there exists a number $\delta > 0$ and a Dirichlet series $F \in S_1(\lambda)$ such that

$$M(\sigma, F) \geq (1+\delta)G_2(\sigma, F) \geq (1+\delta)\max\{\mu(\sigma, F), m(\sigma, F)\} \qquad (\sigma \geq \sigma_0) \tag{5.3}$$

(or $\ln M(\sigma, F) \geq (1+\delta)\ln \mu(\sigma, F)$ for $\sigma \geq \sigma_0$, respectively).

It is proved in [3] that the finiteness of measure is an exact description of the size of exceptional set $E(F)$ in Theorem C.

Theorem E [3]. *For any sequence $\lambda \in \Lambda$ and each function $l \in L$ there exist a number $\delta > 0$ and a Dirichlet series $F \in S_1(\lambda)$ such that for the $E_1 = \{\sigma \geq 0 : M(\sigma, F) \geq (1+\delta)G_2(\sigma, F)\}$ the estimate $\int_{E_1} l(x)dx = +\infty$ is valid.*

Furthermore, in [3] O. B. Skaskiv formulated the following Conjecture.

Conjecture [3]. *For every function $l \in L$ there exist a sequence $\lambda \in L$ for which condition (5.2) holds, a number $\delta > 0$ and a Dirichlet series $F \in S_1(\lambda)$ such that for the set $E_1 = \{\sigma \geq 0 : \ln M(\sigma, F) \geq (1 + \delta)\ln \mu(\sigma, F)\}$ the estimate $\int_{E_1} l(x)dx = +\infty$ is valid.*

Note that in [3] the validity of Conjecture is proved for every function $l \in L$ such that $\int_0^{+\infty} \frac{dx}{xl(x)} < +\infty$.

Below we prove Conjecture in the general case. Moreover, we sharpen and generalize Theorem E.

We first note that the following assertions are elementary corollaries from Theorems C and D.

Theorem F. *Let $q > p > 0$ and $\lambda \in \Lambda$. In order that for every Dirichlet series $F \in S_p(\lambda)$ there exists a set $E(F)$ of finite measure such that $G_p(\sigma, F) \sim G_q(\sigma, F)$ as $\sigma \to +\infty$ and $\sigma \notin E(F)$ it is necessary and sufficient that condition (5.1) holds.*

Theorem G. *Let $q > p > 0$ and $\lambda \in \Lambda$. In order that for every Dirichlet series $F \in S_p(\lambda)$ there exists a set $E(F)$ of finite measure such that $\ln G_p(\sigma, F) \sim \ln G_q(\sigma, F)$ as $\sigma \to +\infty$ and $\sigma \notin E(F)$ it is necessary and sufficient that condition (5.2) holds.*

Remark. For a sequence $\lambda \in \Lambda$ we have (5.1) $\Rightarrow$ (5.2) $\Rightarrow \tau(\lambda) = 0 \Rightarrow (\forall \psi \in L)$ $(\forall p > 0)$ $(\forall q > 0)$: $S_p(\lambda, \psi) = S_q(\lambda, \psi) \Rightarrow (\forall p > 0)(\forall q > 0)$: $S_p(\lambda) = S_q(\lambda)$.

For example, we prove Theorem F (the proof of Theorem G is similar).

Proof of Theorem F. First, we assume that condition (5.1) is valid. Let $F \in S_p(\lambda)$ be a Dirichlet series of form (1.1). Consider the Dirichlet series $F_p(s) = \sum_{n=0}^{\infty} |a_n|^p e^{s\lambda_n p}$ and $F_q(s) = \sum_{n=0}^{\infty} |a_n|^q e^{s\lambda_n q}$. We apply Theorem C to the sequences λp and λq instead of λ. We have

$$G_p^p(\sigma, F) = G_1(\sigma, F_p) \sim \mu(\sigma, F_p) = \mu^p(\sigma, F) \qquad (\sigma \to +\infty, \ \sigma \notin E(F_p)),$$
$$G_q^q(\sigma, F) = G_1(\sigma, F_q) \sim \mu(\sigma, F_q) = \mu^q(\sigma, F) \qquad (\sigma \to +\infty, \ \sigma \notin E(F_q)).$$

Therefore, $G_p(\sigma, F) \sim G_q(\sigma, F)$ as $\sigma \to +\infty$ and $\sigma \notin E(F)$, where $E(F) = E(F_p) \cup E(F_q)$.

Now, we suppose that condition (5.1) is not fulfilled, and show that there exist a number $\varepsilon > 0$ and a Dirichlet series $F^* \in S_p(\lambda)$ such that

$$G_p(\sigma, F^*) \geq (1 + \varepsilon)G_q(\sigma, F^*) \qquad (\sigma \geq \sigma_0),$$

i.e. condition (5.1) is necessary in Theorem F.

Since for the sequence λ condition (5.1) is not fulfilled, the analogous condition is not fulfilled for the sequence λp. Then, as it is noted above, there exist a number $\delta > 0$ and a Dirichlet series $F \in S_1(\lambda p)$, $F(s) = \sum_{n=0}^{\infty} b_n e^{s\lambda_n p}$, such that (5.3) holds. Let $F^*(s) = \sum_{n=0}^{\infty} |b_n|^{1/p} e^{s\lambda_n}$. By (5.3), we obtain

$$G_p^p(\sigma, F^*) = G_1(\sigma, F) \geq M(\sigma, F) \geq (1 + \delta)\mu(\sigma, F) = (1 + \delta)\mu^p(\sigma, F^*) \qquad (\sigma \geq \sigma_0),$$

and hence, by (1.3),

$$G_q^q(\sigma, F^*) \le \mu^{q-p}(\sigma, F^*) G_p^p(\sigma, F^*) \le (1+\delta)^{-\frac{q-p}{p}} G_p^q(\sigma, F^*) \le \frac{1}{1+\varepsilon} G_p^q(\sigma, F^*)$$

for $\sigma \ge \sigma_0$, where $\varepsilon = (1+\delta)^{\frac{q-p}{qp}} - 1 > 0$. Theorem F is proved. □

As it follows from the following assertions, the finiteness of measure is an exact description of the sizes of the exceptional sets in Theorems F and G.

Theorem 4. *Let $q > p > 0$. For any sequence $\lambda \in \Lambda$ and every function $l \in L$ there exist a function $\eta \in L$ and a Dirichlet series $F \in S_p(\lambda)$ such that for the set $E_2 = \{\sigma \ge 0 : G_p(\sigma, F) \ge \eta(\sigma) G_q(\sigma, F)\}$ the estimate $\int_{E_2} l(x)dx = +\infty$ holds.*

Theorem 5. *Let $q > p > 0$. For any sequence $\lambda \in \Lambda$ such that condition (5.2) is valid and every function $l \in L$ there exist a function $\eta \in L$ and a Dirichlet series $F \in S_p(\lambda)$ such that for the set $E_2 = \{\sigma \ge 0 : \ln G_p(\sigma, F) \ge \eta(\sigma) \ln G_q(\sigma, F)\}$ the estimate $\int_{E_2} l(x)dx = +\infty$ holds.*

Note that Theorem E and Conjecture follow from Theorems 4 and 5, respectively.

In order to prove Theorems 4 and 5, we will need some lemmas.

Lemma 2. [7, p. 184] *Let $\lambda \in \Lambda$ and $A > 0$. If $\tau(\lambda) < A$, then for each Dirichlet series $F \in S_1(\lambda)$ the relation $M(\sigma, F) = o(\mu(\sigma + A, F))$ holds as $\sigma \to +\infty$.*

Let Ω be the class of continuously differentiable on $\mathbb{R}$ function Φ such that $\Phi' \in L$. For a given function $\Phi \in \Omega$ let $\Psi(x) = x - \frac{\Phi(x)}{\Phi'(x)}$ be the function associated with Φ in the sense of Newton and let φ be the inverse function to Φ'. It is known [8, p. 18] that $\Psi(x) \nearrow +\infty$ as $x \to +\infty$.

Lemma 3. [8, p. 19] *Let $F \in S_1(\lambda)$ and $\Phi \in \Omega$. In order that*

$$\ln \mu(\sigma, F) \le \Phi(\sigma) \qquad (\sigma \ge \sigma_0) \tag{5.4}$$

it is necessary and sufficient that

$$|a_n| \le e^{-\lambda_n \Psi(\varphi(\lambda_n))} \qquad (n \ge n_0). \tag{5.5}$$

Lemma 4. *Let $\lambda \in \Lambda$, $\tau(\lambda) = 0$, $\Phi \in \Omega$ and $\psi \in L$ be some function such that $\psi(x) = \Psi(\varphi(x))$ for $x \ge x_1$. Then for every number $\delta > 0$ and any Dirichlet series $F \in S_p(\lambda, \psi) = S_1(\lambda)$ the inequality $(\ln G_p(\sigma, F))' \le 2(\sigma+\delta)\Phi'(\sigma+\delta)/\delta$ holds for $\sigma \ge \sigma_1(F, \delta)$. In particular, $(\ln G_p(\sigma, F))' \le 4\Phi'(2\sigma)$ for $\sigma \ge \sigma_2(F)$.*

Proof. Let $F \in S_1(\lambda)$. Then condition (1.5) holds, i.e. condition (5.5) is fulfilled. By Lemma 3, for F we have (5.4). By Lemma 2 and inequality (5.4), for $\varepsilon = \delta/2 > 0$ and $\sigma \ge \sigma_1(F, \delta)$ we obtain

$$\varepsilon(\ln G_p(\sigma, F))' \le \int_\sigma^{\sigma+\varepsilon} (\ln G_p(x, F))' dx \le \ln G_p(\sigma + \varepsilon, F) \le \ln \mu(\sigma + 2\varepsilon, F) - \Phi(0) \le$$

$$\le \Phi(\sigma + 2\varepsilon) - \Phi(0) = \int_0^{\sigma+\delta} \Phi'(x) dx \le (\sigma + \delta)\Phi'(\sigma + \delta),$$

and this proves Lemma 4. □

Lemma 5. [9] *Let $h_1, h_2 \in L$. If $\varlimsup\limits_{x\to+\infty}(h_1(x)/h_2(x)) > \varepsilon > 0$, then for the set $E = \{x \ge 0 : h_1(x) > \varepsilon h_2(x)\}$ the estimates $\int_E d\ln h_1(x) = +\infty$ and $\int_E d\ln h_2(x) = +\infty$ hold.*

Proof of Theorem 4. Let $\lambda \in \Lambda$ and $l \in L$. We can assume, without loss of generality, that $\tau(\lambda) = 0$. Consider a function $\Phi \in \Omega$ such that $\Phi'(2x) = l(x)$ for $x \ge 0$, and put $\psi(x) = \max\{\Psi(\varphi(x)), 1\}$. Then $\psi \in L$ and, by Corollary 2, there exists a Dirichlet series $F \in S_p(\lambda)$ such that relation (1.8) is valid. As follows from (1.8) and Lemma 5, there exists a function $\eta \in L$ such that for the set E_2 the estimate $\int_{E_2}(\ln G_p(x))'dx = \int_{E_2} d\ln G_p(x) = +\infty$ holds. Since, by Lemma 4, $(\ln G_p(x))' \le 4\Phi'(2x) = 4l(x)$ for $x \ge x_2$, we have $\int_{E_2} l(x)dx = +\infty$. □

Proof of Theorem 5. Let $l \in L$. Consider a function $\Phi \in \Omega$ such that $\Phi'(2x) = xl(x)$ for $x \ge 1$, and put $\psi(x) = \max\{\Psi(\varphi(x)), 1\}$. Then $\psi \in L$. Furthermore, since $x = o(\Phi'(x))$ as $x \to +\infty$, we have $\varphi(x) = o(x)$ as $x \to +\infty$. Consequently, $\psi(x) = \Psi(\varphi(x)) = \varphi(x) - \frac{\Phi(\varphi(x))}{x} = o(x)$ as $x \to +\infty$. Hence, there exists a function $\alpha \in L$ such that $\psi(x\alpha(x)) = o(x)$ as $x \to +\infty$.

Let (n_k) be an increasing sequence of integer numbers such that $n_0 = 0$, $n_1 > 1$ and $\alpha(\ln n_k) \ge 2^k$ for any $k \ge 1$. We put $\lambda_{n_0} = 0$ and $\lambda_{n_k} = \alpha(\ln n_k)\ln n_k$ for $k \ge 1$. The other terms of the sequence (λ_n) are defined so that the inequalities $\max\{\lambda_{n_k}, \lambda_{n_{k+1}}/2\} < \lambda_{n_k+1} < \lambda_{n_k+2} < \ldots < \lambda_{n_{k+1}-1} < \lambda_{n_{k+1}}$ hold for any $k \ge 0$. Then $\lambda \in \Lambda$ and

$$\sum_{n=1}^{\infty}\frac{1}{n\lambda_n} = \sum_{k=0}^{\infty}\sum_{n=n_k+1}^{n_{k+1}}\frac{1}{n\lambda_n} \le \sum_{k=0}^{\infty}\frac{2}{\lambda_{n_{k+1}}}\sum_{n=n_k+1}^{n_{k+1}}\frac{1}{n} \le$$
$$\le 2\sum_{k=0}^{\infty}\frac{\ln n_{k+1}}{\lambda_{n_{k+1}}} = 2\sum_{k=0}^{\infty}\frac{1}{\alpha(\ln n_{k+1})} \le 2\sum_{k=0}^{\infty}\frac{1}{2^{k+1}} = 1,$$

i.e. for the sequence λ condition (5.2) is fulfilled.

Now, since

$$\lim_{k\to\infty}\frac{\ln n_k}{\psi(\lambda_{n_k})} = \lim_{k\to\infty}\frac{\ln n_k}{\psi(\alpha(\ln n_k)\ln n_k)} = +\infty,$$

there exist a function $\gamma \in L$ and a subsequence (n_{k_m}) such that $\ln n_{k_m} = \gamma(n_{k_m})^2\psi(\lambda_{n_{k_m}})$ for every $m \ge 0$. Put $h(x) = \exp\left\{\frac{q-p}{2qp}x\gamma(x)\right\}$. Then

$$h(\gamma(n_{k_m})\psi(n_{k_m})) = h\left(\frac{\ln n_{k_m}}{\gamma(n_{k_m})}\right) = \exp\left\{\frac{q-p}{2qp}\frac{\ln n_{k_m}}{\gamma(n_{k_m})}\gamma\left(\frac{\ln n_{k_m}}{\gamma(n_{k_m})}\right)\right\} \le$$
$$\le \exp\left\{\frac{q-p}{2qp}\ln n_{k_m}\right\} = n_{k_m}^{\frac{q-p}{2qp}} = o(n_{k_m}^{\frac{q-p}{qp}}) \qquad (m\to\infty),$$

and we see that for the function h condition (1.6) from Theorem 1 is not fulfilled. By Theorem 1, there exist a Dirichlet series $F \in S_p(\lambda, \psi)$ and an increasing to $+\infty$ sequence (σ_k) such that

$$2G_p(\sigma_k, F) \ge G_q(\sigma_k, F)h(\ln G_q(\sigma_k, F)) \ge h(\ln G_q(\sigma_k, F)) =$$
$$= \exp\left\{\frac{q-p}{2qp}\ln G_q(\sigma_k, F)\gamma(\ln G_q(\sigma_k, F))\right\} \qquad (k \ge 0).$$

Therefore, $\varlimsup\limits_{\sigma\to+\infty} (\ln G_p(\sigma, F)/\ln G_q(\sigma, F)) = +\infty$, and, by Lemma 5, there exists a function $\eta \in L$ such that for the set E_2 we have

$$\int_{E_2} \frac{(\ln G_p(\sigma, F))'}{\ln G_p(\sigma, F)} d\sigma = \int_{E_2} d\ln\ln G_p(\sigma, F) = +\infty.$$

Since, by Lemma 4,

$$\frac{(\ln G_p(\sigma, F))'}{\ln G_p(\sigma, F)} \le \frac{4\Phi'(2\sigma)}{\sigma} = 4l(\sigma) \qquad (\sigma \ge \sigma_2),$$

we obtain $\int_{E_2} l(x)dx = +\infty$. □

6. Relations between the maximum modulus and the Nevanlinna characteristic for power series with gaps. Let N be the class of nonnegative increasing sequences $\lambda = (\lambda_n)_{n=0}^{\infty}$ of integer numbers. For a sequence $\lambda \in N$ by $\mathcal{A}(\lambda)$ we denote the class of entire functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}. \tag{6.1}$$

For function (6.1) let $M(r, f) = \max\{|f(z)| : |z| = r\}$, $T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(re^{i\theta}|d\theta$ and $G_2(r, f) = \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2\lambda_n}\right)^{1/2}$ be its maximum modulus, the Nevanlinna characteristic and the mean, respectively.

A. A. Gol'dberg [4] proved the following assertion.

Theorem H [4]. *For each sequence $\lambda \in N$ there exists an entire function $f \in \mathcal{A}(N)$ such that* $\varlimsup\limits_{r\to+\infty} (\ln M(r, f)/T(r, f)) = +\infty$.

Using Corollary 1, we prove the following stronger assertion.

Theorem 6. *For each sequence $\lambda \in N$ and every function $h \in L$ there exists an entire function $f \in \mathcal{A}(N)$ such that*

$$\varlimsup_{r\to+\infty} \frac{\ln M(r, f)}{h(T(r, f))} = +\infty. \tag{6.2}$$

Proof. As is easily seen from Corollary 1 with $p = 1$ and $q = 2$ it follows that there exists an entire function $f \in \mathcal{A}(N)$ such that

$$\varlimsup_{r\to+\infty} \frac{\ln M(r, f)}{h(\ln G_2(r, f) + 1)} = +\infty. \tag{6.3}$$

Next, we use the following

Lemma 6. [10, p. 116] *Let α be a measurable function on the segment $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b \ln^+ \alpha(x)dx \le \ln^+ \left(\frac{1}{b-a}\int_a^b \alpha(x)dx\right) + \ln 2.$$

By Lemma 6,

$$\begin{aligned} 2T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(re^{i\theta})|^2 d\theta \le \ln^+ \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right) + \ln 2 = \\ = \ln^+ G_2(r, f)^2 + \ln 2 < 2(\ln G_2(r, f) + 1) \qquad (r \ge r_0) \end{aligned}$$

and from (6.3) we see that (6.2) holds. □

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