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ON ASYMPTOTIC BEHAVIOUR OF ENTIRE FUNCTIONS OF ORDER LESS THAN ONE

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For an entire function L of order $\rho \in (0;1)$ conditions on zeros under which $\ln|L(re^{i\varphi})| = \dfrac{\pi\Delta}{\sin\pi\rho} r^{\rho}\cos\rho(\varphi-\pi) + O(1), z = re^{i\varphi} \to +\infty, z \notin E$, are found.

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Для целой функции L порядка $\rho \in (0;1)$ найдены условия на нули, при которых $\ln|L(re^{i\varphi})| = \dfrac{\pi\Delta}{\sin\pi\rho} r^{\rho}\cos\rho(\varphi-\pi) + O(1), z = re^{i\varphi} \to +\infty, z \notin E$.

It is well known [1, 2], an entire function L of order $\rho \in (0;+\infty)$ is of completely regular growth in the sense of Levin-Pfluger if there exists a sequence (r_n) such that

$$0 < r_n \uparrow +\infty, \quad r_{n+1}/r_n \to 1, \quad n \to \infty, \tag{1}$$

and

$$\ln|L(r_n e^{i\varphi})| = r_n^{\rho} h(\varphi) + o(r_n^{\rho}), \quad n \to \infty, \tag{2}$$

uniformly in $\varphi \in [0;2\pi]$, where $h(\varphi)$ is the indicator of L.

Many authors [3–9] were studying subtler asymptotics. In [4, 5] a two-member asymptotics

$$\ln|L(re^{i\varphi})| = r^{\rho}h(\varphi) + r^{\rho_1}h_1(\varphi) + o(r^{\rho_1}), \quad r \to +\infty \tag{3}$$

is considered in the case $0 < \rho_1 < \rho < +\infty$. Here we obtain some results in the case of $\rho_1 = 0$. In particular, we prove the following propositions.

Theorem 1. *Let $0 < \rho < 1$, $0 < \Delta < +\infty$, $0 < \alpha < 1$ and $\lambda_k = \left(\dfrac{k-\alpha}{\Delta}\right)^{1/\rho}$. In order that for an entire L function of the form*

$$L(z) = \prod_{k=1}^{\infty}\Big(1 - \frac{z}{\lambda_k}\Big) \tag{4}$$

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there exists a sequence (r_n) *such that*

$$0 < r_n \uparrow +\infty, \quad r_{n+1}^\rho - r_n^\rho = O(1), \quad n \to \infty, \tag{5}$$

and uniformly in $\varphi \in [0; 2\pi]$

$$\ln |L(r_n e^{i\varphi})| = r_n^\rho h(\varphi) + O(1), \quad n \to \infty, \tag{6}$$

it is necessary and sufficient that $\alpha = 1/2$, *where*

$$h(\varphi) = \frac{\pi\Delta}{\sin \pi\rho} \cos \rho(\varphi - \pi).$$

Theorem 2. *Let* $0 < \rho < 1$, $0 < \Delta < +\infty$ *and* $\lambda_k = \left(\frac{k - \frac{1}{2} + \alpha_k}{\Delta}\right)^{1/\rho}$. *If*

$$\sum_k |\alpha_k| < +\infty, \tag{7}$$

then for an entire function of form (4) there exist a sequence (r_n) *with properties (5) and (6).*

An entire function L of order $\rho \in (0; +\infty)$, for which there exists a sequence (r_n) with properties (5) and (6) will be called an entire function of fine regular growth. For example, the function $L(z) = \sin z$ is an entire function of fine regular growth, for which (6) holds with $\rho = 1$, $h(\varphi) = |\sin \varphi|$ (see [8]). The entire functions with such properties are used in studying functions of bounded l-index [3] and some interpolation problems [8]. If a sequence (r_n) satisfies conditions (5) and (6), then it satisfies conditions (1) and (2). Therefore, every entire function of fine regular growth is an entire function of completely regular growth in the sense of Levin-Pfluger.

For the proof of Theorem 1 and 2 we need some lemmas.

Lemma 1. *Suppose that* (λ_k) *is a non-decreasing to* $+\infty$ *as* $k \to +\infty$ *sequence of positive numbers and for some* ρ, $0 < \rho < 1$, *and* $\Delta \in (0; +\infty)$

$$N(r) = \frac{\Delta}{\rho} r^\rho + O(1), \quad r \to +\infty, \tag{8}$$

where

$$N(r) = \int_0^r \frac{n(t)}{t}\, dt, \quad n(t) = \sum_{|\lambda_k| \le t} 1.$$

Then for entire function (4) we have

$$(\exists c_1)(\forall z \in \mathbb{C}): \ \ln |L(z)| \le \frac{\pi\Delta}{\sin \pi\rho} r^\rho \cos \rho(\varphi - \pi) + c_1 \tag{9}$$

and if $\varphi = \arg z \in (0; 2\pi)$ *then*

$$\ln |L(z)| = \frac{\pi\Delta}{\sin \pi\rho} r^\rho \cos \rho(\varphi - \pi) + \eta(z), \tag{10}$$

moreover (here and so on by c_1, c_2, ... *we denote arbitrary positive constants)*

$$(\forall \delta \in (0; \pi/2))(\exists c_3)(\forall \varphi \in (\delta; 2\pi - \delta))(\forall r \ge 0): \ |\eta(re^{i\varphi})| \le c_3.$$

Proof. First, we show that for an entire function (4) equality (10) holds. Let $z = re^{i\varphi}$. We have

$$\ln |L(re^{i\varphi})| = -r \int_0^{+\infty} \frac{(t^2+r^2)\cos\varphi - 2rt}{(t^2-2rt\cos\varphi+r^2)^2} N(t)\,dt =: I.$$

Using formula (8), we obtain

$$I = -\frac{\Delta}{\rho} r \int_0^{+\infty} t^\rho \frac{(t^2+r^2)\cos\varphi - 2rt}{(t^2-2rt\cos\varphi+r^2)^2}\,dt + I_1 =$$

$$= -\frac{\Delta}{\rho} r^\rho \int_0^{+\infty} u^\rho \frac{(u^2+1)\cos\varphi - 2u}{(u^2-2u\cos\varphi+1)^2}\,du + I_1 = \frac{\pi\Delta}{\sin\pi\rho} r^\rho \cos\rho(\varphi-\pi) + I_1, \tag{11}$$

where

$$I_1 = r \int_0^{+\infty} O(1) \frac{2rt - (t^2+r^2)\cos\varphi}{(t^2-2rt\cos\varphi+r^2)^2}\,dt.$$

We have

$$|I_1| \le c_2 \int_0^{+\infty} \frac{2r^2t + r(t^2+r^2)}{(t^2-2rt\cos\delta+r^2)^2}\,dt =$$

$$= c_2 \int_0^{+\infty} \frac{(u+1)^2}{(u^2-2u\cos\delta+1)^2}\,dt = c_3 < +\infty, \quad r \ge 0.$$

Hence, from (11) we have (10). Now we prove, that function (4) satisfies condition (9). The function

$$\psi(z) = L(z)\cdot f_1(z), \quad f_1(z) = \exp\left(-\frac{\pi\Delta}{\sin\pi\rho}(ze^{-\pi i})^\rho\right),$$

with the natural choice of a holomorphic branch in the angle $\mathbb{C}(-\delta;\delta) = \{z : |\arg z| < \delta\}$ satisfies the condition

$$(\exists c_4)(\forall z \in \mathbb{C}(-\delta;\delta)) : |\psi(z)| \le c_4 e^{\sigma|z|^\rho}, \quad \sigma > 0.$$

Besides, on the boundary of the angle the function ψ is bounded, that is

$$|\psi(z)| \le c_1, \quad z \in \partial\mathbb{C}(-\delta;\delta) = \{z : |\arg z| = \delta\}.$$

By the Phragmen-Lindelöf [1] principle we have $|\psi(z)| \le c_1, z \in \mathbb{C}(-\delta;\delta)$ for all δ small enough. Therefore,

$$|L(z)| \le c_1/|f_1(z)| = c_1 \exp\left(\frac{\pi\Delta}{\sin\pi\rho} r^\rho \cos\rho(\varphi-\pi)\right).$$

Hence we obtain (9). $\square$

Lemma 2. *If L is an entire function of fine regular growth of order $\rho \in (0;+\infty)$ with the indicator $h(\varphi)$ then (8) holds with*

$$\Delta = \frac{\rho}{2\pi}\int\limits_0^{2\pi} h(\varphi)\,d\varphi.$$

Proof. Indeed, we suppose that $L(0)=1$ and, using the Jensen equality [1]

$$N(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|L(re^{i\varphi})|\,d\varphi,$$

we obtain

$$N(r_n) = \frac{r_n^\rho}{2\pi}\int\limits_0^{2\pi} h(\varphi)\,d\varphi + O(1) = \frac{r_n^\rho\Delta}{\rho} + O(1), \quad n \in \mathbb{N}.$$

For every $r > r_1$ there exists n such that $r_n \le r < r_{n+1}$. Since N is a non-decreasing function,

$$N(r) \le N(r_{n+1}) = \frac{r_{n+1}^\rho\Delta}{\rho} + O(1) = \frac{r_{n+1}^\rho - r_n^\rho + r_n^\rho}{\rho}\Delta + O(1) =$$
$$= \frac{r_n^\rho\Delta}{\rho} + O(1) \le \frac{\Delta}{\rho}r^\rho + c_1.$$

On the other hand,

$$N(r) \ge N(r_n) = \frac{r_n^\rho\Delta}{\rho} + O(1) = \frac{r_{n+1}^\rho - r_{n+1}^\rho + r_n^\rho}{\rho}\Delta + O(1) =$$
$$= \frac{r_{n+1}^\rho\Delta}{\rho} + O(1) \ge \frac{\Delta}{\rho}r^\rho - c_2.$$

Thus, (8) is true. □

Lemma 3. *Let $0 < \Delta < +\infty$, $0 < \alpha < 1$, $0 < \rho < +\infty$ and $\lambda_k = \left(\frac{k-\alpha}{\Delta}\right)^{1/\rho}$. Then*

$$N(r) = \frac{\Delta}{\rho}r^\rho + \left(\alpha - \frac{1}{2}\right)\ln r + \frac{1}{\rho}\left(\alpha - \frac{1}{2}\right)\ln\Delta + \frac{1}{\rho}\ln\Gamma(1-\alpha) + O\left(\frac{1}{r^\rho}\right), r \to +\infty.$$

Proof. Let $\lambda_k \le r < \lambda_{k+1}$. Then $\Delta r^\rho + \alpha - 1 < k \le \Delta r^\rho + \alpha$. We have

$$N(r) = \sum_{\lambda_m \le r} \ln\frac{r}{\lambda_m} = \sum_{m=1}^{k} \ln\frac{r}{\lambda_m} = k\ln r - \ln\prod_{m=1}^{k}\left(\frac{m-\alpha}{\Delta}\right)^{1/\rho} =$$

$$= k\ln r - \frac{k}{\rho}\ln\frac{1}{\Delta} - \frac{1}{\rho}\ln\prod_{m=1}^{k}(m-\alpha) = k\ln r - \frac{k}{\rho}\ln\frac{1}{\Delta} - \frac{1}{\rho}\ln\frac{\Gamma(1+k-\alpha)}{\Gamma(1-\alpha)} =$$
$$= k\ln r + \frac{k}{\rho}\ln\Delta - \frac{1}{\rho}\ln\Gamma(1+k-\alpha) + \frac{1}{\rho}\ln\Gamma(1-\alpha).$$

In view of Stirling's formula [1]

$$\ln \Gamma(x) = x\ln x - x - \frac{1}{2}\ln x + \frac{1}{2}\ln 2\pi + \frac{\theta}{x}, \quad |\theta| \le 3, \quad |x| \ge 2.$$

Therefore,

$$N(r) = k\ln r + \frac{k}{\rho}\ln\Delta - \frac{1}{\rho}(1+k-\alpha)\ln(1+k-\alpha) + \frac{1}{\rho}(1+k-\alpha)+$$

$$+\frac{1}{2\rho}\ln(1+k-\alpha) - \frac{\theta}{\rho(1+k-\alpha)} + \frac{1}{\rho}\ln\Gamma(1-\alpha).$$

The function

$$f(x) = x\ln r + \frac{x}{\rho}\ln\Delta - \frac{1}{\rho}(1+x-\alpha)\ln(1+x-\alpha) + \frac{1}{\rho}(1+x-\alpha)+$$

$$+\frac{1}{2\rho}\ln(1+x-\alpha) - \frac{\theta}{\rho(1+x-\alpha)}$$

on $[\Delta r^\rho + \alpha - 1; \Delta r^\rho + \alpha]$ is increasing, and if $r \to +\infty$

$$f(\Delta r^\rho + \alpha - 1) = \left(\alpha - \frac{1}{2}\right)\ln r + \frac{1}{\rho}\left(\alpha - \frac{1}{2}\right)\ln\Delta + \frac{\Delta}{\rho}r^\rho + \frac{c_1}{r^\rho},$$

$$f(\Delta r^\rho + \alpha) = \left(\alpha - \frac{1}{2}\right)\ln r + \frac{1}{\rho}\left(\alpha - \frac{1}{2}\right)\ln\Delta + \frac{\Delta}{\rho}r^\rho + \frac{c_2(1+o(1))}{r^\rho}.$$

Thus,

$$N(r) = \frac{\Delta}{\rho}r^\rho + \left(\alpha - \frac{1}{2}\right)\ln r + \frac{1}{\rho}\left(\alpha - \frac{1}{2}\right)\ln\Delta + \frac{1}{\rho}\ln\Gamma(1-\alpha) + O\left(\frac{1}{r^\rho}\right), \quad r \to +\infty.$$

□

Corollary 3.1 *If* $\lambda_k = \left(\frac{k-\frac{1}{2}}{\Delta}\right)^{1/\rho}$, $0 < \Delta < +\infty$, $0 < \rho < +\infty$, *then*

$$N(r) = \frac{\Delta}{\rho}r^\rho + \frac{1}{\rho}\ln\sqrt{\pi} + O\left(\frac{1}{r^\rho}\right), \quad r \to +\infty.$$

Lemma 4. *Let* $0 < \Delta < +\infty$, $0 < \rho < 1$, *and the sequence of positive numbers* (λ_k) *satisfy the condition*

$$N(r) - N(t) = \frac{\Delta}{\rho}(r^\rho - t^\rho) + q(t;r),$$

where

$$q(t;r) = O\left(\max\left\{\frac{1}{t^\rho}; \frac{1}{r^\rho}\right\}\right), \quad (t;r) \in (0;+\infty)\times(0;+\infty). \tag{12}$$

Then we have (10), moreover, for each $c_1 > 0$ *the function* η *is bounded in the region* $\mathbb{C}\setminus E$, *where* $E = \{z : |\arg z| \le c_1/r^\rho\}$.

Proof. From (12) it follows (8). Therefore, taking into account Lemma 1, the statement of Lemma 4 is sufficient prove for $\{z : |\arg z| < \frac{\pi}{4}\}$. We have

$$\ln|L(re^{i\varphi})| = -\frac{\Delta}{\rho} r \int_0^{+\infty} t^\rho \frac{(t^2+r^2)\cos\varphi - 2rt}{(t^2 - 2rt\cos\varphi + r^2)^2}\, dt +$$

$$+\left(N(r) - \frac{\Delta}{\rho} r^\rho\right) r \int_0^{+\infty} \frac{2rt - (t^2+r^2)\cos\varphi}{(t^2 - 2rt\cos\varphi + r^2)^2}\, dt + I_2 =$$

$$= \frac{\pi\Delta}{\sin\pi\rho} r^\rho \cos\rho(\varphi - \pi) + O(1) + I_2, \tag{13}$$

where

$$I_2 = r \int_0^{+\infty} \frac{(t^2+r^2)\cos\varphi - 2rt}{(t^2 - 2rt\cos\varphi + r^2)^2} q(t;r)\, dt.$$

We have

$$|I_2| \le \int_0^{+\infty} \frac{|(u^2+1)\cos\varphi - 2u|}{(u^2 - 2u\cos\varphi + 1)^2} |q(ur;r)|\, du \le$$

$$\le \frac{c_2}{r^\rho} \left(\int_0^{u_1} + \int_{u_2}^{+\infty} \right) \frac{(u^2+1)\cos\varphi - 2u}{(u^2 - 2u\cos\varphi + 1)^2}\, du \quad +$$

$$+ \frac{c_2}{r^\rho} \int_{u_1}^{u_2} \frac{2u - (u^2+1)\cos\varphi}{(u^2 - 2u\cos\varphi + 1)^2}\, du \le \frac{c_3}{r^\rho \sin\varphi} + \frac{c_4}{r^\rho \sin\varphi} \le c_5,$$

where

$$u_1 = \frac{1 - |\sin\varphi|}{\cos\varphi} \quad , \quad u_2 = \frac{1 + |\sin\varphi|}{\cos\varphi}.$$

For this reason, from (13) we obtain relation (10) . □

Lemma 5. *Let* $0 < \Delta < +\infty$, $0 < \rho < 1$, $r_n = (n/\Delta)^{\frac{1}{\rho}}$, $m_L(r) = \min\{L(z) : |z| = r\}$ *and* $\lambda_k = \left(\frac{k - \frac{1}{2}}{\Delta}\right)^{1/\rho}$, $k \in \mathbb{N}$. *Then for the function of form (4) the inequality*

$$\ln m_L(r_n) \ge \pi\Delta r_n^\rho \operatorname{ctg}\pi\rho + O(1), \quad n \to \infty, \tag{14}$$

holds.

Proof. We have

$$\ln|L(r_n)| = \sum_{k=1}^{n} \ln\left(\frac{r_n}{\left(\frac{k-\frac{1}{2}}{\Delta}\right)^{1/\rho}} - 1 \right) + \sum_{k=n+1}^{\infty} \ln\left(1 - \frac{r_n}{\left(\frac{k-\frac{1}{2}}{\Delta}\right)^{1/\rho}} \right) =$$

$$= \sum_{k=1}^{n} \ln \frac{r_n}{\left(\frac{k-\frac{1}{2}}{\Delta}\right)^{\frac{1}{\rho}}} + \sum_{k=1}^{n} \ln \left(1 - \left(\frac{k-\frac{1}{2}}{n}\right)^{\frac{1}{\rho}}\right) + \sum_{k=n+1}^{\infty} \ln \left(1 - \left(\frac{n}{k-\frac{1}{2}}\right)^{\frac{1}{\rho}}\right). \qquad (15)$$

Using Corollary 3.1, we obtain

$$\sum_{k=1}^{n} \ln \frac{r_n}{\left(\frac{k-\frac{1}{2}}{\Delta}\right)^{\frac{1}{\rho}}} = N(r_n) = \frac{\Delta}{\rho} r_n^{\rho} + O(1), \quad n \to +\infty. \qquad (16)$$

It is easy to show that the functions $\ln(1-x^{\rho})$ and $\ln(1-x^{-\rho})$ are concave on the intervals $(0;1)$ and $(1;+\infty)$, respectively. Therefore, as in [10, p.247] we obtain

$$\ln\left[1-\left(\frac{k-\frac{1}{2}}{n}\right)^{\frac{1}{\rho}}\right] \geq \frac{1}{2}\left\{\ln\left[1-\left(\frac{k-\frac{1}{2}+t}{n}\right)^{\frac{1}{\rho}}\right] + \ln\left[1-\left(\frac{k-\frac{1}{2}-t}{n}\right)^{\frac{1}{\rho}}\right]\right\}, \quad k \leq n,$$

$$\ln\left[1-\left(\frac{n}{k-\frac{1}{2}}\right)^{\frac{1}{\rho}}\right] \geq \frac{1}{2}\left\{\ln\left[1-\left(\frac{n}{k-\frac{1}{2}+t}\right)^{\frac{1}{\rho}}\right] + \right.$$

$$\left. + \ln\left[1-\left(\frac{n}{k-\frac{1}{2}-t}\right)^{\frac{1}{\rho}}\right]\right\}, \quad k \geq n+1.$$

Hence, we can integrate the first of those inequalities with respect to t from 0 to $\frac{1}{2}$,

$$\ln\left[1-\left(\frac{k-\frac{1}{2}}{n}\right)^{\frac{1}{\rho}}\right] \geq \int_{0}^{\frac{1}{2}} \ln\left[1-\left(\frac{k-\frac{1}{2}+t}{n}\right)^{\frac{1}{\rho}}\right] dt+$$

$$+\int_{0}^{\frac{1}{2}} \ln\left[1-\left(\frac{k-\frac{1}{2}-t}{n}\right)^{\frac{1}{\rho}}\right] dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln\left[1-\left(\frac{k-\frac{1}{2}+t}{n}\right)^{\frac{1}{\rho}}\right] dt =$$

$$= \int_{k-1}^{k} \ln\left[1-\left(\frac{x}{n}\right)^{\frac{1}{\rho}}\right] dx, \quad k \leq n,$$

and similarly

$$\ln\left[1-\left(\frac{n}{k-\frac{1}{2}}\right)^{\frac{1}{\rho}}\right] \geq \int_{k-1}^{k} \ln\left[1-\left(\frac{n}{x}\right)^{\frac{1}{\rho}}\right] dx, \quad k \geq n+1.$$

From this it follows

$$\sum_{k=1}^{n} \ln\left[1-\left(\frac{k-\frac{1}{2}}{n}\right)^{\frac{1}{\rho}}\right] + \sum_{k=n+1}^{\infty} \ln\left[1-\left(\frac{n}{k-\frac{1}{2}}\right)^{\frac{1}{\rho}}\right] \geq$$

$$\geq \int_0^n \ln\left[1-\left(\frac{x}{n}\right)^{\frac{1}{\rho}}\right]\,dx + \int_n^{+\infty} \ln\left[1-\left(\frac{n}{x}\right)^{\frac{1}{\rho}}\right]\,dx =$$

$$= \rho\Delta r_n^\rho \left\{\int_0^1 u^{\rho-1}\ln(1-u)\,du + \int_1^{+\infty} u^{\rho-1}\ln\frac{u-1}{u}\,du\right\} = \pi\Delta r_n^\rho \mathrm{ctg}\pi\rho - \frac{\Delta}{\rho} r_n^\rho.$$

Since $m_L(r) = |L(r)|$, from this and using formulas (15), (16) we obtain (14). $\square$

Proof of Theorem 1. The necessity follows directly from Lemmas 2 and 3, because

$$\int_0^{2\pi} \cos\rho(\varphi-\pi)\,d\varphi = \frac{2}{\rho}\sin\rho\pi.$$

We prove the sufficiency. We put $r_n = \left(\frac{n}{\Delta}\right)^{\frac{1}{\rho}}$. If $z = r_n e^{i\varphi}$, $|\varphi| \leq \frac{c_1}{r_n^\rho}$ then by Lemma 5

$$\ln|L(r_n e^{i\varphi}) \geq \ln m_L(r_n) \geq \pi\Delta r_n^\rho \frac{\cos\rho\pi}{\sin\rho\pi} + O(1) =$$

$$= \frac{\pi\Delta}{\sin\pi\rho} r_n^\rho \cos\rho(\varphi-\pi) - \frac{2\pi\Delta}{\sin\pi\rho} r_n^\rho \left|\sin\frac{\rho\varphi}{2}\sin\frac{\rho}{2}(2\pi-\varphi)\right| + O(1) \geq$$

$$\geq \frac{\pi\Delta}{\sin\pi\rho} r_n^\rho \cos\rho(\varphi-\pi) + O(1).$$

Whence by Lemmas 1 and 4 we obtain the required proposition. $\square$

Proof of Theorem 2. Let $\lambda_k^* = \left(\frac{k-\frac{1}{2}}{\Delta}\right)^{1/\rho}$ and $L^*(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_k^*}\right)$. If $r_n = \left(\frac{n}{\Delta}\right)^{\frac{1}{\rho}}$, then

$$\ln|L(r_n)| - \ln|L^*(r_n)| = \sum_{k=1}^{\infty} \ln\left|\frac{\lambda_k - r_n}{\lambda_k^* - r_n}\right| + \sum_{k=1}^{\infty} \ln\frac{\lambda_k^*}{\lambda_k}.$$

Since

$$|\lambda_k^* - r_n| \geq \min\{|\lambda_k^* - r_{k-1}|; |\lambda_k^* - r_k|\} = O(k^{\frac{1}{\rho}-1}), \quad k\to\infty,$$

we have

$$\ln\left|\frac{\lambda_k - r_n}{\lambda_k^* - r_n}\right| \geq \frac{-\left|\frac{\lambda_k-\lambda_k^*}{\lambda_k^*-r_n}\right|}{1-\left|\frac{\lambda_k-\lambda_k^*}{\lambda_k^*-r_n}\right|} \geq -2\left|\frac{\lambda_k-\lambda_k^*}{\lambda_k^*-r_n}\right| \geq -c_1|\alpha_k|, \quad k\geq k_0;$$

and

$$\ln\frac{\lambda_k^*}{\lambda_k} \geq -\frac{\lambda_k-\lambda_k^*}{\lambda_k^*} \geq -c_2\frac{|\alpha_k|}{k} \geq -c_2|\alpha_k|, \quad k\geq k^*.$$

Therefore by Lemma 5

$$\ln m_L(r_n) \geq \pi\Delta r_n^\rho \mathrm{ctg}\,\pi\rho + O(1), \quad n\to\infty.$$

Further,

$$\sum_{\lambda_k\le r}\ln\frac{r}{\lambda_k}-\sum_{\lambda_k\le r}\ln\frac{r}{\lambda_k^*}=\sum_{\lambda_k\le r}\ln\frac{\lambda_k^*}{\lambda_k}=-\frac{1}{\rho}\sum_{\lambda_k\le r}\ln\left(1+\frac{\alpha_k}{k-\frac{1}{2}}\right)=$$

$$=d_0+\frac{1}{\rho}\sum_{\lambda_k\ge r}\ln\left(1+\frac{\alpha_k}{k-\frac{1}{2}}\right)=d_0+\frac{1}{\rho}\sum_{\lambda_k\ge r}\frac{\alpha_k}{k}+O\left(\sum_{\lambda_k\ge r}\frac{1}{k^2}\right)=$$

$$=d_0+\frac{1}{n(r)}\sum_{\lambda_k\ge r}|\alpha_k|+O\left(\frac{1}{r^\rho}\right)=d_0+O\left(\frac{1}{r^\rho}\right),\quad r\to+\infty,$$

where

$$d_0=-\frac{1}{\rho}\sum_{k=1}^{\infty}\ln\left(1+\frac{\alpha_k}{k-\frac{1}{2}}\right).$$

Furthermore,

$$\sum_{\lambda_k\le r}\ln\frac{r}{\lambda_k^*}-\sum_{\lambda_k^*\le r}\ln\frac{r}{\lambda_k^*}=O\left(\max\left\{\left|\ln\frac{\lambda_{m+1}}{\lambda_{m+1}^*}\right|;\left|\ln\frac{\lambda_m}{\lambda_m^*}\right|\right\}\right)=O\left(\frac{1}{r^\rho}\right),r\to+\infty,$$

where m is such that $\lambda_m^*\le r<\lambda_{m+1}^*$. Therefore, the sequence $\lambda_k=\left(\frac{k-\frac{1}{2}+\alpha_k}{\Delta}\right)^{1/\rho}$ satisfies condition (12). Thereupon, in the same way as in the proof of the sufficiency part of Theorem 1, we obtain the required proposition. □

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