УДК 517.5

B. V. Vynnyts'kyi, V. L. Sharan

ON SINGULAR BOUNDARY FUNCTIONS OF A CLASS OF ANALYTIC FUNCTIONS IN THE HALF-PLANE

B. V. Vynnyts'kyi, V. L. Sharan. *On singular boundary functions of a class of analytic functions in the half-plane*, Matematychni Studii, **19** (2003) 89–96.

We describe singular boundary functions of functions $f \not\equiv 0$ which are analytic the in half-plane $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ and satisfy the condition

$$(\exists \tau_1 \in (0;1))\,(\exists c_1 > 0)\,(\forall z \in \mathbb{C}_+) : |f(z)| \le c_1 \exp\left(\eta^{\tau_1}(c_1|z|)\right),$$

where $\eta : [0;+\infty) \to (0;+\infty)$ is an increasing function such that $\ln \eta(r)$ is a convex function in $\ln r$ on $[1;+\infty)$.

Б. В. Винницкий, В. Л. Шаран. *О сингулярных граничных функциях одного класса функций, аналитических в полуплоскости* // Математичні Студії. – 2003. – Т.19, №1. – С.89–96.

Получено описание сингулярных граничных функций аналитических в полуплоскости $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ функцій $f \not\equiv 0$, удовлетворяющих условию

$$(\exists \tau_1 \in (0;1))\,(\exists c_1 > 0)\,(\forall z \in \mathbb{C}_+) : |f(z)| \le c_1 \exp\left(\eta^{\tau_1}(c_1|z|)\right),$$

где $\eta\colon [0;+\infty) \to (0;+\infty)$ — возрастающая функция такая, что функция $\ln \eta(r)$ является выпуклой относительно $\ln r$ на $[1;+\infty)$.

Let a function f be analytic and bounded in each half-disk $Q_R = \{z : |z| < R, \mathrm{Re}\, z > 0\}$, $0 < R < +\infty$. The equality

$$h(t_2) - h(t_1) = \lim_{x \to +0} \int_{t_1}^{t_2} \ln|f(x+iy)|\,dy - \int_{t_1}^{t_2} \ln|f(iy)|\,dy, \quad t_1 < t_2$$

where $f(iy)$ are the angular boundary values of the function f on the imaginary axis, defining [1, p. 23] up to a constant summand values at the points of continuity a nonincreasing on $(-\infty, +\infty)$ function h such that $h'(t) = 0$ almost everywhere. This function h is called a *singular boundary function* of the function f. It is known [1, p. 30], [2, p. 189–190] that a nonincreasing on $(-\infty, +\infty)$ function h such that $h'(t) = 0$ almost everywhere is a singular boundary function of some analytic and bounded in the half-plane $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ function if and only if

$$\int_{-\infty}^{+\infty} \frac{|dh(t)|}{1+t^2} < +\infty. \tag{1}$$

2000 *Mathematics Subject Classification:* 30D40, 30D99.

Let $\eta\colon [0;+\infty)\to(0;+\infty)$ be an increasing function such that $\ln\eta(r)$ is a convex function in $\ln r$ on $[1;+\infty)$. Denote by B_η the class of analytic functions $f\not\equiv 0$ in the half-plane $\mathbb{C}_+$ which satisfy the condition

$$(\exists\tau_1\in(0;1))\,(\exists c_1>0)\,(\forall z\in\mathbb{C}_+):|f(z)|\le c_1\exp\left(\eta^{\tau_1}(c_1|z|)\right). \tag{2}$$

We shall prove here the following statement.

Theorem 1. *A nonincreasing on $(-\infty;+\infty)$ function h such that $h'(t)=0$ almost everywhere is a singular boundary function of some function $f\in B_\eta$ if and only if*

$$(\exists\tau_2\in(0;1))\,(\exists c_2>0)\,(\forall r\ge 1):\ P(r)\le\frac{\eta^{\tau_2}(c_2r)}{r}+c_2, \tag{3}$$

where

$$P(r)=\frac{1}{2\pi}\int\limits_1^r\left(\frac{1}{t^2}-\frac{1}{r^2}\right)(|dh(t)|+dh(-t)).$$

In order to prove Theorem 1 we need the following statements.

Lemma 1. *If a function $\ln\eta(r)$ is convex in $\ln r$ on $[1;+\infty)$, then*

$$\lim_{r\to+\infty}\ln\eta(r)/\ln r=\lim_{r\to+\infty}r\eta'(r)/\eta(r)=\gamma_0\in(0;+\infty], \tag{4}$$

where η' is the right-hand derivative of the function η.

Proof. It follows from the convexity of the function $\ln\eta(r)$ in $\ln r$ on $[1;+\infty)$ that there exist finite or infinite limits

$$\lim_{r\to+\infty}\ln\eta(r)/\ln r\quad\text{and}\quad\lim_{r\to+\infty}r\eta'(r)/\eta(r).$$

Applying l'Hospital's rule we obtain (4). □

Lemma 2. *If a function $\ln\eta(r)$ is convex in $\ln r$ on $[1;+\infty)$, then condition (3) is equivalent to each of the following conditions*

$$(\exists\tau_3\in(0;1))\,(\exists c_3>0)\,(\forall r\ge 1):\ P_0(r)\le\frac{\eta^{\tau_3}(c_3r)}{r}+c_3, \tag{5}$$

$$(\exists\tau_4\in(0;1))\,(\exists c_4>0)\,(\forall r\ge 1):\ p(r)\le\eta^{\tau_4}(c_4r)+c_4, \tag{6}$$

where

$$P_0(r)=\frac{1}{2\pi}\int\limits_1^r\frac{|dh(t)|+dh(-t)}{t^2},\quad p(r)=\frac{1}{2\pi}\int\limits_1^r\frac{|dh(t)|+dh(-t)}{t}.$$

Proof. The equivalence of conditions (3) and (5) follows from the inequalities

$$P(r)\le P_0(r)\le\frac{4}{3}P(2r).$$

Further, prove equivalence of conditions (3) and (6). If condition (3) is fulfilled, then (6) follows from the inequality $P(2r) \geq \frac{3p(r)}{4r}$. Let condition (6) be fulfilled. Since

$$P(r) = \int_1^r \left(\frac{1}{t^2} + \frac{1}{r^2}\right) p(t)\, dt, \tag{7}$$

we have from (6)

$$P(r) \leq \int_1^r \left(\frac{1}{t^2} + \frac{1}{r^2}\right) \eta^{\tau_4}(c_4 t)\, dt + c_5. \tag{8}$$

If $\gamma_0 > 1$, then using the l'Hospital's rule we obtain (3) from (8). If $\gamma_0 \leq 1$, then $\eta(r) = r^{\gamma_0(1+o(1))}$ as $r \to +\infty$. Then condition (6) is equivalent to the condition

$$(\exists c_6 > 0)\,(\exists \gamma_1 < \gamma_0)\,(\forall r \geq 1) : p(r) \leq r^{\gamma_1} + c_6,$$

and condition (3) is equivalent to the condition

$$P(r) = O(1), \quad r \in [1; +\infty).$$

In this case from (7) we obtain that condition (3) follows from (6). Thus, conditions (3) and (6) are equivalent. □

Lemma 3. *Let a function $f \not\equiv 0$ be analytic in $\mathbb{C}_+$ and bounded in each half-disk Q_R. Then for all $r \in [1; +\infty)$ we have*

$$P(r) \leq \frac{1}{\pi r} \int_{-\pi/2}^{\pi/2} \ln\left|f(re^{i\varphi})\right| \cos\varphi d\varphi + \frac{1}{2\pi} \int_1^r \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln|f(it) \cdot f(-it)|\, dt + c_7,$$

where $f(it)$ are the angular boundary values of the function f on the imaginary axis.

This statement follows from the generalized Carleman's formula [1, p. 26–27].

Proof of Theorem 1. Sufficiency. Without loss generality, we may assume that ± 1 are points of continuity of the function h and $h(t) \not\equiv \text{const}$.

1. If $\gamma_0 \leq 1$, then $\eta(r) = r^{\gamma_0(1+o(1))}$ as $r \to +\infty$. Then the class B_η coincides with the class of functions $f \not\equiv 0$ analytic in $\mathbb{C}_+$, for which

$$(\exists \gamma_2 < \gamma_0)\,(\exists c_1' > 0)\,(\forall z \in \mathbb{C}_+) : |f(z)| \leq c_1' \exp\left(|z|^{\gamma_2}\right), \tag{9}$$

and condition (3) is equivalent to the condition $P(r) = O(1)$, $r \in [1; +\infty)$, i.e. the condition (because condition (3) is equivalent to condition (5))

$$\int_{-\infty}^{+\infty} \frac{|dh(t)|}{1 + t^2} < +\infty. \tag{10}$$

If condition (10) holds, then [1, p. 30] the function

$$f(z) = \exp\left(-\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(-itz + 1)\, dh(t)}{(t^2 + 1)\,(it - z)}\right)$$

is analytic in $\mathbb{C}_+$ and satisfies condition (9).

2. If $1 < \gamma_0 < +\infty$, then $\eta(r) = r^{\gamma_0(1+o(1))}$ as $r \to +\infty$. Then the class B_η coincides with the class of functions $f \not\equiv 0$ analytic in $\mathbb{C}_+$, for which

$$(\exists \gamma_3 < \gamma_0)\,(\exists c_1'' > 0)\,(\forall z \in \mathbb{C}_+): \ |f(z)| \le c_1'' \exp\left(|z|^{\gamma_3}\right), \tag{11}$$

and condition (3) is equivalent the condition

$$(\exists \gamma_4 < \gamma_0)\,(\exists c_8 > 0)\,(\forall r \ge 1): \ P(r) \le r^{\gamma_4 - 1} + c_8. \tag{12}$$

If condition (12) holds, then [1, p. 35] the function

$$f(z) = \exp\left(-\frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{(-itz+1)^{q+1}\, dh(t)}{(t^2+1)^{q+1}\,(it-z)} \right), \quad q = [\gamma_4],$$

is analytic in $\mathbb{C}_+$ and satisfies condition (11), where $[x]$ is the integer part of a number x.

3. Let $\gamma_0 = +\infty$. We consider the function

$$f(z) = \exp\left(-\frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{(-itz+1)^{q(|t|)+1}\, dh(t)}{(t^2+1)^{q(|t|)+1}\,(it-z)} \right), \tag{13}$$

where a nondecreasing function $q\colon [0; +\infty) \to \mathbb{Z}_+$ will be chosen below. It is obvious that

$$f(z) = \exp\left(-\frac{1}{\pi} \int\limits_{\{t: |it-z| \ge 1\}} \frac{(-itz+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}\,(it-z)} dh(t) - \right.$$

$$\left. - \frac{1}{\pi} \int\limits_{\{t: |it-z| < 1\}} \frac{(-itz+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}\,(it-z)} dh(t) \right) = \exp\left(I_1 + I_2\right). \tag{14}$$

Since

$$\left| \frac{(-itz+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}\,(it-z)} \right| \le \frac{(|t||z|+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}} \le \frac{\left(2\,(|z|+1)\right)^{q(|t|)+1}}{(|t|+1)^{q(|t|)+1}},$$

for $|it - z| \ge 1$, we have

$$|I_1| \le \frac{1}{\pi} \int\limits_{\{t: |it-z| \ge 1\}} \frac{\left(2\,(|z|+1)\right)^{q(|t|)+1}}{(|t|+1)^{q(|t|)+1}} |dh(t)|. \tag{15}$$

Further,

$$I_2 = -\frac{1}{\pi} \int\limits_{\{t: |it-z| < 1\}} \frac{(-itz+1)^{q(|t|)+1} - (t^2+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}\,(it-z)} dh(t) -$$

$$-\frac{1}{\pi} \int\limits_{\{t: |it-z| < 1\}} \frac{1}{it-z} dh(t) = I_{2,1} + I_{2,2}.$$

Since for $|it-z|<1$

$$\left|\frac{(-itz+1)^{q(|t|)+1}-(t^2+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}(it-z)}\right| \le \frac{4^{q(|t|)+1}(|z|+2)^{q(|t|)}}{(|t|+1)^{q(|t|)+1}},$$

we obtain

$$|I_{2,1}| \le \frac{1}{\pi} \int\limits_{\{t:|it-z|<1\}} \frac{4^{q(|t|)+1}(|z|+2)^{q(|t|)}}{(|t|+1)^{q(|t|)+1}} |dh(t)|. \tag{16}$$

Finally, since the function h is nonincreasing, we have

$$\operatorname{Re} I_{2,1} = \frac{x}{\pi} \int\limits_{\{t:|it-z|<1\}} \frac{dh(t)}{|it-z|^2} \le 0. \tag{17}$$

Therefore, from (14)–(17) we obtain

$$\ln|f(z)| \le \frac{1}{\pi} \int\limits_{\{t:|it-z|\ge 1\}} \frac{(2(|z|+1))^{q(|t|)+1}}{(|t|+1)^{q(|t|)+1}} |dh(t)| + \frac{1}{\pi} \int\limits_{\{t:|it-z|<1\}} \frac{4^{q(|t|)+1}(|z|+2)^{q(|t|)}}{(|t|+1)^{q(|t|)+1}} |dh(t)| \le$$

$$\le \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{(4(|z|+2))^{q(|t|)+1}}{(|t|+1)^{q(|t|)+1}} |dh(t)| = \frac{1}{\pi} \int\limits_{0}^{+\infty} \frac{(4(|z|+2))^{q(t)+1}}{(t+1)^{q(t)+1}} \left(|dh(t)| + dh(-t)\right). \tag{18}$$

Now we construct a function q such that the integral

$$\int\limits_{0}^{+\infty} \frac{(4(|z|+2))^{q(t)+1}}{(t+1)^{q(t)+1}} \left(|dh(t)| + dh(-t)\right)$$

is convergent. Then from (18) it follows that the integral defining the function f converges uniformly in each compact set from $\mathbb{C}_+$ and the function f is analytic in $\mathbb{C}_+$.

We pass to construction of a function q by a method close to [3]. Let $M_\theta(r) = \max\{|\theta(z)| : |z| = r\}$ and $\mu_\theta(r) = \max\{|\theta_n| r^n : n \ge 0\}$ be the maximum modulus and the maximum term of an entire function θ, respectively, with the Taylor coefficients θ_n. Choose numbers Δ_0 and τ_0 such that $\tau_4 < \Delta_0 < \tau_0 < 1$, and let $c_0 = 4c_4$, $\Delta = \tau_0/\Delta_0$. Then $\Delta > 1$. Since the function $\Delta_0 \ln \eta(c_0 r)$ is a convex function in $\ln r$ and $\ln r = o(\Delta_0 \ln \eta(c_0 r))$, there exists (see [4]) an entire function ψ such that

$$\Delta_0 \ln \eta(c_0 r) = (1+o(1)) \ln M_\psi(r), \quad r \to +\infty. \tag{19}$$

It is well known that for each entire function ψ and for any $r>0$ and $\varepsilon>0$

$$\mu_\psi(r) \le M_\psi(r) \le (1+1/\varepsilon)\, \mu_\psi((1+\varepsilon)r). \tag{20}$$

Then from (19) and (20) we have

$$\tau_0 \ln \eta(c_0 r) = \Delta\Delta_0 \ln \eta(c_0 r) = \Delta(1+o(1)) \ln M_\psi(r) \ge \ln \mu_\psi(r), \quad r \to +\infty,$$

$$\tau_0 \ln \eta(c_0 r) = \Delta(1+o(1)) \ln M_\psi(r) \le \tau_0/\tau_5 \ln \mu_\psi(2r), \quad r \to +\infty, \quad \tau_4 < \tau_5 < \Delta_0 < \tau_0 < 1.$$

Thus,

$$\mu_\psi(r) \le \eta^{\tau_0}(c_0 r), \quad \mu_\psi(r) \ge \eta^{\tau_5}(c_0 r/2), \quad r \ge r_0. \tag{21}$$

Let $\hat{\psi}(r) = \sum\limits_{n=0}^{\infty} \hat{\psi}_n z^n$ be Newton's majorant [5, 7] of the function ψ, $\varkappa_n(\hat{\psi}) = |\hat{\psi}_{n-1}|/|\hat{\psi}_n|$ for $n \ge 1$ and $\varkappa_0(\hat{\psi}) = 0$. It is well known [6, 7] that $\mu_{\hat{\psi}}(r) = \mu_\psi(r)$, the sequence $\varkappa_n(\hat{\psi})$ is nondecreasing, $\varkappa_n(\hat{\psi}) \to +\infty$ and $\mu_{\hat{\psi}}(r) = |\hat{\psi}_n| r^n$ for $\varkappa_n\left(\hat{\psi}\right) \le r \le \varkappa_{n+1}\left(\hat{\psi}\right)$. Choose a function q such that $\varkappa_{q(t)}\left(\hat{\psi}\right) \le t+1 < \varkappa_{q(t)+1}\left(\hat{\psi}\right)$. Then $\hat{\psi}_{q(t)}(t+1)^{q(t)} = \mu_{\hat{\psi}}(t+1)$, $\hat{\psi}_{q(t)} r^{q(t)} \le \mu_{\hat{\psi}}(r)$, $r > 0$. Therefore, from (20) and (21) we obtain

$$\int\limits_0^{+\infty} \frac{(4(|z|+2))^{q(t)+1}}{(t+1)^{q(t)+1}} (|dh(t)| + dh(-t)) =$$

$$= \int\limits_0^{1} \frac{(4(|z|+2))^{q(t)+1}}{(t+1)^{q(t)+1}} (|dh(t)| + dh(-t)) + \int\limits_1^{+\infty} \frac{(4(|z|+2))^{q(t)+1}}{(t+1)^{q(t)+1}} (|dh(t)| + dh(-t)) \le$$

$$\le c_8 (|z|+2)^{c_9} + 8\pi(|z|+2) \int\limits_1^{+\infty} \frac{(4(|z|+2))^{q(t)}}{(t+1)^{q(t)}} dp(t) =$$

$$= c_8 (|z|+2)^{c_9} + 8\pi(|z|+2) \int\limits_1^{+\infty} \frac{\hat{\psi}_{q(t)} (4(|z|+2))^{q(t)}}{\hat{\psi}_{q(t)} (t+1)^{q(t)}} dp(t) \le c_8 (|z|+2)^{c_9} +$$

$$+ 8\pi(|z|+2) \int\limits_1^{+\infty} \frac{\mu_{\hat{\psi}}(4(|z|+2))}{\mu_{\hat{\psi}}(t+1)} dp(t) \le c_8 (|z|+2)^{c_9} +$$

$$+ 8\pi(|z|+2) \int\limits_1^{+\infty} \frac{\eta^{\tau_0}(4c_0(|z|+2))}{\eta^{\tau_5}(c_0(t+1)/2)} dp(t). \tag{22}$$

Thus, from (6), (18) and (22) we have

$$\ln|f(z)| \le c_8 (|z|+2)^{c_9} + 8(|z|+2)\eta^{\tau_0}(4c_0(|z|+2)) \int\limits_1^{+\infty} \frac{dp(t)}{\eta^{\tau_5}(c_0(t+1)/2)} \le$$

$$\le c_8 (|z|+2)^{c_9} + 8(|z|+2)\eta^{\tau_0}(16c_4(|z|+2)) \left(c_{10} + c_{11} \int\limits_1^{+\infty} \frac{\eta'(2c_4 t)}{\eta^{\tau_5-\tau_4+1}(2c_2(t+1))} dt \right).$$

Since $\tau_5 > \tau_4$, the last integral converges. Therefore the function f is analytic in $\mathbb{C}_+$ and using that $\lim\limits_{r\to+\infty} \ln\eta(r)/\ln r = +\infty$ we obtain

$$|f(z)| \le c_{12} \exp\left(\eta^{\tau_0}(8c_2|z|)\right), \quad z \in \mathbb{C}_+.$$

Thus, $f \in B_\eta$.

We shall show that h is a singular boundary function of function f (proof of this fact is analogous to [1, p. 25]). First, we shall show that the moduli of angular boundary values of the function f are equal to 1 almost everywhere. It suffices to show that it holds on the each closed interval $[-a;a]\subset\mathbb{R}$. To this end we note that $f(z)=f_1(z)+f_2(z)+f_3(z)$, where

$$f_1(z)=\exp\left(-\frac{1}{\pi}\int\limits_{-a-1}^{a+1}\frac{dh(t)}{it-z}\right),$$

$$f_2(z)=\exp\left(-\frac{1}{\pi}\int\limits_{-a-1}^{a+1}\frac{(-itz+1)^{q(|t|)+1}-(t^2+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}(it-z)}dh(t)\right),$$

$$f_3(z)=\exp\left(-\frac{1}{\pi}\int\limits_{|t|>a+1}\frac{(-itz+1)^{q(|t|)+1}\,dh(t)}{(t^2+1)^{q(|t|)+1}(it-z)}\right).$$

By a property of the Poisson-Stieltjes integral [8, p. 57-62] it holds for f_1. Further, since the real part of the expression

$$\frac{(-itz+1)^{q(|t|)+1}-(t^2+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}(it-z)}$$

contains x, $x=\operatorname{Re}z$, we have $\lim\limits_{\mathbb{C}_+\ni z\to iy}|f_2(z)|=1$. Moreover, using the equality

$$\frac{(-itz+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}(it-z)}=\frac{(x+i(t-y))\,(ty+1)^{q(|t|)+1}}{(t^2+1)^{q(|t|)+1}((t-y)^2+x^2)}+$$

$$+\frac{(x+i(t-y))\left((-itz+1)^{q(|t|)+1}-(ty+1)^{q(|t|)+1}\right)}{(t^2+1)^{q(|t|)+1}((t-y)^2+x^2)},$$

we obtain that $\lim\limits_{\mathbb{C}_+\ni z\to iy}|f_2(z)|=1$.

If $t_1,t_2\in[-a;a]$, then

$$\lim_{x\to+0}\int\limits_{t_1}^{t_2}\ln|f(x+iy)|\,dy-\int\limits_{t_1}^{t_2}\ln|f(iy)|\,dy=$$

$$=\lim_{x\to+0}\int\limits_{t_1}^{t_2}\left(\ln|f_1(x+iy)|\,dy+\int\limits_{t_1}^{t_2}\ln|f_2(x+iy)|\,dy+\int\limits_{t_1}^{t_2}\ln|f_3(x+iy)|\,dy\right).$$

By a property of the Poisson-Stieltjes integral [8, p. 57-62]

$$\lim_{x\to+0}\int\limits_{t_1}^{t_2}\ln|f_1(x+iy)|\,dy=h(t_2)-h(t_1),$$

and the limits of two last integrals are equal to 0.

Necessity. Let h be a singular boundary function of some function $f \in B_\eta$. Then it is bounded in each half-disk Q_R. Thus, [8, p. 182] f has almost everywhere on the imaginary axis the angular boundary values $f(iy)$, and $|f(iy)| \le c_1 \exp(\eta^{\tau_1}(c_1|y|))$ for almost all $y \in \mathbb{R}$. Then from Lemma 1 we obtain

$$P(r) \le \frac{1}{\pi} \int_1^r \frac{\eta^{\tau_1}(c_1 t)}{t^2} dt + \frac{2\eta^{\tau_1}(c_1 r)}{\pi r} + c_{13}. \qquad (23)$$

If $\gamma_0 \le 1$, then $\int_1^r \frac{\eta^{\tau_1}(c_1 t)}{t^2} dt = O(1)$ for $r \in [1; +\infty)$. If $\gamma_0 > 1$, then using l'Hospital's rule and (4), we get

$$\lim_{r\to+\infty} \frac{\int_1^r \frac{\eta^{\tau_1}(c_1 t)}{t^2} dt}{\frac{\eta^{\tau_1}(c_1 r)}{r}} = \lim_{r\to+\infty} \frac{1}{\tau_1 c_1 r \frac{\eta'(c_1 r)}{\eta(c_1 r)} - 1} = \frac{1}{\tau_1 \gamma_0 - 1} = c_{14} < +\infty,$$

because τ_1 can be chosen such that $\tau_1 \gamma_0 > 1$. Hence from (23) we obtain (3). □

REFERENCES

1. Говоров Н. В. Краевая задача Римана с бесконечным индексом. – М.: Наука, 1986. – 240 с.
2. Гофман К. Банаховы пространства аналитических функций. – М.: ИЛ, 1963. – 337 с.
3. Винницький Б. В., Шепарович I. Б. *Про інтерполяційні послідовності деяких класів цілих функцій* // Матем. студії. – 1999. – Т.12, №2. – С.76–84.
4. Clunie J., Kövari I. *On integral functions having prescribed asymptotic growth, II* // Canad. J. Math. – 1968. – V.20, №1. – P.7–20.
5. Валирон Ж. Аналитические функции. – М.: ГИТТЛ, 1957. – 235 с.
6. Полиа Г., Сеге Г. Задачи и теоремы из анализа: В 2-х т. – М.: Наука, 1978. – Т.2. – 432 с.
7. Шеремета М. М. Цілі ряди Діріхле. – К.: ІСДО, 1993. –168 с.
8. Привалов И. И. Граничные свойства аналитических функций. – М;Л.: Гостехтеоретиздат, 1950. – 336 с.

Institute of Physics and Mathematics,
Drohobych State Pedagogical University

Received 20.08.2002