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ON CONNECTION BETWEEN THE GROWTH OF MAXIMUM MODULUS AND MAXIMAL TERM OF ENTIRE DIRICHLET SERIES IN TERMS OF m-TERMED ASYMPTOTICS

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For entire Dirichlet series a condition on exponents is found in order that the logarithms of maximum modulus of its sum and maximal term have the same m-termed exponential asymptotics.

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Для целого ряда Дирихле найдено условие на показатели для того, чтобы логарифмы максимума модуля его суммы и максимального члена имели одинаковую m-членную экспоненциальную асимптотику.

Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an increasing to $+\infty$ sequence of nonnegative numbers ($\lambda_0 = 0$) and $S(\Lambda)$ be the class of entire Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

We put $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ and let $\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$ be the maximal term and $\nu(\sigma, F) = \max\{n : |a_n| \exp(\sigma\lambda_n) = \mu(\sigma, F)\}$ be the central index of series (1).

We suppose that F has Ritt's order $\varrho_R \in (0, +\infty)$ and Ritt's type $T_R \in (0, +\infty)$ and let $n(t) = \sum_{\lambda_n \leq t} 1$ be the counting function of the sequence Λ. In [1] the following theorem is proved.

Theorem A. *Let $0 < \varrho < \varrho_R$ and $\tau \in \mathbb{R}$. In order that*

$$\ln \mu(\sigma, F) = T_R \exp\{\varrho_R \sigma\} + (\tau + o(1)) \exp\{\varrho\sigma\}, \quad \sigma \to +\infty,$$

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it is necessary and sufficient that for every $\varepsilon > 0$ *the inequality*

$$\ln |a_n| \le -\frac{\lambda_n}{\varrho_R} \ln \frac{\lambda_n}{e\varrho_R T_R} + (\tau+\varepsilon)\left(\frac{\lambda_n}{\varrho_R T_R}\right)^{\varrho/\varrho_R}, \quad n \ge n_0(\varepsilon)$$

is valid and there exists a sequence (n_k) *of positive integers such that*

$$\ln |a_{n_k}| \ge -\frac{\lambda_{n_k}}{\varrho_R} \ln \frac{\lambda_{n_k}}{e\varrho_R T_R} + (\tau-\varepsilon)\left(\frac{\lambda_{n_k}}{\varrho_R T_R}\right)^{\varrho/\varrho_R}$$

and

$$\lambda_{n_{k+1}} - \lambda_{n_k} = o(\lambda_{n_k}^{\alpha}) \quad (k \to \infty), \quad \alpha = \frac{\varrho_R + \varrho}{2\varrho_R}.$$

A connection between $\ln \mu(\sigma, F)$ and $\ln M(\sigma, F)$ is established [2] in the following result.

Theorem B. *In order that the relations*

$$\ln \mu(\sigma, F) \le T_R \exp\{\varrho_R\sigma\} + (\tau + o(1)) \exp\{\varrho\sigma\}, \quad \sigma \to +\infty,$$

and

$$\ln M(\sigma, F) \le T_R \exp\{\varrho_R\sigma\} + (\tau + o(1)) \exp\{\varrho\sigma\}, \quad \sigma \to +\infty,$$

be equivalent for each $F \in S(\Lambda)$ *it is necessary and sufficient that*

$$\ln n(t) = o\left(t^{\varrho/\varrho_R}\right), \quad t \to +\infty.$$

Everywhere further for $m \ge 2$ let $0 < \varrho_m < \dots < \varrho_2 < \varrho_1 < +\infty$, $T_1 > 0$, $T_j \in \mathbb{R}\,(j \in \{2, \dots, m-1\})$, $\tau \in \mathbb{R}$ and $\varrho_m + \varrho_1 > 2\varrho_2$ for $m \ge 3$. In [3] the following analogue of Theorem A is proved.

Theorem C. *In order that*

$$\ln \mu(\sigma, F) = \sum_{j=1}^{m-1} T_j \exp\{\varrho_j\sigma\} + (\tau + o(1)) \exp\{\varrho_m\sigma\}, \quad \sigma \to +\infty,$$

it is necessary and sufficient that for every $\varepsilon > 0$ *the inequality*

$$\ln |a_n| \le -\frac{\lambda_n}{\varrho_1} \ln \frac{\lambda_n}{e\varrho_1 T_1} + \sum_{j=2}^{m-1} T_j \left(\frac{\lambda_n}{\varrho_1 T_1}\right)^{\varrho_j/\varrho_1} + (\tau+\varepsilon)\left(\frac{\lambda_n}{\varrho_1 T_1}\right)^{\varrho_m/\varrho_1}, \quad n \ge n_0(\varepsilon)$$

is valid and there exists a sequence (n_k) *of positive integers such that*

$$\ln |a_{n_k}| \ge -\frac{\lambda_{n_k}}{\varrho_1} \ln \frac{\lambda_{n_k}}{e\varrho_1 T_1} + \sum_{j=2}^{m-1} T_j \left(\frac{\lambda_{n_k}}{\varrho_1 T_1}\right)^{\varrho_j/\varrho_1} + (\tau-\varepsilon)\left(\frac{\lambda_{n_k}}{\varrho_1 T_1}\right)^{\varrho_m/\varrho_1},$$

and

$$\lambda_{n_{k+1}} - \lambda_{n_k} = o(\lambda_{n_k}^{\alpha}) \quad (k \to \infty), \quad \alpha = \frac{\varrho_1 + \varrho_m}{2\varrho_1}.$$

Here we will prove the following analogue of Theorem B.

Theorem 1. *In order that the relations*

$$\ln \mu(\sigma, F) \leq \sum_{j=1}^{m-1} T_j \exp\{\varrho_j \sigma\} + (\tau + o(1)) \exp\{\varrho_m \sigma\}, \quad \sigma \to +\infty, \tag{2}$$

and

$$\ln M(\sigma, F) \leq \sum_{j=1}^{m-1} T_j \exp\{\varrho_j \sigma\} + (\tau + o(1)) \exp\{\varrho_m \sigma\}, \quad \sigma \to +\infty, \tag{3}$$

be equivalent for each $F \in S(\Lambda)$ it is necessary and sufficient that

$$\ln n(t) = o\left(t^{\varrho_m/\varrho_1}\right), \quad t \to +\infty. \tag{4}$$

Proof. Let condition (4) hold. Since by Cauchy's inequality $\ln \mu(\sigma, F) \leq \ln M(\sigma, F)$ for every $F \in S(\Lambda)$, (3) implies (2). On the other hand, for $\lambda_n > 2\lambda_{\nu(\sigma+1,F)}$ we have

$$\begin{aligned} |a_n| \exp\{\sigma\lambda_n\} &= |a_n| \exp\{(\sigma+1)\lambda_n\} \exp\{-\lambda_n\} \leq \mu(\sigma+1, F) \exp\{-\lambda_n\} = \\ &= |a_{\nu(\sigma+1,F)}| \exp\{(\sigma+1)\lambda_{\nu(\sigma+1,F)}\} \exp\{-\lambda_n\} = \\ &= |a_{\nu(\sigma+1,F)}| \exp\{\sigma\lambda_{\nu(\sigma+1,F)}\} \exp\{\lambda_{\nu(\sigma+1,F)} - \lambda_n\} \leq \\ &\leq \mu(\sigma, F) \exp\{\lambda_n/2 - \lambda_n\} = \mu(\sigma, F) \exp\{-\lambda_n/2\}. \end{aligned}$$

Therefore,

$$\begin{aligned} M(\sigma, F) &\leq \sum_{n=0}^{\infty} |a_n| \exp\{\sigma\lambda_n\} = \left(\sum_{\lambda_n \leq 2\lambda_{\nu(\sigma+1,F)}} + \sum_{\lambda_n > 2\lambda_{\nu(\sigma+1,F)}} \right) |a_n| \exp\{\sigma\lambda_n\} \leq \\ &\leq \mu(\sigma, F) \left(\sum_{\lambda_n \leq 2\lambda_{\nu(\sigma+1,F)}} 1 + \sum_{\lambda_n > 2\lambda_{\nu(\sigma+1,F)}} \exp\{-\lambda_n/2\} \right) \leq \\ &\leq \mu(\sigma, F) \left(n(2\lambda_{\nu(\sigma+1,F)}) + \int_{2\lambda_{\nu(\sigma+1,F)}}^{\infty} \exp\{-t/2\} dn(t) \right). \end{aligned} \tag{5}$$

Since $\ln n(t) \leq \frac{1}{2} t^{\varrho_m/\varrho_1}$ $(t \geq t_0)$ and $\lambda_{\nu(\sigma+1,F)} \to +\infty$ $(\sigma \to +\infty)$, it follows that

$$\begin{aligned} \int_{2\lambda_{\nu(\sigma+1,F)}}^{\infty} \exp\{-t/2\} dn(t) &\leq \frac{1}{2} \int_{2\lambda_{\nu(\sigma+1,F)}}^{\infty} n(t) \exp\{-t/2\} dt \leq \\ &\leq \frac{1}{2} \int_{2\lambda_{\nu(\sigma+1,F)}}^{\infty} \exp\left\{ -\frac{1}{2}(t - t^{\varrho_m/\varrho_1}) \right\} dt \to 0, \quad \sigma \to +\infty. \end{aligned} \tag{6}$$

On the other hand,

$$\ln \mu(\sigma+2, F) - \ln \mu(\sigma+1, F) = \int_{\sigma+1}^{\sigma+2} \lambda_{\nu(t,F)} dt \geq \lambda_{\nu(\sigma+1,F)}$$

and, thus,

$$\ln\, n(2\lambda_{\nu(\sigma+1,F)}) = o\left(\lambda^{\varrho_m/\varrho_1}_{\nu(\sigma+1,F)}\right) = o\left(\ln^{\varrho_m/\varrho_1}\, \mu(\sigma+2,F)\right), \quad \sigma \to +\infty. \tag{7}$$

From (2), (5), (6) and (7) we obtain

$$\ln\, M(\sigma,F) \le \ln\, \mu(\sigma,F) + \ln\, n(2\lambda_{\nu(\sigma+1,F)}) + o(1) \le \ln\, \mu(\sigma,F) + o\left(\ln^{\varrho_m/\varrho_1}\, \mu(\sigma+2,F)\right) \le$$
$$\le \sum_{j=1}^{m-1} T_j \exp\{\varrho_j\sigma\} + (\tau + o(1))\exp\{\varrho_m\sigma\} + o(\exp\{\varrho_m\sigma\}), \quad \sigma \to +\infty,$$

whence (3) follows. The sufficiency of condition (4) is proved.

For the proof of the necessity of condition (4) we need some lemmas.

Let Ω be a class of positive unbounded on $(-\infty, +\infty)$ functions Φ such that the derivative Φ' is continuous positive and increasing to $+\infty$ on $(-\infty, +\infty)$. By φ we denote the inverse function to Φ', and let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton.

Lemma 1. [3] *Let $\Phi \in \Omega$ and*

$$\Phi(\sigma) = \sum_{j=1}^{m-1} T_j \exp\{\varrho_j\sigma\} + (\tau + o(1))\exp\{\varrho_m\sigma\}, \quad \sigma \to +\infty. \tag{8}$$

Then

$$\varphi(t) = \frac{1}{\varrho_1}\ln\,\frac{t}{\varrho_1 T_1} - \frac{1}{\varrho_1}\sum_{j=2}^{m-1}\frac{T_j\varrho_j}{T_1\varrho_1}\left(\frac{t}{\varrho_1 T_1}\right)^{\varrho_j/\varrho_1 - 1} - \frac{(\tau+o(1))\varrho_m}{\varrho_1^2 T_1}\left(\frac{t}{\varrho_1 T_1}\right)^{\varrho_m/\varrho_1 - 1}, \quad t \to +\infty. \tag{9}$$

Moreover,

$$t\Psi(\varphi(t)) = \frac{t}{\varrho_1}\ln\,\frac{t}{e\varrho_1 T_1} - \sum_{j=2}^{m-1} T_j\left(\frac{t}{\varrho_1 T_1}\right)^{\varrho_j/\varrho_1} - (\tau + o(1))\left(\frac{t}{\varrho_1 T_1}\right)^{\varrho_m/\varrho_1}, \quad t \to +\infty. \tag{10}$$

Lemma 2. [3] *In order that inequality (2) hold it is necessary and sufficient that*

$$\ln\,|a_n| \le -\lambda_n\Psi(\varphi(\lambda_n)) =$$
$$= -\frac{\lambda_n}{\varrho_1}\ln\,\frac{\lambda_n}{e\varrho_1 T_1} + \sum_{j=2}^{m-1} T_j\left(\frac{\lambda_n}{\varrho_1 T_1}\right)^{\varrho_j/\varrho_1} + (\tau + o(1))\left(\frac{\lambda_n}{\varrho_1 T_1}\right)^{\varrho_m/\varrho_1}, \quad n \to \infty.$$

The following lemma is proved in [2].

Lemma 3. *If $0 < \varrho_m,\ \varrho_1 < +\infty$ and*

$$\varlimsup_{n\to\infty}\frac{\ln n}{\lambda_n^{\varrho_m/\varrho_1}} > \beta > 0$$

then there exists a subsequence $(\lambda_k^\star)$ of the sequence (λ_n) such that $k \le \exp\{\beta(\lambda_k^\star)^{\varrho_m/\varrho_1}\} + 1$ $(k \in \mathbb{N})$ and $k_s \ge \exp\{\beta(\lambda_{k_s}^\star)^{\varrho_m/\varrho_1}\}$ for some increasing sequence (k_s) of positive integers.

If condition (4) does not hold then such number β as in Lemma 3 exists. We put $a_n = 0$ for $\lambda_n \neq \lambda_k^\star$ and $a_n = a_k^\star$ for $\lambda_n = \lambda_k^\star$, where

$$\ln\, a_k^\star = -\frac{\lambda_k^\star}{\varrho_1} \ln\, \frac{\lambda_k^\star}{e\varrho_1 T_1} + \sum_{j=2}^{m-1} T_j \left(\frac{\lambda_k^\star}{\varrho_1 T_1}\right)^{\varrho_j/\varrho_1} + \tau \left(\frac{\lambda_k^\star}{\varrho_1 T_1}\right)^{\varrho_m/\varrho_1}.$$

Since $\ln\, k = o(\lambda_k^\star)$, $k \to \infty$, series (1) with such coefficients is entire, and by Lemma 2 relation (2) holds. It is easy to see that the sequence $(a_k^\star)$ is decreasing.

We will prove that (3) is false. We put $m_s = [k_s - \sqrt{k_s}]$. Then

$$\lambda_{m_s}^\star \geq \left(\frac{\ln\,(m_s-1)}{\beta}\right)^{\varrho_1/\varrho_m} \geq \left(\frac{\ln\,(k_s - \sqrt{k_s} - 2)}{\beta}\right)^{\varrho_1/\varrho_m} =$$
$$= \left(\frac{\ln\, k_s - (1+o(1))/\sqrt{k_s}}{\beta}\right)^{\varrho_1/\varrho_m} =$$
$$= \left(\frac{\ln\, k_s}{\beta}\right)^{\varrho_1/\varrho_m} \left(1 - \frac{1+o(1)}{\sqrt{k_s}\ln\, k_s}\right)^{\varrho_1/\varrho_m} \geq \lambda_{k_s}^\star - \lambda_{k_s}^\star \frac{(1+o(1))\varrho_1}{\varrho_m \sqrt{k_s}\ln\, k_s} = \lambda_{k_s}^\star - \delta_s,$$

where

$$\delta_s = \frac{(1+o(1))\varrho_1}{\varrho_m \beta} (\lambda_{k_s}^\star)^{(\varrho_1-\varrho_m)/\varrho_1} \exp\left\{-\frac{\beta}{2}(\lambda_{k_s}^\star)^{\varrho_m/\varrho_1}\right\}, \quad s \to \infty.$$

Therefore, for $\sigma > 0$ we have

$$M(\sigma, F) \geq \sum_{k=m_s}^{k_s} a_k^\star \exp\{\sigma\lambda_k^\star\} \geq (k_s - m_s + 1) a_{k_s}^\star \exp\{\sigma\lambda_{m_s}^\star\} \geq \sqrt{k_s} a_{k_s}^\star \exp\{\sigma(\lambda_{k_s}^\star - \delta_s)\},$$

that is

$$\ln\, M(\sigma, F) \geq \frac{1}{2}\ln\ k_s + \ln\, a_{k_s}^\star + \sigma\lambda_{k_s}^\star - \sigma\delta_s \geq$$
$$\geq \frac{\beta}{2}(\lambda_{k_s}^\star)^{\varrho_m/\varrho_1} - \frac{\lambda_{k_s}^\star}{\varrho_1}\ln\, \frac{\lambda_{k_s}^\star}{e\varrho_1 T_1} + \sum_{j=2}^{m-1} T_j \left(\frac{\lambda_{k_s}^\star}{\varrho_1 T_1}\right)^{\varrho_j/\varrho_1} + \tau\left(\frac{\lambda_{k_s}^\star}{\varrho_1 T_1}\right)^{\varrho_m/\varrho_1} + \sigma\lambda_{k_s}^\star - \sigma\delta_s.1 \quad (1)$$

Taking $\sigma = \sigma_s = \varphi(\lambda_{k_s}^\star)$, where φ is defined by (9) with $\tau + \beta/2(T_1\varrho_1)^{\varrho_m/\varrho_1}$ instead of τ, according to (10), from (11) we obtain

$$\ln\, M(\sigma_s, F) \geq -\lambda_{k_s}^\star \Psi(\varphi(\lambda_{k_s}^\star)) + \sigma_s\lambda_{k_s}^\star - \sigma_s\delta_s =$$
$$= -\lambda_{k_s}^\star \varphi(\lambda_{k_s}^\star) + \Phi(\varphi(\lambda_{k_s}^\star)) + \varphi(\lambda_{k_s}^\star)\lambda_{k_s}^\star - \sigma_s\delta_s = \Phi(\sigma_s) - \varphi(\lambda_{k_s}^\star)\delta_s.$$

It is easy to show that $\varphi(\lambda_{k_s}^\star)\delta_s \to 0$, $s \to \infty$. Therefore, $\ln\, M(\sigma_s, F) \geq \Phi(\sigma_s) + o(1)$, $s \to \infty$, where Φ is defined by (8) with $\tau + \beta/2(T_1\varrho_1)^{\varrho_m/\varrho_1}$ instead of τ, that is inequality (3) does not hold.

The proof of Theorem 1 is complete. □

We remark that using the method of the proof of Lemma 3 in [2] (see also the proof of Lemma 4 in [4]) we may prove the following result.

Proposition 1. *Let $\alpha\colon [1,+\infty) \to [0,+\infty)$ and $\gamma\colon [0,+\infty) \to [0,+\infty)$ be nonnegative continuous increasing to $+\infty$ functions and $\alpha(x+O(1)) \sim \alpha(x)$ as $x \to +\infty$. If $\varlimsup\limits_{n\to\infty} \frac{\alpha(n)}{\gamma(\lambda_n)} > 1$ then there exists a subsequence (λ_k^*) of the sequence (λ_n) such that $k \le \alpha^{-1}(\gamma(\lambda_k^*)) + 1$ $(k \in \mathbb{N})$ and $k_s \ge \alpha^{-1}(\gamma(\lambda_{k_s}))$ for some increasing sequence (k_s) of positive integers.*

Proposition 1 with $\alpha(x) = \ln x$ was used in papers [5–7].

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