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ON TWO CLASSES OF POSITIVE FUNCTIONS AND BELONGING TO THEM OF MAIN CHARACTERISTICS OF ENTIRE FUNCTIONS

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Let L be a class of positive continuous increasing to $+\infty$ functions on $[x_0, +\infty)$. We say that $l \in L^0$ if $l \in L$ and $l(x(1+\gamma(x))) \sim l(x)$ as $x \to +\infty$ for every function $\gamma(x) \to 0$, $x \to +\infty$, and $l \in L_{RO}$ if $l \in L$ and l is an RO-varying function.

Relations between the classes L^0 and L_{RO} are established and conditions under which the main characteristics ($\ln M_f(r)$, $T(r, f)$ etc.) of entire functions belong to the classes L_{RO} and L^0 are indicated.

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Пусть L — класс положительных непрерывных возрастающих к $+\infty$ на $[x_0, +\infty)$ функций. Скажем, что $l \in L^0$, если $l \in L$ и $l(x(1+\gamma(x))) \sim l(x)$ при $x \to +\infty$ для любой функции $\gamma(x) \to 0$, $x \to +\infty$, и $l \in L_{RO}$, если $l \in L$ и l является RO-меняющейся функцией.

Установлены связи между классами L^0 и L_{RO} и указаны условия, при которых основные характеристики ($\ln M_f(r)$, $T(r, f)$ и др.) целых функций принадлежат классам L_{RO} и L^0.

1°. Introduction. A positive measurable function l on $[x_0, +\infty)$ is said to be slowly varying [1, p. 8] if $l(\lambda x) \sim l(x)$ as $x \to +\infty$ for every $\lambda \in (0, +\infty)$, and is said to be RO-varying [1, p. 86] if for every $\lambda \in [1, a]$, $1 < a < +\infty$, and all $x \ge x_0$ the inequalities $0 < m \le l(\lambda x)/l(x) \le M < +\infty$ hold.

Let L be a class of positive continuous increasing to $+\infty$ functions on $[x_0, +\infty)$. As in [2] we denote by L^0 the subclass of functions $l \in L$ such that $l(x(1+\gamma(x))) \sim l(x)$ as $x \to +\infty$ for every function $\gamma(x) \to 0$, $x \to +\infty$. We say that $l \in L_{si}$ if $l \in L$ is slowly varying, and $l \in L_{RO}$ if $l \in L$ is RO-varying.

For an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with zeros $\lambda_k \in \mathbb{C}$ let $M_f(r) = \max\{|f(z)| : |z| = r\}$ and $\mu_f(r) = \max\{|a_n| r^n : n \ge 0\}$ be the maximal term, $\nu_f(r) = \max\{n : |a_n| r^n = \mu_f(r)\}$ be the central index,

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\varphi})| d\varphi$$

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be the Nevanlinna characteristic and

$$N_f(r)=\int_0^r n(t)d\ln t\quad \Big(n(t)=\sum_{\lambda_k\le t}1\Big)$$

be the Nevanlinna counting function of zeros (without loss of generality we assume that $a_0\ne 0$, i.e. $|\lambda_k|>0$ for all $k\ge 1$).

By Borel's theorem, $\ln M_f(r)\in L_{si}$ if and only if $\ln\mu_f(r)\in L_{si}$. In [3] it is proved that $T(r,f)\in L_{si}$ if and only if $\ln\mu_f(r)\in L_{si}$, necessary and sufficient condition on coefficients is indicated in order that $\ln\mu_f(r)\in L_{si}$, a condition on zeros is indicated in order that $N(r,f)\in L_{si}$, and a problem of slow increase of $\nu_f(r)$ is formulated. The problem is solved in [4], where a criterion of slow increase of the central index is obtained in terms of coefficients.

Here we investigate conditions under which the main characteristics of entire functions belong to the classes L_{RO} and L^0.

2°. Relations between classes L_{RO} and L^0. The following theorem was announced in [5].

Theorem 1. *$L^0\subset L_{RO}$, $L^0\ne L_{RO}$ and every function $\alpha\in L_{RO}$ can be represented in the form $\alpha(x)=e^{\eta(x)}\beta(x)$, where $\beta\in L^0$ and the function η is continuous and bounded on $[x_0,+\infty)$.*

Proof. We suppose that $\alpha\in L^0$ and we will show that for every $\lambda\in[1,+\infty)$

$$c(\lambda)\overset{\text{def}}{=\!=}\varlimsup_{x\to+\infty}\frac{\alpha(\lambda x)}{\alpha(x)}<+\infty.$$

Since the function α is increasing, it is sufficient to prove that $c(2)<+\infty$. We suppose, on the contrary, that $c(2)=+\infty$, i. e. there exists an increasing to $+\infty$ sequence (x_k) such that

$$\frac{\alpha(2x_k)}{\alpha(x_k)}=\omega(x_k)\to+\infty,\quad k\to\infty.$$

We may assume that $2x_k<x_{k+1}\,(k\ge 1)$ and $\sqrt[k]{\omega(x_k)}\to+\infty$ as $k\to\infty$.

We divide the interval $[x_k,2x_k]$ into k equal parts by the points $x_k^{(j)}=x_k+\dfrac{j}{k}x_k$, $0\le j\le k$. Then there exists j_k, $0\le j_k\le k-1$, such that $\alpha\left(x_k^{(j_k+1)}\right)/\alpha\left(x_k^{(j_k)}\right)\ge\sqrt[k]{\omega(x_k)}$, because if $\alpha\left(x_k^{(j+1)}\right)/\alpha\left(x_k^{(j)}\right)<\sqrt[k]{\omega(x_k)}$ for all j, $0\le j\le k-1$, then

$$\omega(x_k)=\frac{\alpha(2x_k)}{\alpha(x_k)}=\frac{\alpha(x_k^{(k)})}{\alpha(x_k^{(k-1)})}\cdots\frac{\alpha(x_k^{(1)})}{\alpha(x_k^{(0)})}<\left(\sqrt[k]{\omega(x_k)}\right)^k=\omega(x_k).$$

Thus,

$$\alpha\left(x_k^{(j_k+1)}\right)/\alpha\left(x_k^{(j_k)}\right)\to+\infty,\quad k\to\infty,$$

and

$$\frac{x_k^{(j_k+1)}}{x_k^{(j_k)}}=\frac{1+(j_k+1)/k}{1+j_k/k}=1+\frac{1}{k+j_k}\to 1,\quad k\to\infty.$$

Hence it follows that $\alpha \notin L^0$. Therefore, $c(\lambda) < +\infty$ for every $\lambda \in [1, +\infty)$, and $\alpha \in L_{RO}$. The inclusion $L^0 \subset L_{RO}$ is proved.

We show that $L^0 \neq L_{RO}$. Put $x_0 = 1$, $x_k = 2^{2^k}$, $t_k = x_k - 1$ for $k \geq 1$ and

$$\alpha(x) = \begin{cases} 2^{k+1}, & x = x_k, \\ 2^k + 1, & x = t_k, \\ x, & 0 \leq x \leq 1, \\ \alpha(t_k) + \dfrac{\alpha(x_k) - \alpha(t_k)}{x_k - t_k}(x - t_k), & t_k \leq x \leq x_k, \\ \alpha(x_k) + \dfrac{\alpha(t_{k+1}) - \alpha(x_k)}{t_{k+1} - x_k}(x - x_k), & x_k \leq x \leq t_{k+1}. \end{cases}$$

It is clear that $\alpha \in L$, $x_k/t_k \to 1$ and $\alpha(x_k)/\alpha(t_k) \to 2$ as $k \to \infty$, that is $\alpha \notin L^0$. We fix $\lambda \in (1, +\infty)$ and suppose that $x_k \leq x < x_{k+1}$. Then $\lambda x \in [\lambda x_k, \lambda x_{k+1})$. Since $\lambda x_{k+1}/x_{k+2} = \lambda 2^{-2^{k+1}} \to 0\, (k \to \infty)$, for all $k \geq k_0(\lambda)$ we have $\lambda x \in [x_k, x_{k+2})$ and, therefore,

$$\frac{\alpha(\lambda x)}{\alpha(x)} \leq \frac{\alpha(x_{k+2})}{\alpha(x_k)} = \frac{2^{k+3}}{2^{k+1}} = 4,$$

that is $\alpha \in L_{RO}$ and $L^0 \neq L_{RO}$.

It is known [1, p. 86–87] that every *RO*-varying function α has a representation

$$\alpha(x) = \exp\left\{\eta(x) + \int_a^x \frac{\varepsilon(t)}{t} dt\right\}, \tag{1}$$

where η and ε are measurable functions, $\varepsilon(x) = \dfrac{1}{\ln a} \ln \dfrac{\alpha(ax)}{\alpha(x)}$, $x \geq a$, and

$$\eta(x) = \frac{1}{\ln a} \int_1^a \ln \frac{\alpha(x_0 t)\alpha(x)}{\alpha(tx)} \frac{dt}{t}, \quad x \geq a.$$

Hence it follows that every function $\alpha \in L_{RO}$ (thus, every function $\alpha \in L^0$), in view of its increase, has representation (1), where the function η is continuous and bounded on $[a, +\infty)$, and the function ε is positive, continuous and bounded on $[a, +\infty)$. Therefore, in order to prove the last statement of Theorem 1 we should show that

$$\beta(x) = \exp\left\{\int_a^x \frac{\varepsilon(t)}{t} dt\right\} \in L^0.$$

Let $\delta > 0$. Since the function ε is positive, continuous and bounded on $[a, +\infty)$, we have $\beta(x) \uparrow +\infty\, (x \to +\infty)$ and, for some positive constant K,

$$1 \leq \frac{\beta((1+\delta)x)}{\beta(x)} = \exp\left\{\int_x^{(1+\delta)x} \frac{\varepsilon(t)}{t} dt\right\} \leq \exp\{K \ln(1+\delta)\} \to 1, \quad \delta \to 0.$$

Therefore, the validity of the last statement of Theorem 1 follows from the following proposition: *if* $\beta \in L$ *and* $B(\delta) = \varlimsup_{x\to+\infty} \dfrac{\beta((1+\delta)x)}{\beta(x)}$, $\delta > 0$, *then in order that* $\beta \in L^0$, *it is necessary and sufficient that* $B(\delta) \to 1\,(\delta \to +0)$.

Let us prove the proposition. Suppose that $\beta \in L^0$ and $B(\delta) \not\to 1\,(\delta \to +0)$. Since the function $B(\delta)$ is nondecreasing, there exists $\lim_{\delta\downarrow 0} B(\delta) = b^* > 1$, that is $B(\delta) \ge b^* > 1$. We choose an arbitrary sequence $(\delta_n) \downarrow 0$. For every δ_n there exists a sequence $(x_{n,k})$ such that $x_{n,k} \uparrow +\infty\,(k \to \infty)$ and $\beta((1+\delta_n)x_{n,k}) \ge b\beta(x_{n,k})$, $1 < b < b^*$. We put

$$x_1 = x_{1,1}, \quad x_n = \min\{x_{n,k} : x_{n,k} \ge x_{n-1},\, k \ge n-1\}$$

and construct a function $\gamma(x) \to 0(x \to +\infty)$ such that $\gamma(x_n) = \delta_n$. Then

$$\beta((1+\gamma(x_n))x_n) = \beta((1+\delta_n)x_n) \ge b\beta(x_n).$$

In view of definition of L^0, this is impossible.

On the contrary, let $B(\delta) \to 1\,(\delta \downarrow 0)$ and $\beta \notin L^0$. Then there exist a function $\gamma(x) \to 0\,(x \to +\infty)$ and a sequence $(x_k) \uparrow +\infty\,(k \to \infty)$ such that

$$\lim_{k\to\infty} \frac{\beta((1+\gamma(x_k))x_k)}{\gamma(x_k)} = a \ne 1.$$

Clearly, $a < 1$ provided $\gamma(x_k) < 0$ and $a > 1$ provided $\gamma(x_k) > 0$. We examine, for example, the second case. Let $\delta > 0$ be an arbitrary number. Then $\gamma(x_k) < \delta\,(k \ge k_0)$ and

$$\varlimsup_{x\to+\infty} \frac{\beta((1+\delta)x)}{\gamma(x)} \ge \varlimsup_{k\to\infty} \frac{\beta((1+\delta)x_k)}{\gamma(x_k)} \ge \lim_{k\to\infty} \frac{\beta((1+\gamma(x_k))x_k)}{\gamma(x_k)} = a,$$

that is $B(\delta) \ge a$ and $\lim_{\delta\downarrow 0} B(\delta) > 1$, which is impossible.

The proposition and Theorem 1 are proved. □

We note that if $\alpha \in L_{RO}$ then (1) implies that $\ln \alpha(x) = O(\ln x)$, $x \to +\infty$.

The main growth characteristics of entire functions are logarithmically convex functions and those of entire Dirichlet series are convex functions. Therefore, the following theorem is useful.

Theorem 2. *If a function* $\alpha \in L$ *is logarithmically convex, that is*

$$\alpha(x) = \int_1^x \frac{\nu(t)}{t}dt + C, \quad \nu(t) \nearrow +\infty\,(t \to +\infty), \quad C = const,$$

then the following are equivalent: 1) $\alpha \in L^0$, *2)* $\alpha \in L_{RO}$ *and 3)* $\nu(x)/\alpha(x) = O(1)$ $(x \to +\infty)$.

If $\alpha \in L$ *is convex, that is*

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then the following are equivalent: 1) $\alpha \in L^0$, *2)* $\alpha \in L_{RO}$ *and 3)* $x\nu(x)/\alpha(x) = O(1)$ $(x \to +\infty)$.

Proof. Suppose that a function $\alpha \in L_{RO}$ is logarithmically convex. Then

$$1 \le \alpha(2x)/\alpha(x) \le M < +\infty$$

and

$$\nu(x)\ln 2 \le \int\limits_x^{2x} \frac{\nu(t)}{t}dt = \alpha(2x) - \alpha(x),$$

whence

$$\frac{\nu(x)}{\alpha(x)} \le \frac{1}{\ln 2}\left(\frac{\alpha(2x)}{\alpha(x)} - 1\right) \le \frac{M-1}{\ln 2}.$$

Therefore, for an arbitrary positive function $\gamma(x) \to 0\,(x \to +\infty)$ we have

$$\alpha(x(1+\gamma(x))) - \alpha(x) = \int\limits_x^{x(1+\gamma(x))} \frac{\nu(t)}{t}dt \le \nu(x(1+\gamma(x)))\ln(1+\gamma(x)) \le$$
$$\le \frac{M-1}{\ln 2}\alpha(x(1+\gamma(x)))\gamma(x) = o(\alpha(x(1+\gamma(x)))), \quad x \to +\infty,$$

that is $\alpha(x(1+\gamma(x)))/\alpha(x) \to 1$ $(x \to +\infty)$, and thus, $\alpha \in L^0$. In view of the first statement of Theorem 1 $\alpha \in L^0$ if and only if $\alpha \in L_{RO}$. Simultaneously we can prove that if $\alpha \in L_{RO}$ then $\nu(x) = O(\alpha(x))$ $(x \to +\infty)$, and if $\nu(x) = O(\alpha(x))$ $(x \to +\infty)$, then $\alpha \in L^0 \subset L_{RO}$. The first part of Theorem 2 is proved. The proof of the second part is analogous. □

We remark that if a function $\nu(x)$ is nondecreasing and *RO*-varying then the function $\alpha(x) = \int_1^x \nu(t)t^{-1}\,dt + C$ belongs to L_{RO}, because from the inequality

$$1 \le \nu(\lambda x)/\nu(x) \le M < +\infty$$

we obtain the inequalities

$$1 \le \alpha(\lambda x)/\alpha(x) \le M + o(1), \quad x \to +\infty.$$

However, there exists a nondecreasing function $\nu(x)$ which is not *RO*-varying and $\alpha \in L_{RO}$. Indeed, let $x_k = 2^{2^k}$, $\nu(x) = 1$ on $[1,\, 2x_1)$, $\nu(2x_1) = e$, $\nu(2x_k) = \nu(2x_{k-1})\ln\, x_{k-1}$ $(k \ge 2)$ and $\nu(x) = \nu(2x_k)$ on $[2x_k,\, 2x_{k+1})$. Then $\nu(x_k)/\nu(2x_k) = \nu(2x_{k-1})/\nu(2x_k) \to 0$ $(k \to \infty)$, that is ν is not an *RO*-varying function. At the same time, since $\nu(x) = \nu(2x_k)$ and $\alpha(x) \ge \alpha(2x_k)$ for all $x \in [2x_k,\, 2x_{k+1})$, we have $\nu(x)/\alpha(x) \le \nu(2x_k)/\alpha(2x_k)$, and in order to prove that α belongs to L_{RO} it suffices to show that $\nu(2x_k)/\alpha(2x_k) = O(1)$ $(k \to \infty)$. The latter follows from the inequality

$$\alpha(2x_k) \ge \int\limits_{2x_{k-1}}^{2x_k} \frac{\nu(t)}{t}dt = \nu(2x_{k-1})(\ln\, x_k - \ln\, x_{k-1}) = \nu(2x_{k-1})\ln\, x_{k-1} = \nu(2x_k).$$

In the case when $\alpha(x) = \int_1^x \nu(t)dt + C$ we have a similar situation.

If a function $\alpha \in L$ is logarithmically convex and $l \in L_{si}$, then [6] $\alpha \in L_{si}$ provided $\alpha \asymp l$ (i.e. $0 < A \le \alpha(x)/l(x) \le B < +\infty,\ x \ge x_0$) or provided $l(x^2) = O(l(x))$ $(x \to +\infty)$, and

$\omega(x) \le \alpha(x)/l(x) \le \ln x$, $x \ge x_0$, where $\omega(x)$ is an arbitrary positive function, $\omega(x) \to +\infty$ $(x \to +\infty)$.

For the class L_{RO} we have $\alpha \in L_{RO}$ provided $\alpha \in L$ and $\alpha \asymp l \in L_{RO}$, because, for some positive constants A and B,

$$\frac{A}{B}\frac{l(\lambda x)}{l(x)} \le \frac{\alpha(\lambda x)}{\alpha(x)} \le \frac{B}{A}\frac{l(\lambda x)}{l(x)}.$$

If $\alpha \in L$ is logarithmically convex and $\alpha \asymp l \in L_{RO}$ then by Theorem 2 $\alpha \in L^0$. For the logarithmically convex functions the following result is true.

Theorem 3. *Let a function $l \in L^0$ be such that $l(x^2) = O(l(x))$, $x \to +\infty$, and a function $\alpha \in L$ be logarithmically convex. If $0 < A \le \alpha(x)/l(x) \le B \ln x$, $B < +\infty$ for $x \ge x_0$ then $\alpha \in L^0$.*

Proof. By Theorem 2 it is sufficient to prove that $\alpha \in L_{RO}$. From the logarithmic convexity of α we obtain

$$\alpha(x^2) \ge \int\limits_x^{x^2} \frac{\nu(t)}{t}dt \ge \nu(x)\ln x,$$

that is $\nu(x) \le \alpha(x^2)/\ln x$. Therefore, using the relation $l(2x) \le l(x^2) \le Kl(x)$, $x \ge x_0$, we have

$$\alpha(2x) - \alpha(x) = \int\limits_x^{2x} \frac{\nu(t)}{t}dt \le \int\limits_x^{2x} \frac{\alpha(t^2)}{t\ln t}dt \le \alpha(4x^2)\ln\left(1 + \frac{\ln 2}{\ln x}\right) \le$$
$$\le Bl(4x^2)\ln(4x^2)\ln\left(1 + \frac{\ln 2}{\ln x}\right) \le BK^2 l(x) 3\ln 2 \le \frac{3BK^2\ln 2}{A}\alpha(x),$$

i. e. $\alpha \in L_{RO}$ and Theorem 3 is proved. □

3°. Entire functions. Let f be an entire function. From belonging of one of the functions $\ln M_f(r)$ or $\ln \mu_f(r)$ to L_{RO} (in view of logarithmic convexity of this functions, it means belonging to L^0) it follows that f has a finite order and by Borel's theorem $\ln M_f(r) \sim \sim \ln \mu_f(r)$ $(r \to +\infty)$. Therefore, $\ln M_f(r) \in L_{RO}$ $(\in L^0)$ if and only if $\ln \mu_f(r) \in L_{RO}$.

From the known inequalities [7, p. 54]

$$T(r, f) \le \ln^+ M_f(r) \le \frac{R + r}{R - r}T(R, f), \quad 0 < r < R < +\infty,$$

we obtain the inequalities $T(r, f) \le \ln^+ M_f(r) \le 3T(2r, f)$, whence

$$1 \le \frac{\ln M_f(2r)}{\ln M_f(r)} \le \frac{3T(4r, f)}{T(r, f)}, \quad 1 \le \frac{T(2r, f)}{T(r, f)} \le \frac{3\ln M_f(2r)}{\ln M_f(r/2)},$$

i.e. $T(r, f) \in L^0$ if and only if $\ln M_f(r) \in L^0$.

Thus, as in [3], the investigation reduces to the study of belonging of the function $\ln \mu_f(r)$ to L^0. Since

$$\ln \mu_f(r) = \ln \mu_f(1) + \int\limits_1^r \frac{\nu_f(t)}{t}dt,$$

by Theorem 2 $\ln\,\mu_f(r) \in L^0$ if and only if $\nu_f(r) = O(\ln\,\mu_f(r))$ as $r \to +\infty$.

Let $f^0(z) = \sum_{n=0}^{\infty} a_n^0 z^n$ be a Newton majorant of the entire function f, $r_n^0 = a_n^0/a_{n+1}^0$ and $(n_k, -\ln\,a_{n_k}^0)$ be the peaks of Newton's diagram. Then $a_{n_k}^0 = |a_{n_k}|$, $\nu_f(r) = \nu_{f^0}(r)$, $\mu_f(r) = \mu_{f^0}(r)$, $r_{n_k}^0 \uparrow +\infty\,(k \to \infty)$, and if $r_{n_{k-1}}^0 \le r < r_{n_k}^0$ then $\nu_{f^0}(r) = n_k$ and $\mu_{f^0}(r) = a_{n_k}^0 r^{n_k}$. Hence we easily obtain that $\nu_f(r) = O(\ln\,\mu_f(r))$ $(r \to +\infty)$, if and only if $n_k = O(\ln\,\mu_f(r_{n_{k-1}}^0))$ $(k \to \infty)$. The latter is equivalent to the condition

$$\ln\,r_{n_{k-1}}^0 + \frac{\ln\,a_{n_k}^0}{n_k} \ge h > 0, \quad k \ge 1.$$

But

$$\ln\,r_{n_{k-1}}^0 = \frac{\ln\,a_{n_{k-1}}^0 - \ln\,a_{n_k}^0}{n_k - n_{k-1}} = \frac{\ln\,|a_{n_{k-1}}| - \ln\,|a_{n_k}|}{n_k - n_{k-1}}$$

and, therefore, the following result is proved.

Theorem 4. *For each entire function f the following statements are equivalent:*

1) $T(r,f) \in L^0$,
2) $\ln\,M_f(r) \in L^0$,
3) $\ln\,\mu_f(r) \in L^0$ *and*
4) $\displaystyle\varliminf_{n\to\infty}\left(\frac{\ln\,|a_{n_{k-1}}| - \ln\,|a_{n_k}|}{n_k - n_{k-1}} - \frac{1}{n_k}\ln\,\frac{1}{|a_{n_k}|}\right) > 0$, *where n_k are the abcissas of the peaks of a Newton majorant of f.*

Let $\alpha \in L$ be such that $x\alpha(x)$ is a convex function and B_α be the class of entire transcendental functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\ln\,|a_n| \le -n\alpha(n)$, $n \ge n_0$. In [3] it is proved that in order that $\ln\,M_f(r) \in L_{si}$ for each $f \in B_\alpha$ it is necessary and sufficient that

$$\varliminf_{x\to+\infty} \frac{\alpha(x)}{x} > 0. \tag{2}$$

Since $L_{si} \subset L^0$, we have $\ln\,M_f(r) \in L^0$ for each $f \in B_\alpha$ provided (2) holds.

We will show that if (2) does not hold, then there exists a function $f \in B_\alpha$ such that $\ln\,M_f(r) \notin L^0$. Having this in mind, we consider an entire function

$$f(z) = \sum_{k=1}^{\infty} \exp\{-n_k\alpha(n_k)\} z^{n_k}, \tag{3}$$

where (n_k) is an arbitrary increasing sequence of positive integers. Since $x\alpha(x)$ is a convex function, function (3) coincides with its Newton majorant, and (n_k) coincides with the abscissas of the peaks of Newton's diagram. Therefore, by Theorem 4 $\ln\,M_f(r) \in L^0$ if and only if

$$\varliminf_{n\to\infty}\left(\frac{n_k\alpha(n_k) - n_{k-1}\alpha(n_{k-1})}{n_k - n_{k-1}} - \alpha(n_k)\right) > 0,$$

that is

$$\frac{n_k}{n_{k+1} - n_k}(\alpha(n_{k+1}) - \alpha(n_k)) \ge h > 0, \quad k \ge 1.$$

If condition (2) does not holds then as in [3] we can choose a sequence (n_k) such that

$$\frac{n_k}{n_{k+1}-n_k}(\alpha(n_{k+1})-\alpha(n_k))\to 0, \quad k\to\infty. \tag{4}$$

Therefore, for such function (3) we have $\ln M_f(r)\notin L^0$.

In [3] it is showed that the condition $\overline{\lim}_{x\to+\infty}\alpha(x)/\ln x=+\infty$ is necessary and sufficient for existence of a function f of form (3) such that $\ln f(r)\in L_{si}$. Here we show that in order that among the function of form (3) there exists one such that $\ln f(r)\in L^0$, it is necessary and sufficient that $\overline{\lim}_{x\to+\infty}\alpha(x)/\ln x>0$.

Indeed, we suppose, on the contrary, that $\ln f(r)\notin L^0$ for each function (3), i.e. for every increasing sequence (n_k) of positive integers (4) holds. In particular, for $n_k=k$ we have $k(\alpha(k+1)-\alpha(k))\to 0$, $k\to\infty$, whence we easily obtain the relation $\alpha(x)=o(\ln x)$, $x\to+\infty$, which is impossible. On the other hand, if $\alpha(x)=o(\ln x)$, $x\to+\infty$, then by Hadamard's theorem each function (3) has infinite order and, thus, $\ln f(r)\notin L^0$.

As remarked above, for entire functions of finite order $\ln M_f(r)\sim\ln\mu_f(r)$ as $r\to+\infty$. This relation can be refined if $\ln\mu_f(r)\in L_{RO}$.

Theorem 5. *If* $\ln\mu_f(r)\in L_{RO}$ *then*

$$M_f(r)=O(\mu_f(r)\ln\mu_f(r)), \quad r\to+\infty, \tag{5}$$

and in (5) O cannot be replaced by o.

Proof. Since

$$|a_n|(2r)^n\le\mu_f(2r)=|a_{\nu_f(2r)}|2^{\nu_f(2r)}r^{\nu_f(2r)}\le 2^{\nu_f(2r)}\mu_f(r),$$

by the condition $\ln\mu_f(r)\in L_{RO}$ we have

$$\begin{aligned} M_f(r)&\le\sum_{n<\nu_f(2r)}|a_n|r^n+\sum_{n\ge\nu_f(2r)}|a_n|r^n\le\\ &\le\mu_f(r)\nu_f(2r)+\mu_f(r)\sum_{n\ge\nu_f(2r)}2^{\nu_f(2r)-n}=\mu_f(r)\left(\nu_f(2r)+\sum_{n=0}^{\infty}2^{-n}\right)=\\ &=O(\mu_f(r)\ln\mu_f(2r))=O(\mu_f(r)\ln\mu_f(r)), \quad r\to+\infty.\end{aligned}$$

Relation (5) is proved.

To prove the second part of Theorem 5 we put $n_k=2^{2^k}$, $r_k=e^{n_k}$, $a_{n_0}=1$, $a_{n_k}=\prod_{j=0}^{k-1}r_j^{-(n_{j+1}-n_j)}\,(k\ge 1)$ and $a_n=a_{n_k}r_k^{n_k-n}$ for $n_k<n<n_{k+1}$. Then $a_{n+1}/a_n=1/r_k\to 0\,(k\to\infty)$ for $n_k\le n\le n_{k+1}-1$, whence it follows that the power series with such coefficients represents an entire function f. We easily see that $\nu_f(r)=n_{k+1}$, $\mu_f(r)=a_{n_{k+1}}r^{n_{k+1}}$ for $r_k\le r<r_{k+1}$ and $\mu_f(r_k)=a_{n_k}r^{n_k}$. Hence, first it follows that for

$r_k \le r < r_{k+1}$

$$\frac{\ln \mu_f(r)}{\nu_f(r)} = \frac{\ln a_{n_{k+1}} + n_{k+1}\ln r}{n_{k+1}} \ge \frac{\ln a_{n_{k+1}} + n_{k+1}\ln r_k}{n_{k+1}} =$$
$$= \frac{1}{n_{k+1}}\left(n_{k+1}\ln r_k - \sum_{j=0}^{k}(n_{j+1}-n_j)\ln r_j\right) =$$
$$= \frac{1}{n_{k+1}}\left(\sum_{j=1}^{k} n_j(\ln r_j - \ln r_{j-1}) + n_0 \ln r_0\right) \ge$$
$$\ge \frac{n_k(\ln r_k - \ln r_{k-1})}{n_{k+1}} = \frac{(1+o(1))n_k \ln r_k}{n_{k+1}} = 1 + o(1), \quad k \to \infty,$$

that is by Theorem 2 $\ln \mu_f(r) \in L_{RO}$.

Second,

$$M_f(r_k) \ge \sum_{n=n_k}^{n_{k+1}-1} a_n r_k^n = \sum_{n=n_k}^{n_{k+1}-1} a_{n_k} r_k^{n_k - n} r_k^n =$$
$$= (n_{k+1} - n_k)\mu_f(r_k) = (1+o(1))n_{k+1}\mu_f(r_k) =$$
$$= (1+o(1))\mu_f(r_k) n_k \ln r_k \ge (1+o(1))\mu_f(r_k)(n_k \ln r_k + \ln a_{n_k}) =$$
$$= (1+o(1))\mu_f(r_k)\ln \mu_f(r_k), \quad k \to \infty,$$

that is in (5) we cannot replace O by o. The proof of Theorem 5 is complete. □

Finally, we consider the Nevanlinna counting function. Let (n_k) be an increasing sequence of positive integers such that $|\lambda_{n_k}| < |\lambda_{n_k+1}| = \dots = |\lambda_{n_{k+1}}| < |\lambda_{n_{k+1}+1}|$. We consider an entire function $f^*(z) = a_0 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} 1/|\lambda_k|\right) z^n$. It is known (see, for example, [3]), that $N_f(r) = \ln \mu_{f^*}(r)$. Therefore, applying Theorem 4 to f^*, we obtain that $N_f(r) \in L^0$ if and only if

$$\ln |\lambda_{n_k}| - \frac{1}{n_k}\sum_{j=1}^{n_k} \ln |\lambda_j| \ge h > 0, \quad k \ge 1. \tag{6}$$

By Theorem 4 the functions $T(r,f)$, $\ln M_f(r)$ and $\ln \mu_f(r)$ belong or do not belong to L^0 simultaneously. $N_f(r)$ stands outside this chain. For example, for the entire function $f(z) = e^{e^z}\sin z$ we have $N_f(r) \in L^0$ and $\ln M_f(r) \notin L^0$. On the other hand, if a sequence (λ_k) is such that $\sum_{j=1}^{\infty}(1/|\lambda_k|) < +\infty$ and condition (6) is not valid then for the entire function $f(z) = = e^{z^2}\prod_{k=1}^{\infty}\left(1 - \frac{z}{\lambda_k}\right)$ we have $N_f(r) \notin L^0$ and $\ln M_f(r) \in L^0$.

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