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ON A CENTRAL EXPONENT OF ENTIRE DIRICHLET SERIES

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New estimates for the value of the exceptional set in the asymptotic equality of the central index and logarithmic derivative of the maximum modulus for entire Dirichlet series were found.

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Найдены новые оценки величины исключительного множества в асимптотитическом равенстве центрального показателя и логарифмической производной максимума модуля целого ряда Дирихле.

Let $S(\lambda)$ be the class of entire (absolutely convergent in the complex plane) Dirichlet series of the form

$$F(z) = \sum_{n=0}^{\infty} a_n e^{z\lambda_n}, \tag{1}$$

where $\lambda = (\lambda_n)$, $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leq n \uparrow +\infty)$.

Let us introduce some notations for $F \in S(\lambda)$ and $\sigma \in \mathbb{R}$: $M(\sigma, F) = \sup\{|F(\sigma + iy)| : y \in \mathbb{R}\}$ is the maximum of modulus, $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$ is the maximal term, $\Lambda(\sigma) = \Lambda(\sigma, F) = \max\{\lambda_n : |a_n|e^{\sigma\lambda_n} = \mu(\sigma, F)\}$ is the central exponent.

It is known ([1, p.145], [2]) that the function $\ln M(\sigma, F)$ is convex, that is why in the case $F \in S(\lambda)$ it has a nondecreasing derivative from the right, $L(\sigma, F) = (\ln M(\sigma, F))'_+$.

Let L be the class of positive continuous functions increasing to $+\infty$ on $[0; +\infty)$ and L_0 be the class of positive functions $V(t)$ nondecreasing on $[0; +\infty)$ such that

$$A = \int_0^{+\infty} \frac{dV(t)}{t} < +\infty\,;$$

Let L_1 be the class of continuous positive nondecreasing on $[0, +\infty)$ functions h such that

$$h(x + O(1)) = O(h(x)) \ (x \to +\infty);$$

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Let L_2 be the class of continuous positive nondecreasing on $[0;+\infty)$ functions h such that

$$h\left(x+\frac{1}{h(x)}\right)=O(h(x))\quad (x\to+\infty).$$

For $\varphi_0\in L$ let us denote by $S_{\varphi_0}(\lambda)$ the subclass of the class $S(\lambda)$ which contains the entire Dirichlet series such that

$$(\exists K=K_F>0):\ |a_n|\leq \exp\{-\lambda_n\varphi_0(K\lambda_n)\}\quad (n\geq n_0).$$

For $\Phi\in L$ we introduce the following notations:

$$S^+(\lambda,\Phi)=\{F\in S(\lambda):\ (\exists K>0)(\ln\mu(\sigma,F)\geq K\sigma\Phi(\sigma),\ \sigma\geq\sigma_0)\},$$

$$S^*(\lambda,\Phi)=\{F\in S(\lambda):\ (\exists K>0)(\ln\mu(\sigma,F)\geq K\sigma\Phi(\sigma),\ \sigma=\sigma_j\to+\infty)\}.$$

M. M. Sheremeta proved ([3], Theorem 1) that the condition

$$\sum_{n+1}^{+\infty}\frac{1}{n\lambda_n}<+\infty \tag{2}$$

is necessary and sufficient for the relation

$$L(\sigma,F)=(1+o(1))\Lambda(\sigma,F) \tag{3}$$

to hold when $\sigma\to+\infty$ for every function $F\in S(\lambda)$ outside a certain set $E\subset[0,+\infty)$ of finite Lebesgue measure. Besides, in [3, Theorem 2] it is proved that the conditions $F\in S_{\varphi_0}(\lambda)$, $\varphi_0\in L_{+\infty}$ and the condition

$$(\forall\eta>0):\ \lim_{t\to+\infty}\frac{1}{\varphi_0(\eta t)}\sum_{\lambda_n\leq t}\frac{1}{n\lambda_n}=0 \tag{4}$$

imply relation (3) when $\sigma\to+\infty$ outside a certain set $E\subset[0,+\infty)$ of zero linear density.

In this paper we will supplement these theorems in a part of description of the exceptional set. The following proposition is the basis of the proofs.

Lemma 1. *Let $F\in S(\lambda)$ and there exists a function $C(t)\nearrow+\infty$ $(t\to+\infty)$ such that for all $n\geq 0$ and for all $\sigma\in E$ the inequality*

$$|a_n|e^{\sigma\lambda_n}\leq\mu(\sigma,F)exp\left\{-\int_{\lambda_\nu}^{\lambda_n}\frac{\lambda_n-t}{t^2}C(t)\ln n(2t)dt\right\} \tag{5}$$

holds where $n(t)=\sum_{\lambda_n\leq t}1$ is the counting function of the sequence λ and $\nu=\nu(\sigma,F)=\max\{n:\ |a_n|\exp(\sigma\lambda_n)=\mu(\sigma,F)\}$ is the central index of the Dirichlet series. Then the relation

$$F'(z)=(1+o(1))\Lambda(\sigma,F)F(z) \tag{A}$$

holds when $\sigma\to+\infty$ $(\sigma\in E)$ for all z, $\operatorname{Re}z=\sigma$ such that $|F(z)|=(1+o(1))M(\sigma,F)$ as $\sigma\to+\infty$.

Actually, the proof of Lemma 1 constitutes the second part of the proof of Theorems 1 and 2 in [3], though nowhere this fact is mentioned explicitly. That is why the proof of Lemma 1 is a literal repeating of arguments from [3]. The first part of the proof of Theorems 1 and 2 [3] is the proof of inequality (5) outside exceptional sets (as a matter of fact, in [3] (5) is proved for the function $C(t) = 10l(2t)$, where $l(t) \nearrow +\infty$ $(t \to +\infty)$).

So, taking into account the facts mentioned above, we may consider Lemma 1 proved in [3].

Let h be a positive continuous nondecreasing function and $E \subset [0; +\infty)$ be a locally Lebesgue measurable set of finite measure meas $E = \int_E dx < +\infty$. Then define the h-density of E as

$$D_h(E) = \overline{\lim_{R\to+\infty}}\ h(R)\mathrm{meas}(E \cap [R, +\infty))$$

and by lower h-density of E the value

$$d_h(E) = \underline{\lim}_{R\to+\infty}\ h(R)\mathrm{meas}(E \cap [R, +\infty)).$$

In [4, Corollary 1] the following proposition is proved.

Lemma 2. [4] *Let $h \in L_2$, $\Phi \in L$. If $F \in S^+(\lambda, \Phi)$ and the condition*

$$(\forall b > 0): \ h(\varphi(bx)) \int\limits_x^{+\infty} \frac{\ln n(t)}{t^2} dt \to 0 \ (t \to +\infty) \tag{6}$$

holds then there exists a function $C(t)$ the same as in Lemma 1 such that inequality (5) is true for all $n \geq 0$ and $\sigma \in [0; +\infty) \setminus E$, $D_h(E) = 0$.

Actually, in [4] inequality (5) is proved by taking $n(4t)$ instead of $n(2t)$ but this strengthening is not essential.

Applying Lemmas 1 and 2 we may state that the following proposition is true.

Theorem 1. *Let $h \in L_2$, $\Phi \in L$. If $F \in S^+(\lambda, \Phi)$ and condition (6) holds then relation (A) is true when $\sigma \to +\infty$ outside a certain set E_1 of zero h-density ($D_h E_1 = 0$).*

In order to obtain the similar proposition in the class $S^*(\lambda, \Phi)$, we replace $D_h E_1 = 0$ by the lower h-density $d_h E_1 = 0$ in the formulation of Theorem 1.

Theorem 2. *Let $h \in L_1$, $\Phi \in L$. If $F \in S^*(\lambda, \Phi)$ and condition (6) holds then relation (A) is true when $\sigma \to +\infty$ outside a certain set E_1 of zero lower h–density ($d_h E_1 = 0$).*

In addition to Lemma 1 we need the following lemma in order to prove Theorem 2.

Lemma 3. *Let $h \in L_1$, $\Phi \in L$. If $F \in S^*(\lambda, \Phi)$ and condition (6) holds then there exists a function $C(t)$ the same as in Lemma 1, such that inequality (5) is true for all $n \geq 0$ and $\sigma \in [0; +\infty) \setminus E$, $d_h(E) = 0$.*

The assertion of Lemma 3 will be obtained from the following lemma. In order to prove it we will use a modification of the Wiman–Valiron method proposed by T. Kövari [16,17] and W. Hayman [14] and adapted for gap power series and entire Dirichlet series by M. M. Sheremeta [5,18,19].

Denote by $(R_j(F))_{j=0}^{+\infty}$ the sequence of the jump points of central index of entire Dirichlet series $F(z)$ numbered in such a way that $\nu(\sigma, F) = j$ when $R_j(F) \le \sigma < R_{j+1}(F)$ and if $\nu(R_{j+1}(F), F) = j + p$ then $R_{j+1}(F) = R_{j+2}(F) = \cdots = R_{j+p}(F) < R_{j+p+1}(F)$. Here we avoid the formal definition of normal and exceptional points $\sigma \in \mathbb{R}$ (see the definitions [5], [19, p.35]). However, we take into account the fact that the upper estimates of general term of a Dirichlet series $F \in S(\lambda)$ by the maximal term are obtained from the properties of the so called series of comparison. For a function $F \in S(\lambda)$ of form (1) and for a positive continuous nondecreasing on $[0, +\infty)$ function $\alpha(t)$ this series has the following form:

$$F_\alpha(z) = \sum_{n=0}^{+\infty} a_n \exp\left\{ z\lambda_n + \int_0^{\lambda_n} \alpha(t)dt \right\}.$$

The following simple statements presented in Lemma 4 are basic ones for describing the values of exceptional sets (i.e. the sets of those points σ for which the desired upper estimates of general term by the maximal term of the series may not hold). Remark that these statements were repeatedly used by different authors as an intermediate stage of the reflections (see the proof of Theorem 1 in [5], of Lemma 2 in [6], [7], of Theorems 3 and 4 in [3], of Lemmas 1 and 2 in [8], of Lemma 1 in [9], [10] etc.). We should also mention that similar statements occur in papers on power series by different authors ([11]–[15] etc.). Despite the brevity and elementarity of the proof, we include it for the sake of completeness of exposition.

Lemma 4. *If $F_\alpha \in S(\lambda)$ then for all $j \ge 0$ such that $R_j(F_\alpha) < R_{j+1}(F_\alpha)$ and for all $\sigma \in [R_j(F_\alpha) + \alpha(\lambda_j), R_{j+1}(F_\alpha) + \alpha(\lambda_j))$ and $n \ge 0$*

$$\frac{|a_n|e^{\sigma\lambda_n}}{|a_j|e^{\sigma\lambda_j}} \le \exp\left\{ -\int_{\lambda_j}^{\lambda_n} (\alpha(t) - \alpha(\lambda_j))dt \right\}, \tag{7}$$

moreover, $\nu(\sigma, F_\alpha) = j$.

Proof. For simplicity let us put $\alpha_n = \int_0^{\lambda_n} \alpha(t)dt$ and $R_n = R_n(F_\alpha)$ where $n \ge 0$. According to the definition of the maximal term $\mu(\sigma, F_\alpha)$, for $\sigma \in [R_j, R_{j+1})$ and for all $n \ge 0$ we have

$$|a_n|e^{\sigma\lambda_n + \alpha_n} \le \mu(\sigma, F_\alpha) = |a_j|e^{\sigma\lambda_j + \alpha_j}.$$

Then for $(\sigma - \alpha(\lambda_j)) \in [R_j, R_{j+1})$ and $n \ge 0$ we obtain

$$|a_n|e^{(\sigma - \alpha(\lambda_j))\lambda_n + \alpha_n} \le |a_j| \exp^{(\sigma - \alpha(\lambda_j))\lambda_j + \alpha_j}.$$

That is why for all $n \ge 0$ and $\sigma \in [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j))$ we have

$$\frac{|a_n|e^{\sigma\lambda_n}}{|a_j|e^{\sigma\lambda_j}} \le \exp\{-(\alpha_n - \alpha_j) + \alpha(\lambda_j)(\lambda_n - \lambda_j)\} =$$

$$= \exp\left\{ -\int_{\lambda_j}^{\lambda_n} \alpha(t)dt + \alpha(\lambda_j) \int_{\lambda_j}^{\lambda_n} dt \right\} = \exp\left\{ -\int_{\lambda_j}^{\lambda_n} (\alpha(t) - \alpha(\lambda_j))dt \right\}.$$

Thus, inequality (7) is proved.

Since for $\sigma \in (R_j, R_{j+1})$ and $n \neq j$

$$|a_n|e^{\sigma\lambda_n+\alpha_n} < \mu(\sigma, F_\alpha),$$

inequality (7) for $\sigma \in (R_j(F_\alpha)+\alpha(\lambda_j), R_{j+1}(F_\alpha)+\alpha(\lambda_j))$ when $n \neq j$ is also strict.

The function $\alpha(t)$ is nondecreasing on $[0, +\infty)$, hence, for arbitrary nonnegative integers n and m the inequality

$$\int_{\lambda_m}^{\lambda_n} (\alpha(t) - \alpha(\lambda_m))dt > 0$$

holds when $n \neq m$. Together with inequality (7) for $\sigma \in (R_j(F_\alpha)+\alpha(\lambda_j), R_{j+1}(F_\alpha)+\alpha(\lambda_j))$ in case $R_j(F_\alpha) < R_{j+1}(F_\alpha)$ it implies

$$\frac{|a_n|e^{\sigma\lambda_n}}{|a_j|e^{\sigma\lambda_j}} < 1$$

for all $n \neq j$, i.e. $\nu(\sigma, F) = j$. Thus, the second statement of Lemma 4 is proved, because $\nu(\sigma, F)$ is continuous from the right. □

Lemma 5. *Let $V \in L_0$, $h \in L_1$, $\Phi \in L_\infty$. Suppose that $F \in S^*(\lambda, \Phi)$ and the condition*

$$(\forall b > 0) : \lim_{n\to+\infty} h\left(\varphi(b\lambda_n)\right) \int_{\lambda_n}^{+\infty} \frac{dV(t)}{t} = 0 \tag{8}$$

holds. Then for all $n \geq 0$ and $\sigma \in [0, +\infty)$ the inequality

$$|a_n|e^{\sigma\lambda_n} \leq \mu(\sigma, F) \exp\left\{-\int_{\lambda_\nu}^{\lambda_n} \frac{\lambda_n - t}{t} dV(t)\right\}, \tag{9}$$

where $\nu = \nu(\sigma, F)$ and φ is a function, inverse to Φ, holds outside a set E_1 of zero lower h-density ($d_h E_1 = 0$).

Proof. Let $\alpha(t) = \int_0^t \frac{dV(x)}{x}$. Consider the Dirichlet series $F_\alpha(z)$. Since $V \in L_0$,

$$A \stackrel{\text{def}}{=} \int_0^{+\infty} \frac{dV(t)}{t} < +\infty$$

and for the general term of series $F_\alpha(z)$ we have

$$|a_n| \exp\left\{\sigma\lambda_n + \int_0^{\lambda_n} \alpha(t)dt\right\} \leq$$
$$\leq |a_n| \exp\{(\sigma + \alpha(\lambda_n))\lambda_n\} \leq |a_n| \exp\{(\sigma + A)\lambda_n\},$$

i.e. $F_\alpha \in H(\Lambda)$.

Let $R_n = R_n(F_\alpha)$ where $n \geq 0$. Let us apply Lemma 4 to the function F_α. Then, since

$$\int_{\lambda_j}^{\lambda_n} (\alpha(t) - \alpha(\lambda_j))dt = \int_{\lambda_j}^{\lambda_n} \frac{\lambda_n - t}{t} dV(t),$$

inequality (7) implies

$$\frac{|a_n|e^{\sigma\lambda_n}}{|a_j|e^{\sigma\lambda_j}} \leq \exp\left\{-\int_{\lambda_j}^{\lambda_n} \frac{\lambda_n - t}{t} dV(t)\right\}$$

for all $\sigma \in [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j)]$ and $n \geq 0$. According to the second statement of Lemma 4, the equality $\nu(\sigma, F) = j$ holds for all $\sigma \in [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j))$. Hence, inequality (9) is true for all $n \geq 0$ and $\sigma \in E$ where

$$E \stackrel{\text{def}}{=} \bigcup_{j=1}^{+\infty} [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j)).$$

Let $E_1 = [R_1 + \alpha(\lambda_1); +\infty) \setminus E$. Now we prove that $d_h E_1 = 0$. Remark that if $R_j < R_{j+1} = \cdots = R_{j+p} < R_{j+p+1}$ $(p \geq 1)$ and $\sigma \in [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j))$ then we obtain the following relation for the measure of the set $E_1 \cap [\sigma, +\infty)$:

$$\text{meas}(E_1 \cap [\sigma, +\infty)) = \sum_{k=j}^{+\infty} ((R_{k+1} + \alpha(\lambda_{k+1})) - (R_{k+1} + \alpha(\lambda_k))) =$$

$$= \sum_{k=j}^{+\infty} (\alpha(\lambda_{k+1}) - \alpha(\lambda_k)) = \sum_{k=j}^{+\infty} \int_{\lambda_k}^{\lambda_{k+1}} \frac{dV(t)}{t} = \int_{\lambda_j}^{+\infty} \frac{dV(t)}{t}. \tag{10}$$

If $\sigma \in [R_{j+1} + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_{j+p}))$ then

$$\text{meas}(E_1 \cap [\sigma, +\infty)) \leq \text{meas}(E \cap [R_{j+1} + \alpha(\lambda_j), +\infty)) =$$
$$= \sum_{k=j}^{+\infty} (\alpha(\lambda_{k+1}) - \alpha(\lambda_k)) = \int_{\lambda_j}^{+\infty} \frac{dV(t)}{t}.$$

We have $F \in S^*(\lambda, \Phi)$, thus, there exist a number $K = K_F > 0$ and a sequence $(\sigma_j)_{j=0}^{+\infty}$ increasing to $+\infty$ when $j \to +\infty$ such that

$$K\sigma_j \Phi(\sigma_j) \leq \ln \mu(\sigma_j, F) \qquad (j \geq 1).$$

Note that for all $\sigma \in \mathbb{R}$

$$\ln \mu(\sigma, F_\alpha) = \max\{\ln |a_n| + \int_0^{\lambda_n} \alpha(t)dt + \sigma\lambda_n\} \geq$$
$$\geq \max\{\ln |a_n| + \sigma\lambda_n\} = \ln \mu(\sigma, F).$$

Hence, $F_\alpha \in S^*(\lambda, \Phi)$ and

$$K\sigma_j \Phi(\sigma_j) \leq \ln \mu(\sigma_j, F_\alpha) \qquad (j \geq 1).$$

Since

$$\ln \mu(\sigma, F_\alpha) = \ln \mu(0, F_\alpha) + \int_0^\sigma \lambda_{\nu(t, F_\alpha)} dt \leq 2\sigma\lambda_{\nu(\sigma-0, F_\alpha)},$$

we have

$$\sigma_j \leq \varphi\left(\frac{2}{K}\lambda_{\nu(\sigma_j-0, F_\alpha)}\right) \tag{11}$$

with $j \geq 0$ where $\varphi(t)$ is the function inverse to $\Phi(t)$.

Let the sequence $(\sigma_j^*)_{j=0}^{+\infty}$ be defined by the equality

$$\sigma_j^* = \sigma_j + \alpha(\lambda_{n_j}),$$

where n_j is the number of the interval $[R_n, R_{n+1})$ which contains σ_j. Then $R_{n_j} < R_{n_j+1}$ and $\sigma_j^* \in [R_{n_j} + \alpha(\lambda_{n_j}), R_{n_j+1} + \alpha(\lambda_{n_j}))$. On account of inequality (11) we obtain

$$h(\sigma_j^*) = h(\sigma_j + \alpha(\lambda_{n_j})) \le$$
$$\le h\left(\varphi\left(\frac{2}{K}\lambda_{\nu(\sigma_j-0,F_\alpha)}\right) + \int_0^{\lambda_{n_j}} \frac{dV(x)}{x}\right) \le h\left(\varphi\left(\frac{2\lambda_{n_j}}{K}\right) + A\right),$$

because $\nu(\sigma - 0, F_\alpha) \le j$ with $\sigma \in [R_j, R_{j+1})$.

Taking into account (10) and the assumptions of lemma $h \in L_1$ and (8), we conclude

$$h(\sigma_j^*)\mathrm{meas}(E_1 \cap [\sigma_j^*, +\infty)) = o(1) \qquad (j \to +\infty),$$

i.e. $d_h E_1 = 0$ and this completes the proof of Lemma 5. □

Proof of Lemma 3. Let $\beta(t)$ be a nonnegative function satisfying the condition

$$(\forall b > 0) : h(\varphi(bx)) \int_x^{+\infty} \frac{\beta(t)}{t^2} dt \to 0 \qquad (x \to +\infty), \tag{12}$$

i.e. (12) implies condition (8) for $\beta(t) = \ln n(t)$.

Let

$$l(x) = \int_x^{+\infty} t^{-2}\beta(4t)dt, \qquad l_1(x) = h(\varphi(bx))l(x).$$

Then according to (12),

$$l_1(x) = 4h(\varphi(bx)) \int_{4x}^{+\infty} t^{-2}\beta(t)dt = o(1) \qquad (x \to +\infty)$$

and, therefore, the function $C(x)$ defined by the equality

$$C(x) \overset{\mathrm{def}}{=} (\max\{l_1(t) : t \ge x\})^{-\frac{1}{2}},$$

is nondecreasing and $C(x) \nearrow +\infty$ when $x \to +\infty$.

Remark that

$$h(\varphi(b\lambda_n)) \int_{\lambda_n}^{+\infty} t^{-2}C(t)\beta(4t)dt \le h(\varphi(b\lambda_n)) \int_{\lambda_n}^{+\infty} \frac{\beta(4t)}{t^2(h(\varphi(bt))l(t))^{\frac{1}{2}}} dt \le$$
$$\le -\sqrt{h(\varphi(b\lambda_n))} \int_{\lambda_n}^{+\infty} \frac{dl(t)}{\sqrt{l(t)}} = 2\sqrt{l_1(\lambda_n)} = o(1)$$

when $n \to +\infty$. Hence, for

$$V(x) = \int_0^x t^{-1}C(t)\beta(4t)dt,$$

we have

$$h(\varphi(b\lambda_n)) \int_{\lambda_n}^{+\infty} t^{-2}C(t)\ln n(4t)dt = o(1) \qquad (n \to +\infty).$$

This yields that the function

$$V(x) = \int_0^x t^{-1} C(t) \ln n(4t) dt$$

satisfies the assumptions of Lemma 5 the applying of which completes the proof of Lemma 3. □

Applying Lemmas 1 and 3 we immediately state that the assertion of Theorem 2 is true.

In order to derive the statement on relation (3) from Theorem 1, we will obtain the statement on the following relation:

$$F'(z) = (1 + o(1)) L(\sigma, F) F(z) . \tag{B}$$

The proof will repeat the scheme of the proof of Theorem 1.3.17 from [1].

Theorem 3. *Let $\Phi \in L$, $h \in L_2$ be such that $h(r) = o(\Phi(r))$ $(r \to +\infty)$. If $F \in S(\lambda)$ and $L(\sigma, F) \geq \Phi(\sigma)$ $(\sigma \geq \sigma_0)$ then relation (B) holds when $\sigma \to +\infty$ $(\sigma \notin E, D_h(E) = 0)$ for all z, $\operatorname{Re} z = \sigma$ such that $|F(z)| = (1 + o(1)) M(\sigma, F)$ with $\sigma \to +\infty$.*

In order to prove Theorem 3 we will use the following version of the Borel–Nevanlinna lemma.

Lemma 6. *Let a semicontinuous from the right, increasing on $[r_0, +\infty)$ function $u(r)$ and functions $\Phi \in L$, $h \in L_2$ be such that*

$$u(r) \geq \Phi(r) \, (r \geq r_0), \quad h(r) = o(\Phi(r)) \quad (r \to +\infty).$$

Then there exists a function $\delta(u) \uparrow +\infty$ $(u \to +\infty)$ such that the set

$$E = \left\{ r \geq r_0 : \, u\left(r + \frac{\delta(u(r))}{u(r)}\right) \geq \left(1 + \frac{1}{\delta(u(r))}\right) u(r) \right\}$$

has zero h-density, i.e. $D_h(E) = 0$.

Proof. Let us start with proving that for all $\delta > 0$ the set

$$E(\delta) = \left\{ r \geq r_0 : \, u\left(r + \frac{\delta}{u(r)}\right) \geq \left(1 + \frac{1}{\delta}\right) u(r) \right\}$$

is of zero h–density. There is no loss of generality in assuming that the set $E(\delta)$ is unbounded. In the opposite case the assertion of the lemma is trivial. Let $E(\delta, r) = E(\delta) \cap [r; +\infty)$, $r_1 = \inf\{r : \, r \in E(\delta, r_0)\}$ and $r_1' = r_1 + \delta/u(r_1)$. Suppose that $r_1, ..., r_n$ and $r_1', ..., r_n'$ are defined. Let us define $r_{n+1} = \inf\{r : \, r \in E(\delta, r_n')\}$, $r_{n+1}' = r_{n+1} + \delta/u(r_{n+1})$. It is clear that (see [20, p.67])

$$E(\delta) \subset \bigcup_{n=1}^{+\infty} [r_n, r_n'].$$

Now remark that

$$u(r_{n+1}) \geq u(r_n') = u(r_n + \delta/u(r_n)) \geq (1 + 1/\delta) u(r).$$

This implies $r_n \uparrow +\infty$ $(n \to +\infty)$ and

$$\delta\frac{u(r_{n+1})-u(r_n)}{u(r_n)} \geq 1.$$

Therefore, for $r \in [r_n'; r_{n+1}]$ we obtain

$$\text{meas}E(\delta, r) \leq \sum_{k=n+1}^{+\infty} (r_k' - r_k) \leq \frac{\delta}{u(r_{n+1})} + \delta \sum_{k=n+2}^{+\infty} \frac{u(r_k)-u(r_{k-1})}{u(r_k)u(r_{k-1})} =$$

$$= \frac{\delta}{u(r_{n+1})} + \delta \sum_{k=n+2}^{+\infty} \Big(\frac{1}{u(r_{k-1})} - \frac{1}{u(r_k)}\Big) = \frac{2\delta}{u(r_{n+1})} \leq \frac{2\delta}{\Phi(r_{n+1})}.$$

Similarly, when $r \in [r_n, r_n']$ we get

$$\text{meas}E(\delta, r) \leq \sum_{k=n}^{+\infty} (r_k' - r_k) \leq \frac{\delta}{u(r_n)} + \delta \sum_{k=n+1}^{+\infty} \frac{u(r_k)-u(r_{k-1})}{u(r_k)u(r_{k-1})} =$$

$$= \frac{2\delta}{u(r_n)} \leq \frac{2\delta}{\Phi(r_n)}.$$

Now it remains to note that $h(r) \leq h(r_{n+1})$ $(r \leq r_{n+1})$ and

$$h(r) \leq h(r_n + r_n' - r_n) = h\left(r_n + \frac{\delta}{u(r_n)}\right) = h\left(r_n + o\left(\frac{1}{h(r_n)}\right)\right) = O(h(r_n))$$

when $r \leq r_n'$, $n \to +\infty$. Here we successively applied the conditions $u(r) \geq \Phi(r)$, $h(r) = o(\Phi(r))$ $(r \to +\infty)$ and $h \in L_2$.

Thus, we conclude that for all $r \in [r_n; r_{n+1}]$

$$h(r)\text{meas}E(\delta, r) = O\left(\max\left\{\frac{h(r_n)}{\Phi(r_n)};\ \frac{h(r_{n+1})}{\Phi(r_{n+1})}\right\}\right) = o(1) \quad (n \to +\infty),$$

i.e. $D_h(E(\delta)) = 0$.

Now let $\delta_n = n$ and let R_n be a sequence increasing to $+\infty$ such that for $r \geq R_n$

$$h(r)\text{meas}E(\delta_n, r) \leq \frac{1}{n^2}.$$

Let $\delta(u(r)) = n$ for $r \in [R_n; R_{n+1}]$, $E_0 = \cup_{n=1}^{+\infty}(E(\delta_n, R_n) \cap [R_n; R_{n+1}))$. Then from what has been already proved above it follows that for all $r \in [R_n; R_{n+1}) \setminus E_0$

$$u\left(r + \frac{\delta(u(r))}{u(r)}\right) = u\left(r + \frac{\delta_n}{u(r)}\right) < \left(1 + \frac{1}{\delta_n}\right)u(r) = \left(1 + \frac{1}{\delta(u(r))}\right)u(r).$$

Thus, for the set E defined in the formulation of Lemma 6 we have $E \subset E_0$. Then for $r \in [R_n, R_{n+1})$ with $n \to +\infty$ we obtain

$$h(r)\text{meas}\,(E_0 \cap [r; +\infty)) \leq h(r)\text{meas}E(\delta_n, r) +$$

$$+ \sum_{k=n+1}^{+\infty} \frac{h(r)}{h(R_k)} h(R_k)\text{meas}E(\delta_k, R_k) \leq \sum_{k=n}^{+\infty} \frac{1}{n^2} = o(1),$$

i.e. $D_h(E_0) = 0$. This is the desired conclusion. □

Proof of Theorem 3. Applying Lemma 6 to the function $u(r) = L(r, F)$ we deduce that for all $\sigma \in [0; +\infty) \setminus E$ $(D_h(E) = 0)$ the relation

$$L\left(\sigma + \frac{\delta(\sigma)}{L(\sigma, F)}, F\right) < \left(1 + \frac{1}{\delta(\sigma)}\right) L(\sigma, F)$$

holds where $\delta(\sigma) = \delta(L(\sigma, F))$. From what is written above it follows that for all $|\tau| \leq \psi(\sigma) = \delta(\sigma)/L(\sigma, F)$ and $\sigma \in [0; +\infty) \setminus E$

$$|L(\sigma + \tau, F) - L(\sigma, F)| < L(\sigma, F)/\delta(\sigma).$$

Let $\varepsilon(\sigma) \to +0\,(\sigma \to +\infty)$ and let $w = \sigma + it$ be an arbitrary point such that $|F(w)| \geq M(\sigma, F)(1 + \varepsilon(\sigma))^{-1}$. From the convexity of $\ln M(\sigma, F)$ it follows that for all $\{\sigma, h\} \subset \mathbb{R}$

$$\ln M(\sigma + h, F) - \ln M(\sigma, F) \leq hL(\sigma + h, F),$$

thus,

$$\ln M(\sigma + h, F) - \ln M(\sigma, F) - hL(\sigma, F) \leq |h||L(\sigma + h, F) - L(\sigma, F)|.$$

Hence, for all $|\tau| \leq \psi(\sigma)$ and $\sigma \in [0; +\infty) \setminus E$

$$\ln M(\sigma + \tau, F) - \ln M(\sigma, F) - \tau L(\sigma, F) \leq 1.$$

Therefore, for all $\sigma \in [0; +\infty) \setminus E$ and $\eta \in \mathbb{C}$, $|\operatorname{Re} \eta| \leq \psi(\sigma)$

$$\left|\frac{F(w + \eta)}{F(w)} e^{-\eta L(\sigma, F)}\right| \leq$$

$$\leq (1 + \varepsilon(\sigma)) \exp\{\ln M(\sigma + \operatorname{Re} \eta, F) - \ln M(\sigma, F) - \operatorname{Re} \eta L(\sigma, F)\} \leq (1 + \varepsilon(\sigma))e.$$

Obviously, this implies that the same inequality holds for all $\eta \in \mathbb{C}$, $|\eta| \leq \psi(\sigma)$, $\sigma \notin E$, because $\{\eta \in \mathbb{C} : |\eta| \leq a\} \subset \{\eta \in \mathbb{C} : |\operatorname{Re} \eta| \leq a\}$. Now let us consider the function

$$g(\eta) = \frac{F(w + \eta)}{F(w)} e^{-\eta L(\sigma, F)} - 1,$$

where $|\eta| < \psi(\sigma)$, $\sigma \notin E$. According to Schwartz' lemma, for all $|\eta| < \psi(\sigma)$

$$|g(\eta)| \leq (1 + e(1 + \varepsilon(\sigma))) \frac{|\eta|}{\psi(\sigma)} = c(\sigma) \frac{|\eta|}{\psi(\sigma)}.$$

This yields

$$\left|\frac{F(w + \eta)}{F(w)} e^{-\eta L(\sigma, F)}\right| \geq 1 - |g(\eta)| > 0,$$

where $|\eta| < \psi(\sigma)/c(\sigma)$, $\sigma \notin E$, i.e. $|F(w + \eta)| > 0$ for $|\eta| < \psi(\sigma)/c(\sigma)$.

Consider the function

$$G(\eta) = \int_0^\eta \frac{F'(w + \tau)}{F(w + \tau)} d\tau - \eta L(\sigma, F), \quad G(0) = 0,$$

analytic in the disc $\{\eta : |\eta| < \psi(\sigma)/c(\sigma)\}$. Remark that $G'(0) = \frac{F'(w)}{F(w)} - L(\sigma, F)$ and for all $|\eta| \leq q < \psi(s)/c(\sigma)$

$$\operatorname{Re} G(\eta) = \ln \left|\frac{F(w + \eta)}{F(w)} e^{-\eta L(\sigma, F)}\right| = \ln |1 + g(\eta)| \leq$$

$$\leq \ln(1+|g(\eta)|) \leq \ln\left(1+\frac{qc(\sigma)}{\psi(\sigma)}\right).$$

By the modified Cauchy inequality,

$$|G'(0)|q \leq 2\sup\{\operatorname{Re} G(\eta):\ |\eta|\leq q\} \leq 2\ln\left(1+\frac{qc(\sigma)}{\psi(\sigma)}\right).$$

Therefore, with $\sigma\to+\infty$ $(\sigma\notin E)$

$$\left|\frac{F'(w)}{F(w)}\frac{1}{L(\sigma,F)}-1\right| \leq \frac{2}{qL(\sigma,F)}\ln\left(1+\frac{qc(\sigma)}{\psi(\sigma)}\right) \leq \frac{2c(\sigma)}{L(\sigma,F)\psi(\sigma)} = \frac{c(\sigma)}{\delta(L(\sigma,F))} = o(1)$$

for all $w=\sigma+it$ such that $|F(w)|\geq M(\sigma,F)(1+\varepsilon(\sigma))^{-1}$. The proof of Theorem 3 is complete. □

From Theorems 1 and 3 we derive the following proposition.

Theorem 4. *Suppose that the assumptions of Theorem 1 hold. Then relation (3) is true when* $\sigma\to+\infty\,(\sigma\notin E, D_h(E)=0)$.

Proof. It is sufficient to prove that the assumptions of Theorem 3 follow from those of Theorem 1. Note that

$$\sigma\Phi(\sigma)\leq \ln\mu(\sigma,F)\leq \ln M(\sigma,F) = \ln M(\sigma_0,F)+\int_{\sigma_0}^{\sigma} L(t,F)dt \leq$$

$$\leq \ln M(\sigma_0,F)+(\sigma-\sigma_0)L(\sigma,F).$$

Now if $\sigma_1>\sigma_0$ is such that $\ln M(\sigma_0,F)-\sigma_0 L(\sigma_1,F)\leq 0$ then for $\sigma\geq\sigma_1$ we have $L(\sigma,F)\geq \Phi(\sigma)$.

Further, condition (6) implies

$$\frac{h(r)\ln n(\Phi(r))}{\Phi(r)} \leq h(r)\int_{\Phi(r)}^{+\infty}\frac{\ln n(t)}{t^2} = o(1)$$

when $r\to+\infty$. Hence, $h(r)=o(\Phi(r))\,(r\to+\infty)$, i.e. the assumptions of Theorem 3 hold. It remains to note that if E_1, E_2 are two sets such that $D_h(E_j)=0$, $j\in\{1,2\}$ then $D_h(E_1\cup E_2)=0$. Theorem 4 is proved. □

Remark. Analyzing the proofs of Theorems 2 and 3 it is easy to notice that the proposition similar to Theorem 4 can be obtained under the conditions of Theorem 2. Moreover, it is clear that the only estimate that may be obtained for the exceptional set is $d_h(E)=0$.

REFERENCES

1. Стрелиц Ш. И. Асимптотические свойства аналитических решений дифференциальных уравнений. – Вильнюс: Минтис, 1972.–468 с.

2. Doetsch G. *Über die obere Grenze des absoluten Betrages einer analytischen Function auf Geraden* // Math. Zeitschr. – 1920. – Bd.8. – S.237–240.

3. Шеремета М.Н. *О производной целого ряда Дирихле* // Мат. сб. – 1988. – Т.137(179), №1(9). – С.128–139.

4. Скасків О.Б., Сало Т.М. *Цілі ряди Діріхле швидкого зростання і нові оцінки міри виняткових множин в теоремах типу Вімана-Валірона* // Укр. мат. журн. – 2001. – Т.53, №6. – С.830–839.

5. Шеремета М.Н. *Метод Вимана-Валирона для рядов Дирихле* // Укр. мат. журн. – 1978. – Т.30, №4. – С.488–497.

6. Скаскив О.Б. *О поведении максимального члена ряда Дирихле, задающего целую функцию* // Мат. заметки. – 1985. – Т.37, №1. – С.41–47.

7. Скаскив О.Б., Шеремета М.Н. *Об асимптотическом поведении целых рядов Дирихле* // Мат. сб. – 1986. – Т.131(173), №3(11).–С.385–402.

8. Шеремета М.Н. *О поведении максимума модуля целого ряда Дирихле вне исключительного множества* // Мат. заметки. – 1995. – Т.57, №2. – С.283–296.

9. Шеремета М.Н. *Об эквивалентности логарифмов максимума модуля и максимального члена целого ряда Дирихле* // Мат. заметки. – 1987. – Т.42, №2. – С.215–226.

10. Хомяк М.М. *Метод Вимана-Валирона для целых функций, заданных рядами Дирихле, с условием на рост на некоторой последовательности* // Укр. мат. журн. – 1983. – Т.35, №4. – С.527–533.

11. Fenton P.C. *Some results of Wiman-Valiron type for integral functions of finite lower order* // Ann. Math. – 1976. – V.103, №2. – P.237–252.

12. Fenton P.C. *Wiman-Valiron theory for entire functions of finite lower growth* // Trans. Amer. Math. Soc. – 1979. – V.252. – P.221–232.

13. Fuchs W.H.J. *Asymptotic evalution of integrals and Wiman-Valiron theory* // Complex anal. and appl. Int. semin. course. Trieste, 1975. – V.1. – Vienna: 1976. – P.235–283.

14. Hayman W.K. *The local growth of power series: a survey of the Wiman-Valiron method* // Canad. Math. Bull. – 1974. – V.17, №3. – P.317–358.

15. Skaskiv O.B. *On the Polya conjecture concerning the maximum and minimum of the modulus of an entire functions of finite order given by a lacunary power series* // Anal. Math. – 1990. – V.16, №2. – P.143–157.

16. Kövari T. *On the Borel exeptional values of lacunary integral functions* // J. Anal. Math. – 1961. – V.9. – P.71–109.

17. Kövari T. *A gap-theorem for entire functions of infinite order* // Michigan Math. J. – 1965. – V.12. – P.133–140.

18. Шеремета М.Н. *Аналоги теоремы Вимана для рядов Дирихле* // Мат. сб. – 1979. – Т.110, №1. – С.102–116.

19. Шеремета М.М. Цілі ряди Діріхле. – К.: ІСДО, 1993.–168 с.

20. Хейман У.К. Мероморфные функции. – М.: Мир, 1968.–288 с.

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