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ON THE REMAINDER OF DIRICHLET SERIES

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For Dirichlet series $\sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ with nonnegative exponents an asymptotic behaviour of the remainder $\sum_{k=n}^{\infty} |a_k| \exp\{\sigma\lambda_k\}$ as $n \to \infty$ is investigated.

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Для ряда Дирихле $\sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ с неотрицательными показателями исследовано асимптотическое поведение при $n \to \infty$ остатка $\sum_{k=n}^{\infty} |a_k| \exp\{\sigma\lambda_k\}$.

1. Let (λ_n) be an increasing to $+\infty$ sequence of nonnegative numbers and a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\},\ s = \sigma + it, \tag{1}$$

has an abscissa of absolute convergence $\sigma_a \in (-\infty, +\infty]$.

In the note we investigate an asymptotic behaviour of the remainder

$$R_n(F,\sigma) := \sum_{k=n}^{\infty} |a_k| \exp\{\sigma\lambda_k\}$$

for fixed $\sigma < \sigma_a$ and as $n \to \infty$. We put $\varkappa_n = \dfrac{\ln\,|a_n| - \ln\,|a_{n+1}|}{\lambda_{n+1} - \lambda_n}$.

2. We begin from the case $\sigma_a = 0$ and prove the following

Lemma 1. *Let $\sigma_a = 0$, $|a_n| \to \infty\,(n \to \infty)$, $\varkappa_n \to 0\,(n \to \infty)$ and $\lambda_{n+1} - \lambda_n \geq 1$ for all $n \geq n_0$. Then for every $\sigma < 0$*

$$1 \leq \varliminf_{n\to\infty} \frac{R_n(F,\sigma)e^{|\sigma|\lambda_n}}{|a_n|} \leq \varlimsup_{n\to\infty} \frac{R_n(F,\sigma)e^{|\sigma|\lambda_n}}{|a_n|} \leq 1 + \frac{1}{e^{|\sigma|} - 1}. \tag{2}$$

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Proof. Since $R_n(F,\sigma)\ge |a_n|e^{\sigma\lambda_n}$, we have left inequality (2). On the other hand,

$$R_n(F,\sigma)=|a_n|e^{\sigma\lambda_n}\left(1+\sum_{k=n+1}^{\infty}\exp\{\ln|a_k|-\ln|a_n|+\sigma(\lambda_k-\lambda_n)\}\right)=$$
$$=|a_n|e^{\sigma\lambda_n}\left(1+\sum_{k=n+1}^{\infty}\exp\left\{-|\sigma|(\lambda_k-\lambda_n)\left(1+\frac{\ln|a_k|-\ln|a_n|}{-|\sigma|(\lambda_k-\lambda_n)}\right)\right\}\right)=$$
$$=|a_n|e^{\sigma\lambda_n}\left(1+\sum_{k=n+1}^{\infty}\exp\left\{-|\sigma|(\lambda_k-\lambda_n)\left(1+o(1)\right)\right\}\right)\le$$
$$\le|a_n|e^{\sigma\lambda_n}\left(1+\sum_{k=n+1}^{\infty}\exp\left\{-|\sigma|(k-n)\left(1+o(1)\right)\right\}\right)=$$
$$=|a_n|e^{\sigma\lambda_n}\left(1+\frac{1}{\exp\{(1+o(1)|\sigma|\}-1}\right),\quad n\to\infty.$$

Hence right inequality (2) follows. □

Now, we put $\Delta n(t):=\sum_{t\le\lambda_n<t+1}1$ for $t\ge 0$.

Theorem 1. *Let $\sigma_a=0$, $|a_n|\to\infty$ and $\varkappa_n\to 0$ as $n\to\infty$. Then for every fixed $\sigma<0$*

$$\overline{\lim_{n\to\infty}}\frac{\ln\left(R_n(F,\sigma)e^{|\sigma|\lambda_n}\right)}{\ln|a_n|}=1+\overline{\lim_{n\to\infty}}\frac{\ln\Delta n(\lambda_n)}{\ln|a_n|}\tag{3}$$

and

$$\underline{\lim_{n\to\infty}}\frac{\ln\left(R_n(F,\sigma)e^{|\sigma|\lambda_n}\right)}{\ln|a_n|}\ge 1+\underline{\lim_{n\to\infty}}\frac{\ln\Delta n(\lambda_n)}{\ln|a_n|}.\tag{4}$$

Proof. Let $\varepsilon>0$ be an arbitrary number and $n_0=n_0(\varepsilon)$ be such that $|a_n|\ge e$ and

$$|\ln|a_n|-\ln|a_{n+1}||<\varepsilon(\lambda_{n+1}-\lambda_n)\ \text{ for }\ n\ge n_0.$$

Then

$$|\ln|a_n|-\ln|a_p||<\varepsilon|\lambda_n-\lambda_p|\ \text{ for }\ n\ge n_0,\ p\ge n_0,$$

and if $|\lambda_n-\lambda_p|<1$ then

$$|a_n|^{1-\varepsilon}\le|a_p|\le|a_n|^{1+\varepsilon},\quad n\ge n_0,\ p\ge n_0.\tag{5}$$

Therefore,

$$R_n(F,\sigma)\ge\sum_{\lambda_n\le\lambda_p<\lambda_n+1}|a_p|e^{\sigma\lambda_p}\ge|a_n|^{1-\varepsilon}e^{\sigma(\lambda_n+1)}\Delta n(\lambda_n),\quad n\ge n_0,$$

and

$$\frac{\ln\left(R_n(F,\sigma)e^{|\sigma|\lambda_n}\right)}{\ln|a_n|}\ge\frac{\ln\Delta n(\lambda_n)}{\ln|a_n|}+1-\varepsilon+\frac{\sigma}{\ln|a_n|},$$

whence in view of arbitrariness of ε we obtain (4) and

$$\overline{\lim_{n\to\infty}}\frac{\ln\left(R_n(F,\sigma)e^{|\sigma|\lambda_n}\right)}{\ln|a_n|}\ge 1+\overline{\lim_{n\to\infty}}\frac{\ln\Delta n(\lambda_n)}{\ln|a_n|}.\tag{6}$$

Now, let

$$\varlimsup_{n\to\infty}\frac{\ln\,\Delta n(\lambda_n)}{\ln\,|a_n|}=\tau<+\infty.$$

We choose a subsequence (λ_{n_k}) of sequence (λ_n) such that the intervals $[\lambda_{n_k},\lambda_{n_k}+1)$ are mutually disjoint and the set $\{\lambda_n\}$ is contained in $\bigcup_k[\lambda_{n_k},\lambda_{n_k}+1)$. For $\varepsilon>0$ let $n_0=n_0(\varepsilon)$ be such that (5) holds provided $|\lambda_n-\lambda_p|<1$ and $\dfrac{\ln\,\Delta n(\lambda_n)}{\ln\,|a_n|}\le\tau+\varepsilon$. Then

$$R_{n_k}(F,\sigma)=\sum_{j=k}^{\infty}\sum_{\lambda_{n_j}\le\lambda_p<\lambda_{n_j}+1}|a_p|e^{\sigma\lambda_p}\le\sum_{j=k}^{\infty}|a_{n_j}|^{1+\varepsilon}e^{\sigma\lambda_{n_j}}\Delta n(\lambda_{n_j})\le\sum_{j=k}^{\infty}|a_{n_j}|^{1+\tau+2\varepsilon}e^{\sigma\lambda_{n_j}}.$$

Since $\dfrac{\ln\,|a_{n_k}|-\ln\,|a_{n_{k+1}}|}{\lambda_{n_{k+1}}-\lambda_{n_k}}\to 0$, $k\to\infty$, and $\lambda_{n_{k+1}}-\lambda_{n_k}>1$, by Lemma 1 in view of arbitrariness of ε we have

$$\varlimsup_{k\to\infty}\frac{\ln\left(R_{n_k}(F,\sigma)\exp\{|\sigma|\lambda_{n_k}\}\right)}{\ln\,|a_{n_k}|}\le\tau+1. \tag{7}$$

Let $n\in\mathbb{N}$ be an arbitrary number and $\lambda_{n_k}\le\lambda_n<\lambda_{n_k}+1$ for some k. Then

$$R_n(F,\sigma)e^{|\sigma|\lambda_n}\le e^{|\sigma|}e^{|\sigma|\lambda_{n_k}}\sum_{j=n_k}^{\infty}|a_j|e^{\sigma\lambda_j}=e^{|\sigma|}R_{n_k}(F,\sigma)e^{|\sigma|\lambda_{n_k}}$$

and

$$\frac{\ln\left(R_n(F,\sigma)\exp\{|\sigma|\lambda_n\}\right)}{\ln\,|a_n|}\le\frac{|\sigma|}{\ln\,|a_n|}+\frac{\ln\left(R_{n_k}(F,\sigma)\exp\{|\sigma|\lambda_{n_k}\}\right)}{\ln\,|a_{n_k}|}\frac{\ln\,|a_{n_k}|}{\ln\,|a_n|}.$$

Hence in view of (5), (7) and arbitrariness of ε we obtain

$$\varlimsup_{n\to\infty}\frac{\ln\left(R_n(F,\sigma)\exp\{|\sigma|\lambda_n\}\right)}{\ln\,|a_n|}\le\tau+1. \tag{8}$$

For $\tau=+\infty$ inequality (8) is obvious. From (6) and (8) we have (3). □

Corollary 1. *Let $\sigma_a=0$, $|a_n|\to\infty$ and $\varkappa_n\to 0$ as $n\to\infty$. In order that*

$$\lim_{n\to\infty}\frac{\ln\left(R_n(F,\sigma)e^{|\sigma|\lambda_n}\right)}{\ln\,|a_n|}=1,$$

it is necessary and sufficient that $\displaystyle\lim_{n\to\infty}\frac{\ln\,\Delta n(\lambda_n)}{\ln\,|a_n|}=0.$

Let a_n^0 be coefficients of Newton's majorant [1] of an arbitrary Dirichlet series (1) with $\sigma_a=0$ and $|a_n|\to\infty$ as $n\to\infty$. Then $a_n^0\to+\infty\,(n\to\infty)$, $|a_n|\le a_n^0$ for all n, $|a_{n_k}|=a_{n_k}^0$ for some increasing sequence (n_k) and

$$\varkappa_n^0(F)=\frac{\ln\,|a_n^0|-\ln\,|a_{n+1}^0|}{\lambda_{n+1}-\lambda_n}\nearrow 0,\quad n\to\infty.$$

We put $R_n^0(\sigma)=\sum_{k=n}^{\infty}a_k^0\exp\{\sigma\lambda_k\}$. If $\lim\limits_{n\to\infty}\dfrac{\ln\,\Delta n(\lambda_n)}{\ln\,|a_n|}=0$, then $\lim\limits_{n\to\infty}\dfrac{\ln\,\Delta n(\lambda_n)}{\ln\,|a_n^0|}=0$, and by Corollary 1 $R_n^0(\sigma)e^{|\sigma|\lambda_n}=(1+o(1))a_n^0, n\to\infty$. Since $R_n(F,\sigma)\le R_n^0(\sigma)$ and $|a_{n_k}|=a_{n_k}^0$, we have $R_{n_k}(F,\sigma)\exp\{|\sigma|\lambda_{n_k}\}\le(1+o(1))|a_{n_k}|$, $k\to\infty$, that is

$$\varliminf_{n\to\infty}\frac{\ln\left(R_n(F,\sigma)e^{|\sigma|\lambda_n}\right)}{\ln\,|a_n|}\le 1.$$

Therefore, in view of (4) we obtain the following

Corollary 2. *Let $\sigma_a=0$, $|a_n|\to\infty$ as $n\to\infty$. If $\ln\,\Delta n(\lambda_n)=o(\ln\,|a_n|)$, $n\to\infty$, then*

$$\varliminf_{n\to\infty}\frac{\ln\left(R_n(F,\sigma)e^{|\sigma|\lambda_n}\right)}{\ln\,|a_n|}=1.$$

3. Now we consider entire Dirichlet series (i.e. $\sigma_a=+\infty$). In this case the following lemma is true.

Lemma 2. *Let $\sigma_a=+\infty$, $\varkappa_n\to+\infty\,(n\to\infty)$ and $\ln\,\dfrac{1}{|a_{n+1}|}-\ln\,\dfrac{1}{|a_n|}\ge 1$ for all $n\ge n_0$. Then*

$$1\le\varliminf_{n\to\infty}\frac{R_n(F,\sigma)e^{-\sigma\lambda_n}}{|a_n|}\le\varlimsup_{n\to\infty}\frac{R_n(F,\sigma)e^{-\sigma\lambda_n}}{|a_n|}\le 1+\frac{1}{e-1}. \tag{9}$$

Proof. We need to prove right inequality (9). As above we have

$$R_n(F,\sigma)=|a_n|e^{\sigma\lambda_n}\left(1+\sum_{k=n+1}^{\infty}\exp\left\{-\left(\ln\,\frac{1}{|a_k|}-\ln\,\frac{1}{|a_n|}\right)\left(1+\sigma\frac{\lambda_k-\lambda_n)}{\ln\,|a_k|-\ln\,|a_n|}\right)\right\}\right)\le$$

$$\le|a_n|e^{\sigma\lambda_n}\left(1+\sum_{k=n+1}^{\infty}\exp\left\{-(\lambda_k-\lambda_n)\left(1+o(1)\right)\right\}\right)=$$

$$=|a_n|e^{\sigma\lambda_n}\left(1+\frac{1}{\exp\{(1+o(1)\}-1}\right)$$

as $n\to\infty$, whence right inequality (9) follows. □

For $t\ge 0$ now we put $\Delta k(t):=\sum\limits_{t\le A_n<t+1}1$, where $A_k=\ln\,\dfrac{1}{|a_k|}\to+\infty$, $k\to\infty$.

Theorem 2. *Let $\sigma_a=+\infty$ and $\varkappa_n\to+\infty$ as $n\to\infty$. Then for every fixed $\sigma\in\mathbb{R}$*

$$\varlimsup_{n\to\infty}\frac{\ln\,R_n(F,\sigma)}{\ln\,(1/|a_n|)}=\varlimsup_{n\to\infty}\frac{\ln\,\Delta k(\ln\,(1/|a_n|))}{\ln\,(1/|a_n|)}-1 \tag{10}$$

and

$$\varliminf_{n\to\infty}\frac{\ln\,R_n(F,\sigma)}{\ln\,(1/|a_n|)}\ge\varliminf_{n\to\infty}\frac{\ln\,\Delta k(\ln\,(1/|a_n|))}{\ln\,(1/|a_n|)}-1. \tag{11}$$

Proof. We note that $\ln\,(1/|a_n|)\uparrow+\infty$, $n\to\infty$, because $\sigma_a=+\infty$ and $\varkappa_n\to+\infty$ as $n\to\infty$. Since $\lambda_{n+1}-\lambda_n<\varepsilon(A_{n+1}-A_n)$ for every $\varepsilon>0$ and all $n\ge n_0=n_0(\varepsilon)$, we have $|\lambda_k - \lambda_n| < \varepsilon |A_k - A_n|$ for all $n \geq n_0$ and $k \geq n_0$. Therefore, if $|A_k - A_n| < 1$ then $\frac{|a_n|}{e} \leq |a_k| \leq |a_n| e$,

$$R_n(F,\sigma) \geq \sum_{A_n \leq A_p < A_n+1} |a_p| e^{\sigma\lambda_p} \geq \frac{|a_n|}{e} e^{\sigma\lambda_n - \varepsilon|\sigma|} \Delta k(A_n)$$

and

$$\frac{\ln\, R_n(F,\sigma)}{\ln(1/|a_n|)} \geq \frac{\ln\, \Delta k(A_n)}{\ln(1/|a_n|)} + \frac{\sigma\lambda_n}{\ln(1/|a_n|)} - \frac{\varepsilon|\sigma|+1}{\ln(1/|a_n|)} - 1. \tag{12}$$

Since $\sigma_a = +\infty$, we have $\frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \to +\infty$, $n \to \infty$, and from (12) we obtain inequality (11) and

$$\overline{\lim_{n\to\infty}} \frac{\ln\, R_n(F,\sigma)}{\ln(1/|a_n|)} \geq \overline{\lim_{n\to\infty}} \frac{\ln\, \Delta k(\ln(1/|a_n|))}{\ln(1/|a_n|)} - 1 \tag{13}$$

Since $R_n(F,\sigma) \to 0$, $n \to \infty$, from (13) we have

$$\overline{\lim_{n\to\infty}} \frac{\ln\, \Delta k(\ln(1/|a_n|))}{\ln(1/|a_n|)} = \beta \leq 1.$$

We suppose that $\beta < 1, 0 < \varepsilon < 1 - \beta$ and choose a subsequence (A_{n_k}) of sequence (A_n) such that the intervals $[A_{n_k}, A_{n_k}+1)$ are mutually disjoint and the set $\{A_n\}$ is contained in $\bigcup_k [A_{n_k}, A_{n_k}+1)$. Let $n_0 = n_0(\varepsilon)$ be such that $|\lambda_n - \lambda_p| < \varepsilon$ provided $|A_n - A_p| < 1$ and $\ln\, \Delta k(A_n) \leq (\beta+\varepsilon)A_n$ for $n \geq n_0$ and $p \geq n_0$. Then

$$R_{n_k}(F,\sigma) = \sum_{j=k}^{\infty} \sum_{A_{n_j} \leq A_p < A_{n_j}+1} |a_p| e^{\sigma\lambda_p} \leq$$

$$\leq \sum_{j=k}^{\infty} e|a_{n_j}| e^{\sigma\lambda_{n_j}+\varepsilon|\sigma|} \Delta k(A_{n_j}) \leq e^{\varepsilon|\sigma|} \sum_{j=k}^{\infty} |a_{n_k}|^{1-\beta-\varepsilon} e^{\sigma\lambda_{n_k}}.$$

Hence, as in the proof of Theorem 1, by Lemma 2 we obtain first

$$\overline{\lim_{k\to\infty}} \frac{\ln(R_{n_k}(F,\sigma)\exp\{-\sigma\lambda_{n_k}\})}{\ln(1/|a_{n_k}|)} \leq \beta + \varepsilon - 1.$$

and again

$$\overline{\lim_{n\to\infty}} \frac{\ln(R_n(F,\sigma)\exp\{-\sigma\lambda_n\})}{\ln(1/|a_n|)} \leq \beta - 1. \tag{14}$$

Since $\frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \to +\infty$, $n \to \infty$, from (13) and (14) we obtain (10). □

Corollary 3. *Let $\sigma_a = +\infty$ and $\varkappa_n \to \infty$ as $n \to \infty$. In order that* $\lim_{n\to\infty} \frac{\ln\, R_n(F,\sigma)}{\ln(1/|a_n|)} = 1$, *it is necessary and sufficient that*

$$\lim_{n\to\infty} \frac{\ln\, \Delta k(\ln(1/|a_n|))}{\ln(1/|a_n|)} = 0.$$

Let a_n^0 be the coefficients of Newton's majorant [2, p. 180–182] of an arbitrary entire Dirichlet series (1). As above, from Corollary 3 we obtain the following

Corollary 4. *Let* $\sigma_a = \infty$, $|\varkappa_n| \to \infty$ *as* $n \to \infty$. *If* $\ln \Delta k(\ln(1/|a_n^0|)) = o(\ln(1/|a_n^0|))$, $n \to \infty$, *then*

$$\varliminf_{n\to\infty} \frac{\ln R_n(F,\sigma)}{\ln(1/|a_n|)} = 1.$$

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