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ON BELONGING OF NAFTALEVICH-TSUJI PRODUCTS TO A GENERALIZED CONVERGENCE CLASS

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In terms of the zeros distribution, a class of Naftalevich-Tsuji products π defined by the convergence of the integral $\int\limits_0^1 \dfrac{\alpha(\ln^+ M_\pi(r))}{\beta(1/(1-r))}dr$ is described, where α and β are positive continuous increasing to $+\infty$ functions.

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В терминах распределения нулей описан класс произведений Нафталевича-Цудзи π, определенный сходимостью интеграла $\int\limits_0^1 \dfrac{\alpha(\ln^+ M_\pi(r))}{\beta(1/(1-r))}dr$, где α и β — положительные непрерывные возрастающие к $+\infty$ функции.

1. Introduction. For an analytic in $\mathbb{D}_R = \{z : |z| < R\}$, $0 < R \le +\infty$, function f let $M_f(r) = \max\{|f(z)| : |z| = r < R\}$, and let z_k be the zeros of f lying in $\mathbb{D}_R$. G. Valiron [1] defined for entire functions ($R = +\infty$) of order $\varrho \in (0, +\infty)$ the convergence class by the condition $\int_1^\infty \ln M_f(r) r^{-\varrho-1}\,dr < +\infty$ and proved that if f belongs to this convergence class then $\sum_{k=1}^\infty |z_k|^{-\varrho} < +\infty$. If ϱ is a noninteger number then the last condition is also sufficient in order that f belongs to the convergence class.

Let L be a class of positive continuous increasing to $+\infty$ functions on $[x_0, +\infty)$. For canonical products of infinite order in [2] the following theorem is proved.

Theorem A. *Let a continuously differentiable function $\alpha \in L$ be such that $\alpha'(x) x \ln x \ln\ln x = O(1)$ as $x \to +\infty$, $\alpha'(x)e^{\alpha(x)}$ is a nonincreasing function on $[x_0, +\infty)$ and $\alpha'(x-1)e^{\alpha(x-1)} = O\left(\alpha'(x)e^{\alpha(x)}\right)$ as $x \to +\infty$.*

In order that a canonical product $\pi(z) = \prod_{n=1}^\infty E(z/z_n, [\ln n])$ belongs to the convergence α-class $\left(i.e.\ \int\limits_{r_0}^\infty \exp\{\alpha(\ln M_f(r))\} r^{-\varrho-1}\,dr < +\infty\right)$ it is necessary and sufficient that

$$\sum_{k=k_0}^\infty \frac{\alpha'(k)\exp\{\alpha(k)\}}{|z_k|^\varrho} < +\infty.$$

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We remark that in the proof of Theorem A in [2] the condition $\varliminf\limits_{r\to+\infty} \ln n(r)/\ln r > 0$ is used, where $n(r) = \sum_{|z_k|\le r} 1$ is the counting function of the sequence (z_k), but this condition is unnecessary.

The following theorem is proved in [3].

Theorem B. *Let $\alpha \in L$, $\beta \in L$, $\omega \in L$, $\alpha(x^2) = O(\alpha(x))$ and $\beta(2x) = O(\beta(x))$ as $x \to +\infty$, $\int\limits_{x_0}^{\infty} \frac{dx}{\beta(x)} < +\infty$ and $\ln n(r) \sim \omega(r) \ln r$ as $r \to +\infty$.*

In order that a canonical product $\pi(z) = \prod_{n=1}^{\infty} E(z/z_n, [2\omega(|z_n|)])$ belongs to the convergence $\alpha\beta$-class, i.e.

$$\int\limits_{r_0}^{\infty} \frac{\alpha(\ln M_f(r))}{r\beta(\ln r)} dr < +\infty,$$

it is necessary and sufficient that

$$\sum_{k=k_0}^{\infty} (\alpha(k) - \alpha(k-1)) B(\ln |z_k|) < +\infty,$$

where $B(x) = \int\limits_{x}^{\infty} \frac{dt}{\beta(t)}$.

Here we obtain analogues of Theorem A and B for Naftalevich-Tsuji canonical products.

So, let (a_k) be a sequence of numbers from $\mathbb{D} = \mathbb{D}_1$ such that $|a_k| \le |a_{k+1}|$, $\sum_{k=1}^{\infty}(1 - |a_k|)^p = +\infty$ and $\sum_{k=1}^{\infty}(1-|a_k|)^{p+1} < \infty$. Studying a factorization of meromorphic functions in $\mathbb{D}$, A. G. Naftalevich [4–5] and M. Tsuji [6] constructed and used the following canonical product

$$\pi(z) = \prod_{k=1}^{\infty} E\left(\frac{1-|a_k|^2}{1-\overline{a}_k z}, p\right), \tag{1}$$

where

$$E(u, p) = (1-u)\exp\left\{u + \frac{u^2}{2} + \dots + \frac{u^p}{p}\right\}$$

is a primary Weierstrass factor. Product (1) is absolutely and uniformly convergent on each compact set from $\mathbb{D}$ and, thus, π is an analytic function in $\mathbb{D}$. A. G. Naftalevich [5] proved that if ρ is a noninteger number then product (1) belongs to a convergence class (i.e. $\int_0^1 (1-r)^{\varrho-1} T(r,\pi) dr < +\infty$, where $T(r,\pi)$ is the Nevanlinna characteristic) if and only if $\sum_{k=1}^{\infty}(1-|a_k|)^{\varrho} < +\infty$.

We construct Naftalevich-Tsuji products which belong to a generalized convergence class in the case when $\sum_{k=1}^{\infty}(1-|a_k|)^p = +\infty$ for each $p \in \mathbb{N}$.

2. Auxiliary results. We use the Dirichlet series theory. Let $0 < \lambda_n \nearrow \infty\, (n \to \infty)$ and a Dirichlet series

$$F(\sigma) = \sum_{n=1}^{\infty} a_n \exp\{\sigma\lambda_n\} \quad (a_n > 0) \tag{2}$$

be convergent for all $\sigma \in \mathbb{R}$. Let $\mu(\sigma) = \max\{a_n \exp\{\sigma\lambda_n\} : n \geq 1\}$ be the maximal term and $\nu(\sigma) = \max\{n : a_n \exp\{\sigma\lambda_n\} = \mu(\sigma)\}$ be the central index of series (2). The following lemma is well known [7; 8, p. 23] in the case when $\lambda_n \uparrow \infty\,(n \to \infty)$. In virtue of simplicity we give its proof.

Lemma 1. *If* $\overline{\lim\limits_{n\to\infty}} \dfrac{\ln n}{\ln(1/a_n)} = h < 1$, *then* $F(\sigma) \leq K(\varepsilon)\mu\left(\dfrac{\sigma}{1-h-\varepsilon}\right)$ *for every* $\varepsilon \in (0, 1-h)$ *and all* $\sigma \geq \sigma_0(\varepsilon)$, *where* $K(\varepsilon) = const > 0$.

Proof. Since $\sum_{n=1}^{\infty} a_n \exp\{\sigma\lambda_n\}$ is convergent for all $\sigma \in \mathbb{R}$, we have $\dfrac{1}{\lambda_n} \ln \dfrac{1}{a_n} \to +\infty$ $(n \to \infty)$. We put

$$N_1 = \left\{n : \frac{1}{\lambda_n} \ln \frac{1}{a_n} \leq \frac{\sigma}{1-h-\varepsilon}\right\} \text{ and}$$

$$N_2 = \left\{n : \frac{1}{\lambda_n} \ln \frac{1}{a_n} > \frac{\sigma}{1-h-\varepsilon}\right\}.$$

Then

$$\begin{aligned} F(\sigma) &= \sum_{n\in N_1} a_n \exp\left\{\frac{\sigma\lambda_n}{1-h-\varepsilon}\right\} \exp\left\{-\frac{(h+\varepsilon)\sigma\lambda_n}{1-h-\varepsilon}\right\} + \sum_{n\in N_2} a_n \exp\{\sigma\lambda_n\} \leq \\ &\leq \mu\left(\frac{\sigma}{1-h-\varepsilon}\right) \sum_{n\in N_1} \exp\left\{-\frac{(h+\varepsilon)\lambda_n}{1-h-\varepsilon} \frac{(1-h-\varepsilon)}{\lambda_n} \ln \frac{1}{a_n}\right\} + \\ &\quad + \sum_{n\in N_2} a_n \exp\left\{\lambda_n \frac{1-h-\varepsilon}{\lambda_n} \ln \frac{1}{a_n}\right\} \leq \\ &\leq 2\mu\left(\frac{\sigma}{1-h-\varepsilon}\right) \sum_{n=1}^{\infty} \exp\left\{-(h+\varepsilon)\ln \frac{1}{a_n}\right\}. \end{aligned}$$

Since $\ln n \leq (h + \varepsilon/2)\ln(1/a_n)$, $n \geq n_0(\varepsilon)$, we obtain the required inequality. □

Lemma 2. *For all* $\sigma > \sigma_0$

$$\ln \mu(\sigma) = \ln \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t)} dt.$$

In the case when $\lambda_n \uparrow \infty\,(n \to \infty)$ Lemma 2 is well known [9, p. 184; 8, p. 17]. For the case when $\lambda_n \nearrow \infty\,(n \to \infty)$ its proof is the same as in [8].

We need also the following

Lemma 3. [10, p.18] *The inequality*

$$\ln |E(z,p)| \leq 3e(2 + \ln p)|z|^{p+1}/(1 + |z|)$$

holds for all $z \in \mathbb{C}$.

Now, let $z_k \in \mathbb{D}$, $0 < |z_k| \nearrow 1\,(k \to \infty)$ and (p_k) be a sequence of positive integers, $1 \le p_k \nearrow \infty\,(k \to \infty)$. Consider the canonical product

$$\pi(z) = \prod_{k=1}^{\infty} E\left(\frac{1-|z_k|^2}{1-\overline{z}_k z},\, p_k\right), \quad |z| = r \in [0,\, 1). \tag{3}$$

In view of Lemma 3

$$\ln\,|\pi(z)| = \sum_{k=1}^{\infty} \ln\left|E\left(\frac{1-|z_k|^2}{1-\overline{z}_k z},\, p_k\right)\right| \le$$

$$\le \sum_{k=1}^{\infty} 3e(2+\ln\,p_k)\frac{((1-|z_k|^2)/|1-\overline{z}_k z|)^{p_k+1}}{1+(1-|z_k|^2)/|1-\overline{z}_k z|} \le \tag{4}$$

$$\le \sum_{k=1}^{\infty} \left(\frac{K_1(1-|z_k|)}{|1-\overline{z}_k z|}\right)^{p_k+1} \le \sum_{k=1}^{\infty} \left(\frac{K_1(1-|z_k|)}{1-r}\right)^{p_k+1}, \quad K_1 > 1. \tag{5}$$

Here and further by K_j we denote positive constants.

It is easy to see that if

$$\varliminf_{n\to\infty} \frac{p_k}{\ln\,k} \ln\frac{1}{1-|z_k|} > 1 \tag{6}$$

then the last series in (4) is convergent and, thus, π is an analytic function in $\mathbb{D}$. Condition (5) holds if, for example, $p_k = [\ln\,k]$.

For $\alpha \in L$ and $\beta \in L$ we say that π belongs to the convergence $\alpha\beta$-class if

$$\int_{r_0}^{1} \frac{\alpha(\ln\,M_\pi(r))}{(1-r)\beta(1/(1-r))}dr < +\infty. \tag{7}$$

Proposition 1. *Let $\alpha \in L$, $\beta \in L$, $\alpha(x^2) = O(\alpha(x))$ and $\beta(2x) = O(\beta(x))$ as $x \to +\infty$ and $\int_{x_0}^{\infty} \frac{\alpha(x)dx}{x\beta(x)} < +\infty$. Then (6) implies*

$$\int_{r_0}^{1} \frac{\alpha(n(r))}{(1-r)\beta(1/(1-r))}dr < +\infty. \tag{8}$$

Proof. Indeed, by the Jensen inequality

$$\ln\,M_\pi\left(\frac{1+r}{2}\right) \ge \int_0^{(1+r)/2} \frac{n(t)}{t}dt + \ln\,|\pi(0)| \ge \int_r^{(1+r)/2} \frac{n(t)}{t}dt + \ln\,|\pi(0)| \ge$$

$$\ge \frac{2n(r)}{1+r}\int_r^{(1+r)/2} dt + \ln\,|\pi(0)| = \frac{1-r}{1+r}n(r) + \ln\,|\pi(0)|,$$

that is

$$n(r) \le \frac{3}{1-r}\ln\,M_\pi\left(\frac{1+r}{2}\right), \quad r \in [r_0,\, 1).$$

Therefore, in view of the condition $\alpha(x^2) = O(\alpha(x))$ $(x \to +\infty)$ we have

$$\alpha(n(r)) = \alpha\left(\exp\left\{\ln\frac{3}{1-r} + \ln\ln M_\pi\left(\frac{1+r}{2}\right)\right\}\right) \le$$
$$\le \alpha\left(\exp\left\{2\max\left\{\ln\frac{3}{1-r},\ \ln\ln M_\pi\left(\frac{1+r}{2}\right)\right\}\right\}\right) \le$$
$$\le K_2\alpha\left(\exp\left\{\max\left\{\ln\frac{3}{1-r},\ \ln\ln M_\pi\left(\frac{1+r}{2}\right)\right\}\right\}\right) =$$
$$= K_2\max\left\{\alpha\left(\frac{3}{1-r}\right),\ \alpha\left(\ln M_\pi\left(\frac{1+r}{2}\right)\right)\right\} \le$$
$$\le K_2\left(\alpha\left(\frac{3}{1-r}\right) + \alpha\left(\ln M_\pi\left(\frac{1+r}{2}\right)\right)\right). \tag{9}$$

Since $\alpha(3x) = O(\alpha(x))$ $(x \to +\infty)$, the condition $\int\limits_{x_0}^{\infty}\frac{\alpha(x)}{x\beta(x)}dx < +\infty$ implies

$$\int\limits_{r_0}^{1}\frac{\alpha(3/(1-r))}{(1-r)\beta(1/(1-r))}dr < +\infty.$$

Clearly,

$$\int\limits_{r_0}^{1}\frac{\alpha(\ln M_\pi((1+r)/2))}{(1-r)\beta(1/(1-r))}dr = \int\limits_{(1+r_0)/2}^{1}\frac{\alpha(\ln M_\pi(r))}{(1-r)\beta(1/2(1-r))}dr.$$

Therefore, in view of the condition and $\beta(2x) = O(\beta(x))$ $(x \to +\infty)$, (6) and (8) imply (7). $\square$

Proposition 2. *Let $\alpha \in L$, $\beta \in L$, $\int\limits_{x_0}^{\infty}\frac{\alpha(x)dx}{x\beta(x)} < +\infty$ and $B(x) = \int\limits_{x}^{\infty}\frac{dt}{t\beta(t)}$. Then (6) holds if and only if*

$$\sum_{n=n_0}^{\infty}(\alpha(n) - \alpha(n-1))B\left(\frac{1}{1-|z_n|}\right) < +\infty. \tag{10}$$

Proof. Indeed,

$$\int\limits_{r_0}^{1}\frac{\alpha(n(r))}{(1-r)\beta(1/(1-r)}dr = K_3 + \sum_{n=n_0}^{\infty}\int\limits_{|z_n|}^{|z_{n+1}|}\frac{\alpha(n(r))}{(1-r)\beta(1/(1-r))}dr =$$
$$= K_3 + \sum_{n=n_0}^{\infty}\alpha(n)\int\limits_{|z_n|}^{|z_{n+1}|}\frac{dr}{(1-r)\beta(1/(1-r))} = K_3 + \sum_{n=n_0}^{\infty}\alpha(n)\int\limits_{1/(1-|z_n|)}^{1/(1-|z_{n+1}|)}\frac{dt}{t\beta(t} =$$

$$= K_3 + \sum_{n=n_0}^{\infty} \alpha(n) \left(B\left(\frac{1}{1-|z_n|}\right) - B\left(\frac{1}{1-|z_{n+1}|}\right) \right) =$$

$$= K_4 + \sum_{n=n_0}^{\infty} (\alpha(n) - \alpha(n-1)) B\left(\frac{1}{1-|z_n|}\right).$$

□

Proposition 3. *If* $p_k = [\ln k]$ *then there exist constants* $K_2 > 1$ *and* $K_3 > 1$ *such that for canonical product (3)*

$$\ln\ln M_\pi(r) \le K_3 \int\limits_0^{1-(1-r)/K_2} \frac{\ln n(t)}{1-t} dt. \tag{11}$$

Proof. We put $\sigma = \ln \dfrac{1}{1-r}$. Then for $\eta > 0$ we obtain from (4)

$$\ln M_\pi(r) \le \sum_{n=1}^{\infty} (K_1(1-|z_n|)e^\sigma)^{[\ln n]+1} =$$

$$= \sum_{n=1}^{\infty} (K_1(1-|z_n|)e^{\sigma+1+\eta})^{[\ln n]+1} e^{-(1+\eta)([\ln n]+1)} \le$$

$$\le \mu(\sigma+1+\eta) \sum_{n=1}^{\infty} e^{-(1+\eta)\ln n} \le \frac{K_4}{\eta} \mu(\sigma+1+\eta), \tag{12}$$

where $\mu(\sigma)$ is the maximal term of Dirichlet series (2) with $a_n = (K_1(1-|z_n|))^{[\ln n]+1}$. Since $\mu(\sigma) \to +\infty$, $\sigma \to +\infty$, for central index of this series we have $K_1(1-|z_{\nu(\sigma)}|)e^\sigma > 1$ that is $|z_{\nu(\sigma)}| \le 1 - 1/(K_1 e^\sigma)$, whence $\nu(\sigma) \le n(1-1/(K_1 e^\sigma))$. Therefore, by Lemma 2

$$\ln \mu(\sigma) \le \ln \mu(0) + \int\limits_0^\sigma (\ln \nu(t) + 1) dt \le \ln \mu(0) + \int\limits_0^\sigma (\ln n(1-1/(K_1 e^t)) + 1) dt \le$$

$$\le \ln \mu(0) + \int\limits_0^{1-1/(K_1 e^\sigma)} \frac{\ln n(t) + 1}{1-t} dt. \tag{13}$$

From (11) and (12) we obtain

$$\ln\ln M_\pi(r) \le \ln \mu \left(\ln \frac{1}{1-r} + 1 + \eta \right) + \ln \frac{K_4}{\eta} \le$$

$$\le \ln \frac{K_4}{\eta} + K_5 + \int\limits_0^{1-(1-r)/(K_1 e^{1+\eta})} \frac{\ln n(t) + 1}{1-t} dt,$$

whence (10) follows. □

Proposition 4. *Let* $\ln\, n(r) \sim \omega(r)\ln\frac{1}{1-r}$, $r\uparrow 1$, *where* ω *is a positive continuous decreasing function on* $[0,\,1)$. *If* $p_k = 2[\omega(|z_k|)]$ *then there exist constants* $K_2 > 1$ *and* $K_3 > 1$ *such that for canonical product (3)*

$$\ln\,\ln\, M_\pi(r) \le K_3 \ln\, n(1-(1-r)^3/K_2). \tag{14}$$

Proof. Clearly, condition (5) holds and for Dirichlet series (2) with $a_n = (K_1(1-|z_n|))^{p_n+1}$ we have $\varlimsup\limits_{n\to\infty}\frac{\ln n}{\ln(1/a_n)} = \frac{1}{2}$. Therefore, by Lemma 1 from (4) we obtain $\ln\, M_\pi(r) \le K_6\mu(3\sigma)$, $\sigma = \ln\frac{1}{1-r}$. By Lemma 2, $\ln\,\mu(\sigma) \le \sigma\lambda_\nu(\sigma)$ for greater σ. As in the proof of Proposition 3, $|z_{\nu(\sigma)}| \le 1 - 1/(K_1e^\sigma)$ and, thus,

$$\lambda_\nu(\sigma) \le p_\nu(\sigma) + 1 \le 2\omega(|z_{\nu(\sigma)}|) + 1 \le 2\omega(1-1/(K_1e^\sigma) + 1 \le \frac{3}{\sigma + \ln\, K_1}\ln\, n(1-1/(K_1e^\sigma)).$$

Therefore,

$$\ln\,\ln\, M_\pi(r) \le \ln\, K_6 + \ln\,\mu\left(3\ln\frac{1}{1-r}\right) \le$$
$$\le \ln\, K_6 + 3\ln\frac{1}{1-r}\frac{\ln\, n(1-1/(K_1\exp\{3\ln(1/(1-r))\})}{\ln(1/(1-r)) + \ln\, K_1} \le$$
$$\le \ln\, K_6 + 3\ln\, n(1-(1-r)^3/K_1),$$

whence (13) follows. □

3. Main results. First, we consider the case when $p_k = [\ln\, k]$.

Theorem 1. *Let* $\alpha \in L$ *and* $\beta \in L$ *be continuosly differentiable functions such that*

$$\frac{\alpha'(x)}{\alpha(x)}x\ln\, x\ln\,\ln\, x = O(1),\ \frac{\beta'(x)}{\beta(x)}x = O(1), \quad x\to+\infty \text{ and } \int\limits_{x_0}^{\infty}\frac{\alpha(x)dx}{x\beta(x)} < +\infty.$$

In order that product (3) with $p_k = [\ln\, k]$ *belongs to the convergence* $\alpha\beta$*-class it is necessary and sufficient that its zeros satisfy condition (9).*

Proof. From the conditions on α and β it follows that α and β satisfy the conditions of Proposition 1. Therefore, in view of Propositions 1 and 2, we need to prove only that (7) implies (6).

From (10) we obtain

$$\ln\,\ln\, M_\pi(r) \le K_3\ln\, n(R)\ln\frac{K_2}{1-r},$$

where $R = 1-(1-r)/K_2$. We put $\exp_2 x = \exp\{e^x\}$. Then in view of the condition

$\frac{\alpha'(x)}{\alpha(x)} x \ln x \ln \ln x = O(1)$, $x \to +\infty$, we have

$$\begin{aligned} \alpha(\ln M_\pi(r)) = \alpha(\exp_2\{\ln \ln \ln M_\pi(r)\}) \le \\ \le \alpha\left(\exp_2\left\{\ln K_3 + \ln \ln \frac{K_2}{1-r} + \ln \ln n(R)\right\}\right) \le \\ \le \alpha\left(\exp_2\left\{3 \max\left\{\ln \ln \frac{1}{1-r}, \ln \ln n(R)\right\}\right\}\right) \le \\ \le K_6 \max\left\{\alpha\left(\frac{1}{1-r}\right), \alpha(n(R))\right\} \le \\ \le K_6\left(\alpha\left(\frac{1}{1-r}\right) + \alpha(n(R))\right). \end{aligned}$$

Therefore, in view of the conditions $\int\limits_{x_0}^{\infty} \frac{\alpha(x)dx}{x\beta(x)} < +\infty$ and $\frac{\beta'(x)}{\beta(x)} x = O(1)$, $x \to +\infty$, we obtain

$$\begin{aligned} \int\limits_{r_0}^{1} \frac{\alpha(\ln M_\pi(r))}{(1-r)\beta(1/(1-r))} dr \le K_7 + K_6 \int\limits_{r_0}^{1} \frac{\alpha(n(R))}{(1-r)\beta(1/(1-r))} dr = \\ = K_7 + K_6 \int\limits_{R_0}^{1} \frac{\alpha(n(R))dR}{K_2(1-R)\beta(1/(K_2(1-R)))} \le \\ \le K_7 + K_8 \int\limits_{R_0}^{1} \frac{\alpha(n(R))dR}{(1-R)\beta(1/(1-R))}, \end{aligned}$$

whence it follows that (7) implies (6). Theorem 1 is proved. □

If we choose $\beta(x) = x^\varrho$, $\varrho > 0$, then $B(x) = 1/(\varrho x^\varrho)$ and from Theorem 2 we obtain

Corollary 1. *Let $\alpha \in L$ be a continuosly differentiable function, $\frac{\alpha'(x)}{\alpha(x)} x \ln x \ln \ln x = O(1)$ $(x \to +\infty)$. Then for product (3) with $p_k = [\ln k]$*

$$\left(\int\limits_{r_0}^{1} (1-r)^{\varrho-1} \alpha(\ln M_\pi(r)) dr < +\infty\right) \Longleftrightarrow \left(\sum_{n=n_0}^{\infty} (\alpha(n) - \alpha(n-1))(1-|z_n|)^\varrho < +\infty\right).$$

The function $\alpha(x) = \ln x$ does not satisfy the condition of Corollary 1. However, the following result is true.

Theorem 2. *For product (3) with $p_k = [\ln k]$*

$$\left(\int\limits_{r_0}^{1} (1-r)^{\varrho-1} \ln \ln M_\pi(r) dr < +\infty\right) \Longleftrightarrow \left(\sum_{n=n_0}^{\infty} \frac{(1-|z_n|)^\varrho}{n} < +\infty\right).$$

Proof. From (10) we obtain

$$\int_{r_0}^{1}(1-r)^{\varrho-1}\ln\ln M_\pi(r)dr \le \frac{K_3}{\varrho}\int_{r_0}^{1}\int_{0}^{1-(1-r)/K_2}\frac{\ln n(t)}{1-t}\,dt\,d(-(1-r)^{\varrho}) \le$$

$$\le K_4 + K_5\int_{r_0}^{1}(1-r)^{\varrho-1}\ln n(1-(1-r)/K_2)dr =$$

$$= K_6 + K_7\int_{r_0}^{1}(1-r)^{\varrho-1}\ln n(r)dr$$

and, thus, (7) implies (6). In view of Propositions 1 and 2, Theorem 2 is proved. □

Now, we examine the case when $\ln n(r) \sim \omega(r)\ln\frac{1}{1-r}$, $r\uparrow 1$. The following theorem is true.

Theorem 3. *Let* $\ln n(r) \sim \omega(r)\ln\frac{1}{1-r}$, $r\uparrow 1$, *where* ω *is a positive continuous decreasing function on* $[0, 1)$. *Let* $\alpha\in L$ *and* $\beta\in L$ *be continuously differentiable functions such that* $\frac{\alpha'(x)}{\alpha(x)}x\ln x = O(1)$, $\frac{\beta'(x)}{\beta(x)}x\ln x = O(1)$ *as* $x\to+\infty$ *and* $\int\limits_{x_0}^{\infty}\frac{\alpha(x)dx}{x\beta(x)} < +\infty$.

In order that product (3) with $p_k = 2[\omega(|z_k|)]$ *belongs to the convergence* $\alpha\beta$*-class it is necessary and sufficient that its zeros satisfy condition (9).*

Proof. In view of the condition $\frac{\alpha'(x)}{\alpha(x)}x\ln x = O(1)$, $x\to+\infty$, from (13) we obtain

$$\alpha(\ln M_\pi(r)) \le K_4\alpha(n(1-(1-r)^3/K_2))$$

and in view of the condition $\frac{\beta'(x)}{\beta(x)}x\ln x = O(1)$ $(x\to+\infty)$, we have

$$\int_{r_0}^{1}\frac{\alpha(\ln M_\pi(r))}{(1-r)\beta(1/(1-r))}dr \le K_5\int_{R_0}^{1}\frac{\alpha(n(R))}{(1-R)\beta(1/(K_2(1-R)^{1/3})}dr \le$$

$$\le K_6\int_{R_0}^{1}\frac{\alpha(n(R))}{(1-R)\beta(1/(1-R))}dr,$$

that is (7) implies (6). In view of Propositions 1 and 2, Theorem 3 is proved. □

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