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A REMARK ON THE DERIVATIVE OF ENTIRE DIRICHLET SERIES

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For an entire Dirichlet series with arbitrary exponents an estimate of the maximum modulus of the derivative is obtained.

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Для целого ряда Дирихле с произвольными показателями получена оценка максимума модуля производной.

Refining a result of S. Bernstein [1], T. Kövari [2] proved for an entire function of order $\rho \in (0, +\infty)$ and type $T \in (0, +\infty)$ the following sharp estimate holds:

$$\overline{\lim}_{r \rightarrow +\infty} \frac{M_{f'}(r)}{M_f(r) T \rho r^{\rho-1}} \leq e \quad (M_f(r) = \max\{|f(z)| : |z| = r\}).$$

M. M. Sheremeta and S. I. Fedynyak [3] extended the result on the Dirichlet series

$$F(z) = \sum_{n=1}^{\infty} a_n \exp\{z\lambda_n\}, \quad z = x + iy, \quad (1)$$

with exponents $0 = \lambda_0 < \lambda_n \uparrow +\infty$ and arbitrary abscissa of absolute convergence $A \in (-\infty, +\infty]$. In particular, they proved that if $A = +\infty$, $\ln n = o(\lambda_n)$ ($n \rightarrow \infty$), and F has R -order $\rho \in (0, +\infty)$ and R -type $T \in (0, +\infty)$ then

$$\overline{\lim}_{x \rightarrow +\infty} \frac{M(x, F')}{M(x, F) T \rho e^{\rho x}} \leq e. \quad (2)$$

Here

$$M(x, F) = \sup\{|F(x + iy)| : y \in \mathbb{R}\},$$

$$\rho = \overline{\lim}_{x \rightarrow +\infty} \frac{\ln \ln M(x, F)}{x}, \quad T = \overline{\lim}_{x \rightarrow +\infty} \frac{\ln M(x, F)}{e^{\rho x}}.$$

In the note we consider the case when (λ_n) is an arbitrary sequence (without the condition $\ln n = o(\lambda_n)$, $n \rightarrow \infty$) and obtain an estimate, which is somewhat worse than (2).

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Theorem. *If an entire Dirichlet series (1) has R-order $\rho \in (0, +\infty)$ and R-type $T \in (0, +\infty)$ then*

$$\overline{\lim}_{x \rightarrow +\infty} \frac{M(x, F')}{M(x, F)T\rho e^{\rho x}} \leq \frac{e^2}{2}. \quad (3)$$

Proof. Since F is bounded in each half-plane $\{z : \operatorname{Re} z \leq \sigma\}$, $0 < \sigma < +\infty$, the function $(F(z) - F(-1))/(z + 1)$ belongs to the Hardy class \mathbb{H}^2 in $\{z : \operatorname{Re} z \leq \sigma\}$ and, therefore [4],

$$\frac{F(z) - F(-1)}{z + 1} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{F(t) - F(-1)}{(t + 1)(t - z)} dt, \quad \operatorname{Re} z = x < 0,$$

whence

$$F(z) = F(-1) + \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} F(t) \left(\frac{1}{t - z} - \frac{1}{t + 1} \right) dt$$

and

$$F'(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{F(t)}{(t - z)^2} dt, \quad \operatorname{Re} z < 0.$$

Hence

$$M(x, F') \leq \frac{M(0, F)}{2\pi} \int_{-\infty}^{+\infty} \frac{d\tau}{|i\tau - x - iy|^2} d\tau = \frac{M(0, F)}{2\pi} \int_{-\infty}^{+\infty} \frac{d\tau}{(\tau - y)^2 + |x|^2} d\tau = \frac{M(0, F)}{2|x|}$$

and, thus, $M(\sigma + x, F') \leq M(\sigma, F)/(2|x|)$ for $x < 0$ and $0 < \sigma < +\infty$. Therefore, for every $\delta > 0$ and all $x \in \mathbb{R}$,

$$M(x, F') \leq M(x + \delta, F)/(2\delta). \quad (4)$$

Since $\ln M(x, F)$ is a convex function,

$$\ln M(x + \delta, F) - \ln M(x, F) = \int_x^{x+\delta} l(t) dt,$$

where l is a positive nondecreasing function. Hence

$$\begin{aligned} \ln M(x + \delta, F) - \ln M(x, F) &\leq \delta l(x + \delta) \leq \delta \rho \int_{x+\delta}^{x+\delta+1/\rho} l(t) dt \leq \\ &\leq \delta \rho \ln M(x + \delta + 1/\rho, F) \leq \delta \rho \tau \exp\{\rho x + \delta \rho + 1\} \end{aligned} \quad (5)$$

for each $\tau > T$ and all $x \geq x_0(\tau)$. From (4) and (5) we have

$$\frac{M(x, F')}{M(x, F)\tau\rho e^{\rho x}} \leq \frac{\exp\{\delta \rho \tau \exp\{\rho x + \delta \rho + 1\}\}}{2\delta \tau \rho e^{\rho x}},$$

whence for $\delta = 1/(\rho \tau \exp\{\rho x + \delta \rho + 1\})$ we obtain

$$\overline{\lim}_{x \rightarrow +\infty} \frac{M(x, F')}{M(x, F)\tau\rho e^{\rho x}} \leq \frac{e}{2} \overline{\lim}_{x \rightarrow +\infty} \exp\{\delta \rho + 1\} = \frac{e^2}{2}.$$

In view of arbitrariness of τ Theorem is proved. \square

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