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A REMARK ON THE DERIVATIVE OF ENTIRE DIRICHLET SERIES

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For an entire Dirichlet series with arbitrary exponents an estimate of the maximum modulus of the derivative is obtained.

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Для целого ряда Дирихле с произвольными показателями получена оценка максимума модуля производной.

Refining a result of S. Bernstein [1], T. Kövari [2] proved for an entire function of order $\varrho \in (0,+\infty)$ and type $T \in (0,+\infty)$ the following sharp estimate holds:

$$\varlimsup_{r\to+\infty} \frac{M_{f'}(r)}{M_f(r)T\varrho r^{\varrho-1}} \le e \quad (M_f(r) = \max\{|f(z)|:\ |z|=r\}).$$

M. M. Sheremeta and S. I. Fedynyak [3] extended the result on the Dirichlet series

$$F(z) = \sum_{n=1}^{\infty} a_n \exp\{z\lambda_n\}, \quad z = x+iy, \tag{1}$$

with exponents $0=\lambda_0<\lambda_n \uparrow +\infty$ and arbitrary abscissa of absolute convergence $A \in (-\infty,+\infty]$. In particular, they proved that if $A=+\infty$, $\ln n = o(\lambda_n)$ $(n\to\infty)$, and F has R-order $\varrho\in(0,+\infty)$ and R-type $T\in(0,+\infty)$ then

$$\varlimsup_{x\to+\infty} \frac{M(x,F')}{M(x,F)T\varrho e^{\varrho x}} \le e. \tag{2}$$

Here

$$M(x,F) = \sup\{|F(x+iy)|:\ y\in\mathbb{R}\},$$
$$\varrho = \varlimsup_{x\to+\infty} \frac{\ln\ln M(x,F)}{x}, \quad T = \varlimsup_{x\to+\infty} \frac{\ln M(x,F)}{e^{\varrho x}}.$$

In the note we consider the case when (λ_n) is an arbitrary sequence (without the condition $\ln n = o(\lambda_n)$, $n\to\infty$) and obtain an estimate, which is somewhat worse than (2).

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Theorem. *If an entire Dirichlet series (1) has R-order $\varrho \in (0,+\infty)$ and R-type $T \in (0,+\infty)$ then*

$$\overline{\lim_{x\to+\infty}} \frac{M(x,F')}{M(x,F)T\varrho e^{\varrho x}} \le \frac{e^2}{2}. \tag{3}$$

Proof. Since F is bounded in each half-plane $\{z : \operatorname{Re} z \le \sigma\}$, $0<\sigma<+\infty$, the function $(F(z)-F(-1))/(z+1)$ belongs to the Hardy class $\mathbb{H}^2$ in $\{z : \operatorname{Re} z \le \sigma\}$ and, therefore [4],

$$\frac{F(z)-F(-1)}{z+1} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{F(t)-F(-1)}{(t+1)(t-z)}dt, \quad \operatorname{Re} z = x < 0,$$

whence

$$F(z) = F(-1) + \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} F(t)\left(\frac{1}{t-z} - \frac{1}{t+1}\right)dt$$

and

$$F'(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{F(t)}{(t-z)^2}dt, \quad \operatorname{Re} z < 0.$$

Hence

$$M(x,F') \le \frac{M(0,F)}{2\pi}\int_{-\infty}^{+\infty}\frac{d\tau}{|i\tau - x - iy|^2}d\tau = \frac{M(0,F)}{2\pi}\int_{-\infty}^{+\infty}\frac{d\tau}{(\tau-y)^2+|x|^2}d\tau = \frac{M(0,F)}{2|x|}$$

and, thus, $M(\sigma+x,F') \le M(\sigma,F)/(2|x|)$ for $x<0$ and $0<\sigma<+\infty$. Therefore, for every $\delta>0$ and all $x\in\mathbb{R}$,

$$M(x,F') \le M(x+\delta,F)/(2\delta). \tag{4}$$

Since $\ln\, M(x,F)$ is a convex function,

$$\ln\, M(x+\delta,F) - \ln\, M(x,F) = \int_x^{x+\delta} l(t)dt,$$

where l is a positive nondecreasing function. Hence

$$\ln\, M(x+\delta,F) - \ln\, M(x,F) \le \delta l(x+\delta) \le \delta\varrho \int_{x+\delta}^{x+\delta+1/\varrho} l(t)dt \le$$

$$\le \delta\varrho \ln\, M(x+\delta+1/\varrho,F) \le \delta\varrho\tau\exp\{\varrho x + \delta\varrho + 1\} \tag{5}$$

for each $\tau > T$ and all $x \ge x_0(\tau)$. From (4) and (5) we have

$$\frac{M(x,F')}{M(x,F)\tau\varrho e^{\varrho x}} \le \frac{\exp\{\delta\varrho\tau\exp\{\varrho x+\delta\varrho+1\}\}}{2\delta\tau\varrho e^{\varrho x}},$$

whence for $\delta = 1/(\varrho\tau\exp\{\varrho x+\delta\varrho+1\})$ we obtain

$$\overline{\lim_{x\to+\infty}} \frac{M(x,F')}{M(x,F)\tau\varrho e^{\varrho x}} \le \frac{e}{2}\overline{\lim_{x\to+\infty}} \exp\{\delta\varrho+1\} = \frac{e^2}{2}.$$

In view of arbitrariness of τ Theorem is proved. □

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