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ON PALEY'S EFFECT FOR ENTIRE FUNCTIONS

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Let $M_f(r)$ be the maximum modulus of an entire function f, $T_f(r)$ be its Nevanlinna characteristic, Φ be a convex function such that $\Phi(x)/x \to +\infty$ as $x \to +\infty$, and $\mathcal{A}(\Phi)$ be the class of entire functions f such that $\ln M_f(r) \le \Phi(\ln r)$ as $r \ge r_0(f)$. It is shown that for the relation $\ln M_f(r) \sim T_f(r)$ (or $\ln M_f(r) = O(T_f(r))$) as $r \to +\infty$ for each entire function $f \in \mathcal{A}(\Phi)$ it is necessary and sufficient that $\Phi(x) = O(x^2)$ as $x \to +\infty$.

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Пусть $M_f(r)$ — максимум модуля целой функции f, $T_f(r)$ — её характеристика Неванлинны, Φ — выпуклая функция такая, что $\Phi(x)/x \to +\infty$, $x \to +\infty$, и $\mathcal{A}(\Phi)$ — класс целых функций f таких, что $\ln M_f(r) \le \Phi(\ln r)$, $r \ge r_0(f)$. Показано, что для того, чтобы соотношение $\ln M_f(r) \sim T_f(r)$ (или $\ln M_f(r) = O(T_f(r))$), $r \to +\infty$, выполнялось для любой целой функции $f \in \mathcal{A}(\Phi)$, необходимо и достаточно, чтобы $\Phi(x) = O(x^2)$ при $x \to +\infty$.

1. Introduction. Let $\mathcal{A}$ be the class of transcendental entire functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

Put $M_f(r) = \max\{|f(z)| : |z| = r\}$ and let $T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta$ be the Nevanlinna characteristic of entire function (1) ($u^+ := \max\{u, 0\}$).

Denote by Ω the class of convex on $(-\infty, +\infty)$ functions Φ such that $\Phi(x)/x \to +\infty$, $x \to +\infty$. It is known that $\ln M_f(e^x), T_f(e^x) \in \Omega$ and $T_f(r) \le \ln^+ M_f(r)$ for each entire function $f \in \mathcal{A}(\Phi)$.

For $\Phi \in \Omega$ let $\mathcal{A}(\Phi)$ be the class of entire functions $f \in \mathcal{A}$ such that $\ln M_f(r) \le \Phi(\ln r)$, $r \ge r_0(f)$.

For every $\rho \ge 0$, Paley [1] has constructed an example of entire function f of order ρ (i.e. $\overline{\lim\limits_{r\to+\infty}} \ln\ln M_f(r)/\ln r = \rho$) such that

$$\overline{\lim_{r\to+\infty}} \frac{\ln M_f(r)}{T_f(r)} = +\infty. \tag{2}$$

As in [2], we say that f has the Paley effect, if (2) holds.

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It is known (see for example [3]) that if $f \in \mathcal{A}$ and $\ln M_f(r) = O(\ln^2 r)$, $r \to +\infty$, then

$$\ln M_f(r) \sim T_f(r), \quad r \to +\infty.$$

Therefore, the condition

$$\Phi(x) = O(x^2), \quad x \to +\infty, \tag{3}$$

is sufficient in order that each function $f \in \mathcal{A}(\Phi)$ has not the Paley effect. In this paper we prove the necessity of condition (3) in the last assertion.

Theorem. *Let $\Phi \in \Omega$. If condition (3) is not fulfilled, then there exists an entire function $f \in \mathcal{A}(\Phi)$ such that (2) holds.*

2. Proof of Theorem. Let $\mu_f(r) = \max\{|a_n|r^n : n \geq 0\}$ be the maximal term, and $\nu_f(r) = \max\{n \geq 0 : |a_n|r^n = \mu_f(r)\}$ be the central index of entire function (1).

In order to prove Theorem, we will need two lemmas.

Lemma 1. [4] *Let (n_k) be an increasing sequence of nonnegative integers, and (c_k) be a sequence of positive numbers increasing to $+\infty$. If (a_n) is a complex sequence such that $a_n = 0$ for $n < n_0$, $a_{n_0} \neq 0$ and for every $k \geq 0$*

$$|a_{n_{k+1}}| = |a_{n_0}| \prod_{j=0}^{k} \frac{1}{c_j^{n_{j+1}-n_j}}; \qquad |a_n| \leq |a_{n_k}| c_k^{n_k - n}, \text{ when } n \in (n_k, n_{k+1}),$$

then power series (1) with the coefficients a_n defines an entire function such that:

(i) $\nu_f(r) = n_0$ *for* $r \in (0, c_0)$;

(ii) $\nu_f(r) = n_{k+1}$ *for* $r \in [c_k, c_{k+1})$ *and* $k \geq 0$.

Lemma 2. *Let $c = \text{const} \geq e^2$. Then for every integer $n \geq 0$ there exist nonnegative numbers $\delta(0, n), \ldots, \delta(n, n)$ and ε_n such that:*

(i) $\max\{\delta(0, n), \ldots, \delta(n, n)\} = 1$;

(ii) $\varepsilon_n = o(1)$, $n \to \infty$;

(iii) $\left|\sum_{j=0}^{n} \delta(j, n) e^{ij\theta}\right| \leq \exp\left\{\frac{n}{c}(\cos\theta - 1 + \varepsilon_n)\right\}$ *for every* $\theta \in [0, 2\pi]$.

Proof. We put $\delta(0, n) = \ldots = \delta(n, n) = 1$ and $\varepsilon_n = 2 + c$ for $n \in [0, c)$.

Let $n \geq c$ and $r = n/c$. Consider the entire function

$$g(z) = e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}.$$

Using the inequality

$$m^m e^{-m} \sqrt{2\pi m} < m! < m^m e^{-m + \frac{1}{12m}} \sqrt{2\pi m}, \quad m \geq 1, \tag{4}$$

we obtain

$$\sum_{j>n} \frac{r^j}{j!} < \sum_{j>n} \left(\frac{j}{c}\right)^j \frac{1}{j^j e^{-j}} = \sum_{j>n} \left(\frac{e}{c}\right)^j \leq \sum_{j>n} \frac{1}{e^j} < \frac{1}{e^n}.$$

Consequently,

$$\left|\sum_{j\le n}\frac{r^j e^{ij\theta}}{j!}\right| = \left|e^{re^{i\theta}} - \sum_{j>n}\frac{r^j e^{ij\theta}}{j!}\right| < e^{r\cos\theta} + \frac{1}{e^n} < 2e^{\frac{n}{c}\cos\theta}. \qquad (5)$$

It is known [5] that $\mu_g(r) = r^{[r]}/[r]!$. Put

$$\delta(j,n) = \frac{r^j}{j!\mu_g(r)},\ \text{if } j\in\{0,\dots,n\};\quad \varepsilon_n = \frac{c}{n}\left(\frac{n}{c} - [r] + \frac{1}{12[r]} + \ln\sqrt{2\pi[r]} + \ln 2\right).$$

Then (i) and (ii) are valid. Further, we prove (iii). Using (5) and (4) we have

$$\left|\sum_{j=0}^{n}\delta(j,n)e^{ij\theta}\right| \le \frac{2}{\mu_g(r)}e^{\frac{n}{c}\cos\theta} \le \frac{2[r]!}{[r]^{[r]}}e^{\frac{n}{c}\cos\theta} <$$
$$< 2e^{\frac{n}{c}\cos\theta - [r] + \frac{1}{12[r]} + \ln\sqrt{2\pi[r]}} = e^{\frac{n}{c}(\cos\theta - 1 + \varepsilon_n)}.$$

Lemma 2 is proved. □

Now, we will prove Theorem. Let $\Phi\in\Omega$ be such that condition (3) is not fulfilled. We take $\Psi(x) = \frac{1}{4}\Phi(x-1)$ and $l(x) = \Psi'_+(x)$. Then $\Psi\in\Omega$ and the function l are nondecreasing. Furthermore, since

$$\overline{\lim_{x\to+\infty}}\frac{\Psi(x)}{x^2} = +\infty,$$

then in view of l'Hospital's rule we obtain that

$$\overline{\lim_{x\to+\infty}}\frac{l(x)}{x} = +\infty.$$

Therefore, there exists an increasing to $+\infty$ sequence (c_k) such that $c_0 > 1$, $[l(\ln c_0)] \ge 1$ and

$$[l(\ln c_{k+1})] \ge (k+1)[l(\ln c_k)]\ln c_{k+1},\quad k\ge 0. \qquad (6$$

Let $n_0 = 0$ and $n_{k+1} = [l(\ln c_k)]$ for $k\ge 0$.

We put $b_0 = b_{n_0} = 1$ and let

$$b_{n_{k+1}} = \prod_{j=0}^{k}\frac{1}{c_j^{n_{j+1}-n_j}};\quad b_n = b_{n_k}c_k^{n_k - n},\ \text{if } n\in(n_k, n_{k+1}), \qquad (7)$$

for $k\ge 0$. Consider the auxiliary power series

$$h(z) = \sum_{n=0}^{\infty} b_n z^n. \qquad (8)$$

By Lemma 1 this series defines an entire function such that

$$\nu_h(r) = n_{k+1} = [l(\ln c_k)] \le l(\ln r),\quad r\in[c_k, c_{k+1}),\ k\ge 0.$$

Since [5],

$$\ln\mu_h(r_2) - \ln\mu_h(r_1) = \int_{r_1}^{r_2}\frac{\nu_f(t)}{t}\,dt,\quad r_2 > r_1 > 0,$$

then for $r > c_0$

$$\ln \mu_h(r) - \ln \mu_h(c_0) = \int_{c_0}^{r} \frac{\nu_f(t)}{t} dt \le \int_{c_0}^{r} \frac{l(\ln t)}{t} dt = \Psi(\ln r) - \Psi(\ln c_0),$$

and we see that $\ln \mu_h(r) \le 2\Psi(\ln r)$ for $r \ge r_3$. Therefore, for every $r \ge r_4$

$$M_h(r) = \sum_{n=0}^{\infty} b_n r^n = \sum_{n=0}^{\infty} b_n (2r)^n \frac{1}{2^n} \le \mu_h(2r) \sum_{n=0}^{\infty} \frac{1}{2^n} = 2\mu_h(2r) \le \mu_h(2r)^2 \le$$
$$\le \exp\{4\Psi(\ln 2r)\} \le \exp\{4\Psi(\ln r + 1)\} = \exp\{\Phi(\ln r)\},$$

i.e. $h \in \mathcal{A}(\Phi)$.

Further, from (7) we obtain $\mu_h(c_k) = b_{\nu_h(c_k)} c_k^{\nu_h(c_k)} = b_n c_k^n$ for each $n \in [n_k, n_{k+1}]$ and $k \ge 0$. Thus, by (6),

$$0 \le \ln \mu_h(c_k) = \ln b_{n_k} + n_k \ln c_k \le n_k \ln c_k \le n_{k+1}/k, \quad k \ge 1,$$

and, moreover, $N_k := n_{k+1} - n_k \sim n_{k+1}$ as $k \to \infty$. Hence, $\alpha_k := c \ln \mu_h(c_k)/N_k \to 0$ as $k \to \infty$, where $c = \text{const} \ge e^2$.

Put $k_0 = 0$. Since $\ln r = o(\ln \mu_h(r))$ as $r \to +\infty$, and series (8) converges for every $z = r > 0$, we can define

$$k_{p+1} = \min \left\{ m \ge k_p + 2 : \ \ln \sum_{n \le n_{k_p+1}} b_n c_{k_m}^n \le \frac{1}{p+1} \ln \mu_h(c_{k_m}), \ \sum_{n \ge n_{k_m}} b_n c_{k_p}^n \le 1 \right\} \tag{9}$$

for each $p \ge 0$.

Choose $a_n = b_n \delta(n - n_{k_p}, N_{k_p})$ for every $n \in [n_{k_p}, n_{k_p+1}]$ and $p \ge 0$, where $\delta(0, N_{k_p}), \ldots, \delta(N_{k_p}, N_{k_p})$ are numbers from Lemma 2, and let $a_n = 0$ in other cases. Consider power series (1) with the coefficients a_n and note that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{p=0}^{\infty} \sum_{n=n_{k_p}}^{n_{k_p+1}} a_n z^n.$$

Since $a_n \le b_n$ for $n \ge 0$, then $f \in \mathcal{A}(\Phi)$.

Now, we prove (2). From (i) of Lemma 2 we see that $\mu_f(c_{k_p}) = \mu_h(c_{k_p})$ for $p \ge 0$. Thus, for every $n \in [n_{k_p}, n_{k_p+1})$ and $p \ge 0$,

$$a_n c_{k_p}^n = b_n \delta(n - n_{k_p}, N_{k_p}) c_{k_p}^n = \mu_h(c_{k_p}) \delta(n - n_{k_p}, N_{k_p}) = \mu_f(c_{k_p}) \delta(n - n_{k_p}, N_{k_p}).$$

Applying (iii) of Lemma 2 we find that

$$\left| \sum_{n=n_{k_p}}^{n_{k_p+1}} a_n c_{k_p}^n e^{in\theta} \right| = \mu_f(c_{k_p}) \left| \sum_{j=0}^{N_{k_p}} \delta(j, N_{k_p}) e^{ij\theta} \right| \le$$
$$\le \exp\left\{ \frac{N_{k_p}}{c} (\cos\theta - 1 + \varepsilon_{N_{k_p}} + \alpha_{k_p}) \right\}, \quad \theta \in [0, 2\pi]. \tag{10}$$

Consider

$$f_p(z)=\sum_{n\le n_{k_{p-1}+1}} a_n z^n,\qquad g_p(z)=\sum_{n=n_{k_p}}^{n_{k_p+1}} a_n z^n,\qquad h_p(z)=\sum_{n\ge n_{k_{p+1}}} a_n z^n,$$

for $p\ge 2$. Since $f(z)=f_p(z)+g_p(z)+h_p(z)$, then [6, §1.2]

$$T_f(r)\le T_{f_p}(r)+T_{g_p}(r)+T_{h_p}(r)+\ln 3$$

for every $r\ge 0$. From (9) we get

$$T_{f_p}(c_{k_p})\le \ln^+ M_{f_p}(c_{k_p})\le \ln \sum_{n\le n_{k_{p-1}+1}} b_n c_{k_p}^n \le \frac{1}{p}\ln \mu_h(c_{k_p})=$$

$$=\frac{1}{p}\ln \mu_f(c_{k_p})=o(\ln M_f(c_{k_p}))\quad \text{as } p\to\infty; \tag{11}$$

$$T_{h_p}(c_{k_p})\le \ln^+ M_{h_p}(c_{k_p})\le \ln^+ \sum_{n\ge n_{k_{p+1}}} b_n c_{k_p}^n=0. \tag{12}$$

Further, since $0\le \varepsilon_{N_{k_p}}+\alpha_{k_p}\to 0$ as $p\to\infty$, then for $p\ge p_0$ we can define $\theta_p=\arccos(1-\varepsilon_{N_{k_p}}-\alpha_{k_p})$. It is clear that $\theta_p\to 0$ as $p\to\infty$. From (10) we have $|g_p(c_{k_p}e^{i\theta})|\le 1$, if $\theta\in[\theta_p,2\pi-\theta_p]$ and $p\ge 0$. Thus,

$$T_{g_p}(c_{k_p})=\left(\int_0^{\theta_p}+\int_{2\pi-\theta_p}^{2\pi}\right)\ln^+|g_p(c_{k_p}e^{i\theta})|d\theta\le 2\theta_p\ln^+ M_{g_p}(c_{k_p})\le$$

$$\le 2\theta_p \ln^+ M_f(c_{k_p})=o(\ln M_f(c_{k_p})),\quad \text{as } p\to\infty. \tag{13}$$

From (11), (12) and (13) we obtain $T_f(c_{k_p})=o(\ln M_f(c_{k_p}))$ as $p\to\infty$. So, the proof of Theorem is complete.

REFERENCES

1. Paley R. E. A. C. *A note on integral functions*, Proc. Cambridge Philos. Soc. **28** (1932), 262–265.
2. Гольдберг А. А., Островский И. В. *Об эффекте Пейли для целых характеристических функций и целых функций, представленных рядами Дирихле*, Теория функц., функц. анализ и их прил. **43** (1985), 18–23.
3. Zabolotskii N. V., Sheremeta M. N. *On the slow growth of the main characteristics of entire functions*, Math. Notes **65** (1999), No. 2, 168–174. Trans. from Mat. Zametki **65** (1999), No. 2, 206–214.
4. Filevych P. V. *On the slow growth of power series convergent in the unit disk*, Mat. Studii, **16** (2001), No. 2, 217–221.
5. Pólya G., Szegő G. Aufgaben und Lehrsätze aus der Analysis, Zweiter Band, Springer-Verlag, Berlin-Göttingen-Heidelberg-New York (1964).
6. Hayman W. K. Meromorphic functions, Clarendon Press, Oxford (1964).

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