УДК 517.53

M. T. Bordulyak, I. E. Chyzhykov*, M. M. Sheremeta

PRESERVATION OF l-INDEX BOUNDEDNESS UNDER ZEROS SHIFTS

M. T. Bordulyak, I. E. Chyzhykov, M. M. Sheremeta. *Preservation of l-index boundedness under zeros shifts*, Matematychni Studii, **19** (2003) 21–30.

For entire functions represented by canonical products with zeros on a finite system of curves of regular rotation conditions on shifts of zeros preserving boundedness of l-index are found.

М. Т. Бордуляк, И. Э. Чижиков, М. Н. Шеремета. *Сохранение ограниченности l-индекса при смещениях нулей* // Математичні Студії. – 2003. – Т.19, №1. – С.21–30.

Для целых функций, заданных каноническими произведениями с нулями на конечной системе кривых правильного вращения, найдены условия на смещения нулей, при которых сохраняется ограниченность l-индекса этих функций.

1. Introduction. Let Λ be the class of positive continuous functions l on $[0,+\infty)$ and Q be the class of functions $l\in\Lambda$ such that $l(r+O(1/l(r)))=O(l(r))$ $(r\to+\infty)$.

For $l\in\Lambda$ an entire function f is said to be of bounded l-index [1] if there exists $N\in\mathbb{Z}_+$ such that $\dfrac{|f^{(n)}(z)|}{n!l^n(|z|)}\le\max\left\{\dfrac{|f^{(k)}(z)|}{k!l^k(|z|)}:\,0\le k\le N\right\}$ for all $n\in\mathbb{Z}_+$ and $z\in\mathbb{C}$. For $l(x)\equiv1$ we obtain the definition of an entire function of bounded index (see [2]).

If $a_k\in\mathbb{C}$ are zeros of an entire function f, we put $n(r,z_0,1/f)=\sum\limits_{|a_k-z_0|\le r}1$, and $G_q(f)=\bigcup\limits_k\left\{z:\,|z-a_k|\le\frac{q}{l(|a_k|)}\right\}$ for $l\in\Lambda$, $q\in(0,+\infty)$.

G. Frike [3], [1, p.128] has proved that an entire function f of exponential type is a function of bounded index if and only if $|f'(z)/f(z)|\le M(\rho)<+\infty$ for arbitrary $\rho>0$ and all $z\in\mathbb{C}\setminus G_\rho(f)$ with $l(x)\equiv1$.

In the general case we have the following criterion ([4], [1, p.27]).

Lemma 1. *If $l\in Q$ then an entire function f is of bounded l-index if and only if*

1) *for every $q>0$ there exists $P(q)>0$ such that $|f'(z)/f(z)|\le P(q)l(|z|)$ for all $z\in\mathbb{C}\setminus G_q(f)$ and*
2) *for every $q>0$ there exists $n^*(q)\in\mathbb{N}$ such that $n(q/l(|z_0|),z_0,1/f)\le n^*(q)$ for each $z_0\in\mathbb{C}$.*

2000 *Mathematics Subject Classification:* 30D15.

* The investigation of the second author was partially supported by INTAS, project 99-00089.

Suppose that f is of bounded index. Then ([5], [1, p. 69]) $\ln M_f(r) = O(r)$, $r \to +\infty$, $M_f(r) = \max\{|f(z)| : |z| = r\}$. Therefore, by the Hadamard representation theorem either

$$f(z) = Az^m \pi(z), \quad \pi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right), \quad \text{if} \quad \int_1^{\infty} \frac{\ln M_f(r)}{r^2} dr < +\infty, \tag{2}$$

or

$$f(z) = Az^m e^{az} \pi(z), \quad \pi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{z/a_k}, \quad \text{if} \quad \int_1^{\infty} \frac{\ln M_f(r)}{r^2} dr = +\infty, \tag{3}$$

where $A \in \mathbb{C}$, $m \in \mathbb{Z}_+$, $a \in \mathbb{C}$, $a_k \in \mathbb{C} \setminus \{0\}$ and $|a_k| \nearrow +\infty$ $k \to +\infty$.

For a sequence $\psi = (\psi_k)$, $k \geq 0$, we define

$$f_\psi(z) = A(z - \psi_0)^m \pi_\psi(z), \quad \pi_\psi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k + \psi_k}\right), \quad \text{if} \quad \int_1^{\infty} \frac{\ln M_f(r)}{r^2} dr < +\infty, \tag{4}$$

and

$$f_\psi(z) = A(z - \psi_0)^m e^{az} \pi_\psi(z), \quad \pi_\psi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k + \psi_k}\right) e^{z/(a_k + \psi_k)},$$
$$\text{if} \quad \int_1^{\infty} \frac{\ln M_f(r)}{r^2} dr = +\infty. \tag{5}$$

In [6] M. M. Sheremeta has proved that if zeros (a_k) of the entire function f, which is necessary of the exponential type, lay on a finite number of rays gone from the origin, and $\psi_k = O(1)$, then the entire function f_ψ is also of bounded index. On the other hand, *for every positive continuous nondecreasing to $+\infty$ function ξ on $[0, +\infty)$ there exist an entire function f of bounded index and a sequence ψ such that $|\psi_k| \leq \xi(k)$ and f_ψ is of unbounded index.* It was conjectured [7] that it should be true without any restriction on zeros.

We are going to generalize the mentioned result of Sheremeta on boundedness of index for the functions f_ψ in two directions. First, we consider functions f with zeros on a finite number of so called *curves of regular rotation* introduced and investigated by Balashov (see for example [8]). Second, we provide unimprovable sufficient conditions for l-index boundedness of the functions f_ψ.

2. Functions with zeros on curves of regular rotation.

Recall the notion of the regular rolling curve ([8]). Let

$$L^\gamma = (z = re^{i\gamma(r)} : 0 \leq r_0 \leq r < \infty),$$

where $r_0 \in \mathbb{R}$, $\gamma \colon [r_0, \infty) \to \mathbb{R}$. The curve L^γ is called a *curve of regular rotation* if $\gamma \in C^1[r_0, \infty)$, and there exists

$$\lim_{r \to +\infty} r\gamma'(r) = c \in [0, +\infty).$$

Theorem 1. *Let f be a function of bounded index, and all its zeros, except, possibly, a finite number, lie on a finite number of curves of regular rotation. If $\sup_k |\psi_k| < +\infty$, then the function f_ψ is also of bounded index.*

Remark 1. Let p be an entire function, q be a function of bounded index and $f(z) = p(z)q(z)$. By the multiplication theorem [3], [1, p.34] f is of bounded index if and only if p is of bounded index. Since $q(z) = P(z)e^{az}$, where $a \in \mathbb{C}$, $P(z)$ is a polynomial, is a function of bounded index, in order to prove Theorem 1 it is sufficient to consider canonical products (2) and (3).

Remark 2. The condition $\sup_k |\psi_k| < +\infty$ cannot be improved.

Corollary 1. *Let f satisfy the conditions of Theorem 1, $\varphi_k \in [-\pi, \pi)$, $\psi_k = a_k e^{i\varphi_k} - a_k$. Then f_ψ with zeros $b_k = a_k e^{i\varphi_k}$ is of bounded index provided that $\sup_k |a_k \varphi_k| < +\infty$.*

In particular, if we rotate an infinite number of zeros of a function f, which is of bounded index, on a fixed angle, then f_ψ can be of unbounded index.

Example 1. Consider the function

$$\pi(z) = \frac{\sin \pi z}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \left(1 + \frac{z}{n}\right) e^{-z/n} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{6}$$

Let $\varphi = (\varphi_n)$ be a sequence of positive numbers. Rotate every negative zero $-n$ on the angle φ_n. We obtain the canonical product

$$\pi_\varphi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) \exp\left\{\frac{z}{n}\right\} \left(1 + \frac{z}{ne^{i\varphi_n}}\right) \exp\left\{\frac{-z}{ne^{i\varphi_n}}\right\}. \tag{7}$$

For this product we define as in [9, p. 42] the value

$$\delta(r) = \left| \sum_{n \le r} \left(\frac{1}{n} - \frac{1}{ne^{i\varphi_n}}\right) \right| = \sum_{n \le r} \frac{|1 - e^{i\varphi_n}|}{n}.$$

If $\varphi_n \equiv \varphi > 0$, then $\delta(r) = +\infty$, and by the Lindelöf theorem [9, p.42] π_φ has order 1 and maximal type, thus cannot be of bounded index. If $\varphi_n \to 0$ $(n \to +\infty)$, then $\delta(r) = +\infty$ as far as $\sum_n \varphi_n/n$ diverges. And again π_φ is of unbounded index.

To prove Theorem 1 we need the following lemma.

Lemma 2. *Let (a_k) be a sequence of complex numbers lying on a curve of regular rotation L^γ_φ ordered by increasing moduli, and satisfying the condition $|a_{k+1} - a_k| > h > 0$ $(k \ge 1)$. Then for arbitrary $\alpha \in (0, 1)$*

$$|a_{k+1}| - |a_k| > \frac{\alpha h}{\sqrt{1 + c^2}}, \quad k \to +\infty,$$

where $c = c(\gamma)$ is a constant from the definition of L^γ_φ.

Proof. Consider L^γ. Since $a_k = |a_k| e^{i\gamma(|a_k|)}$ and $r\gamma'(r) \to c$ $(r \to +\infty)$, for sufficiently large k we have

$$\gamma(|a_{k+1}|) - \gamma(|a_k|) = \int_{|a_k|}^{|a_{k+1}|} \gamma'(t)\, dt = (c + o(1)) \int_{|a_k|}^{|a_{k+1}|} \frac{dt}{t}.$$

Suppose that $|a_{k+1}| - |a_k| \le \alpha h/\sqrt{1 + c^2}$ for some $\alpha \in (0, 1)$ and a sequence of values k tending to $+\infty$. Then $|a_k| \sim |a_{k+1}|$ and

$$\gamma(|a_{k+1}|) - \gamma(|a_k|) = |c + o(1)|(|a_{k+1}| - |a_k|)/|a_k| \to 0$$

on this sequence of $k \uparrow +\infty$. Hence,

$$a_{k+1} - a_k = |a_{k+1}|e^{i\gamma(|a_{k+1}|)} - |a_k|e^{i\gamma(|a_{k+1}|)} + |a_k|e^{i\gamma(|a_{k+1}|)} - |a_k|e^{i\gamma(|a_k|)} =$$
$$= (|a_{k+1}| - |a_k|)e^{i\gamma(|a_{k+1}|)} + |a_k|e^{i\gamma(|a_k|)}(e^{i(\gamma(|a_{k+1}|)-\gamma(|a_k|))} - 1) =$$
$$= (|a_{k+1}| - |a_k|)e^{i\gamma(|a_k|)}e^{i(\gamma(|a_{k+1}|)-\gamma(|a_k|))} + |a_k|e^{i\gamma(|a_k|)}(i(\gamma(|a_{k+1}|) - \gamma(|a_k|))(1 + o(1)) =$$
$$= e^{i\gamma(|a_k|)}(|a_{k+1}| - |a_k|)(1 + ic + o(1)).$$

Consequently,

$$h \le |a_{k+1} - a_k| \le (1 + o(1))\sqrt{1 + c^2}(|a_{k+1}| - |a_k|) \le (1 + o(1))\alpha h,$$

which is impossible, because $\alpha < 1$. This contradiction proves the lemma. □

Proof of Theorem 1. Let $\pi(z)$ be a canonical product of form (2) or (3) of bounded index with zeros lying on a finite number of curves of regular rotation $L_j = L^{\gamma_j}$, $j \in \{1, \dots, m\}$, $c = \max_j c_j$, and $\sup_{k\ge1} |\psi_k| = H < +\infty$. Set $h = 3H$. Then, $h/2 - |\psi_k| \ge H/2 > 0$, $k \ge 1$. On the other hand, by Lemma 1 for any $s > 0$ there exists $n^*(s) \in \mathbb{N}$ such that for all $z_0 \in \mathbb{C}$ we have $n(s, z_0, 1/\pi) \le n^*(s)$. Put $s = 3H\sqrt{1 + c^2}$, thus, $n(3H\sqrt{1 + c^2}, z_0, 1/\pi) \le n^*(3H\sqrt{1 + c^2})$, $z_0 \in \mathbb{C}$, i.e. for any zero a_k there are at most $n^*(3H\sqrt{1 + c^2}) - 1$ zeros a_l with

$$|a_l - a_k| \le 3H\sqrt{1 + c^2}, \quad l \ne k. \tag{8}$$

Obviously, every sequence $(a_l^{(j)})$ of those zeros (a_l) which lie in L^{γ_j} still satisfies (8) (and every a_n belongs only to one sequence $(a_l^{(j)})$, $j \in \{1, \dots, m\}$). Therefore, every $(a_l^{(j)})$ can be represented as a union of at most $n^*(3H\sqrt{1 + c^2})$ sequences $(a_l^{(j,k)})$ ordered by increasing moduli and satisfying $|a_{l+1}^{(j,k)} - a_l^{(j,k)}| > 3H\sqrt{1 + c^2}, l \ge 1, j \in \{1, \dots, m\}$. By Lemma 2 $|a_{l+1}^{(j,k)}| - |a_l^{(j,k)}| \ge 3H = h$. By Theorem 3 from [6] the canonical product $\pi_\psi^{j,k}(z)$ with zeros $\{a_l^{(j,k)}\}$ is of bounded index. Now, by the multiplication theorem $\pi_\psi(z) = \prod_{j,k} \pi_\psi^{j,k}$ is also of bounded index. □

3. l-index boundedness of π_ψ.

We shall consider cases (2) and (3) separately. First, let $f(z)$ be of form (2). It is natural to consider l-index boundedness of f with $l(r) = o(1)$ $(r \to +\infty)$, $l \in Q$ (see, for example [10]).

Theorem 2. *Let $l(r)$ be a nonincreasing function on $[0, +\infty)$ such that $rl(r) \nearrow +\infty$ as $r \to +\infty$, and $f(z)$ of form (2) be of bounded l-index with positive zeros. If $|\psi_k| \le \frac{K_1}{l(a_k)}$ $(k \ge 1)$ where K is a constant, then $\pi_\psi(z)$ of form (4) is of bounded l-index.*

Proof of Theorem 2. Remark that our assumptions on $l(r)$ imply ([10]) that $l \in Q$. Moreover, it is easy to see that $2l(2r) \ge l(r) \ge l(2r)$, $r \ge 0$.

By Lemma 1 we have

$$(\forall z_0 \in \mathbb{C}) : \ n(q/l(|z_0|), z_0, 1/f) \le n^*(q). \tag{9}$$

Define the sequence R_n, $n \in \mathbb{Z}_+$, by the equalities $R_0 = 0$, $R_n = R_{n-1} + 6K_1/l(R_{n-1})$, $n \in \mathbb{N}$. Since $l(r)$ nonincreasing, $R_n \uparrow +\infty$ $(n \uparrow +\infty)$. By (9) a number of zeros a_n on $[R_{2(n-1)}, R_{2n}]$ does not exceed $n(6K_1/l(R_{2n-1}), R_{2n-1}, 1/f) \le n^*(6K_1)$. Set $I_n = (R_n, R_{n+1}]$.

From each interval I_{2m} we choose one from zeros (if there exists) of π and construct a canonical product π_1^* by such zeros, then we choose another second zero (if there exists) of π and construct a canonical product π_2^* by such zeros etc. So we construct the $n_1 \le n^*$ canonical products π_j^* with zeros $a_k^{(j)}$ satisfying the condition

$$a_{k+1}^{(j)} - a_k^{(j)} \ge \frac{6K_1}{l(a_{k+1}^{(j)} - 2/l(a_{k+1}^{(j)}))} \ge \frac{3K_1}{l(a_{k+1}^{(j)})}, \quad k \ge k_0.$$

Choosing by analogy zeros of π from each interval I_{2m+1}, we construct $n_2 \le n^*$ canonical products π_j^{**} with zeros satisfying the same condition.

Hence, $\pi(z) = \prod_{j=1}^{n_1+n_2} \pi_j$, where π_j are canonical products with zeros $a_k^{(j)}$ satisfying the condition $a_{k+1}^{(j)} - a_k^{(j)} \ge \frac{3K_1}{l(a_{k+1}^{(j)})}$. Let $(b_k^{(j)})$ be the corresponding sequence constructed by (b_k), $b_k = a_k + \psi_k$. Then, we have $(j \in \{1, \dots, n_1 + n_2\})$

$$|b_{k+1}^{(j)}| - |b_k^{(j)}| \ge a_{k+1}^{(j)} - \frac{K_1}{l(a_{k+1}^{(j)})} - a_k^{(j)} - \frac{K_1}{l(a_k^{(j)})} \ge \frac{K_1}{l(a_{k+1}^{(j)})} \ge \frac{K_1}{2l(|b_{k+1}^{(j)}|)}, \quad k \ge k_0. \qquad (10)$$

Clearly,

$$\left| \frac{\pi_\psi'(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)} \right| \le \sum_{j=1}^{n_1+n_2} \left| \frac{\pi_{j,\psi}'(z)}{\pi_{j,\psi}(z)} - \frac{\pi_j'(z)}{\pi_j(z)} \right|. \qquad (11)$$

Let $q \in (0, 3K_1/2)$, $\rho \in (0, \sigma)$, where $0 < \sigma < K_1/2$, $z \in \mathbb{C} \setminus (G_\rho(\pi) \cup G_q(\pi_\psi).)$ We need two lemmas.

Lemma 3. *Let $l(r)$ be positive and either $l(r)$ is nonincreasing and $rl(r) \nearrow +\infty$, or $l(r)$ is nondecreasing and $l(r)r^{-\kappa} \searrow 0$ as $r \to +\infty$, for some $\kappa \ge 1$, (d_k) a sequence of positive numbers such that $d_{k+1} - d_k \ge h/l(d_{k+1})$ $(k \ge 1)$, and $s \ge 0$. Then for every $r > 0$*

a)

$$\sum_{d_k > r} \frac{1}{d_k^s l(d_k)} = O\Big(\frac{1}{r^{s-1}}\Big), \quad s > 1;$$

b)

$$\sum_{d_k \le r} \frac{d_k^s}{l(d_k)} = O(r^{s+1}), \quad s \ge 0.$$

Remark 3. Under the assumptions of Lemma 3 $l \in Q$ (cf. [10, p.124]) and $\frac{1}{C_1} l(r) \le l(2r) \le C_1 l(r)$, $r > 0$ where C_1 is a positive constant.

Lemma 4. *Let $l(r)$ satisfy the conditions of Theorem 2, $|c_{k+1}| - |c_k| \ge 3K_1/l(|c_{k+1}|)$, $|\psi_k| \le K_1/l(|c_k|)$ $(k \ge 1)$. Then for every $\rho \in (0, 3K_1/2)$, $q \in (0, \sigma)$, where $\sigma < K_1/2$, there exists $P = P(\rho, q) > 0$ such that*

$$(\forall z \in \mathbb{C} \setminus (G_\rho \cup G_{q_\psi}) : \ \Big| \sum_{k=1}^{+\infty} \Big(\frac{1}{z - c_k} - \frac{1}{z - c_k - \psi_k} \Big) \Big| \le P(p, q) l(|z|),$$

where

$$G_\rho = \bigcup_k \left\{ z : \ |z - c_k| \le \frac{\rho}{l(|c_k|)} \right\}, \quad G_{q\psi} = \bigcup_k \left\{ z : \ |z - c_k - \psi_k| \le \frac{q}{l(|c_k + \psi_k|)} \right\}.$$

Proof of Lemma 3. Define inductively $r_{n+1} = r_n + 1/l(r_n)$, $n \in \mathbb{N}$, where $r_1 \in (0, d_1)$.

From the condition $d_{k+1} - d_k \geq h/l(d_{k+1})$ $(k \geq 1)$ it follows that $n_d(r_{n+1} - r_n) \leq n^*(h)$, where $n_d(r) = \sum_{d_k \leq r} 1$ is the counting function of the sequence (d_k). If l is nonincreasing, we have $r_n \geq r_1 + (n-1)/l(r_1) \to +\infty$ $(n \to +\infty)$. Otherwise l is nondecreasing, and in this case if (r_n) is bounded above, say by $r_* \in (0, +\infty)$, then $l(r_n) \leq l(r^*)$, $n \in \mathbb{N}$. Therefore, $r_{n+1} \geq r_1 + n/l(r^*)$, $n \in \mathbb{N}$, that contradicts the assumption on boundedness of (r_n). Hence, $r_n \uparrow +\infty$ $(n \to +\infty)$ in any case.

Further arguments concern the case when $l(r)$ is nonincreasing and $rl(r) \nearrow +\infty$. In the other case arguments are similar. Differences can be overcome using Remark 3.

Let $r \geq r_1$, then $r \in [r_m, r_{m+1})$ for some $m \in \mathbb{N}$. Using the definition of (r_k), and $l \in Q$ we obtain

$$\sum_{d_k > r} \frac{1}{d_k^s l(d_k)} \leq \sum_{k=m}^{+\infty} \sum_{r_k < d_j \leq r_{k+1}} \frac{1}{d_j^s l(d_j)} \leq$$
$$\leq \sum_{k=m}^{+\infty} \frac{n_d(r_{k+1}) - n_d(r_k)}{r_k^s l(r_{k+1})} \leq n^*(h) \sum_{k=m}^{+\infty} \frac{r_{k+2} - r_{k+1}}{r_k^s} =$$
$$= O\Big(\sum_{k=m}^{+\infty} \int_{r_{k+1}}^{r_{k+2}} \frac{dt}{t^s}\Big) = O\Big(\int_r^{+\infty} \frac{dt}{t^s}\Big) = O\Big(\frac{1}{r^{s-1}}\Big), \quad r \to +\infty.$$

Assertion a) is proved. Let us prove b). Similarly,

$$\sum_{d_k \leq r} \frac{d_k^s}{l(d_k)} \leq \sum_{k=1}^{m} \sum_{r_k < d_j \leq r_{k+1}} \frac{d_j^s}{l(d_j)} \leq$$
$$\leq \sum_{k=1}^{m} r_{k+1}^s \frac{n_d(r_{k+1}) - n_d(r_k)}{l(r_{k+1})} \leq n^*(h) \sum_{k=1}^{m} r_k^s (r_{k+2} - r_{k+1}) =$$
$$= O\Big(\sum_{k=1}^{m} \int_{r_{k+1}}^{r_{k+2}} t^s dt\Big) = O\Big(\int_{r_2}^{r_{m+2}} t^s dt\Big) = O\Big(r_{m+2}^{s+1}\Big) = O(r^{s+1}), \quad r \to +\infty.$$

□

Proof of Lemma 4. Let $z \in \mathbb{C} \setminus (G_\rho(\pi) \cup G_q(\pi_\psi))$ and $|c_n| \leq |z| \leq |c_{n+1}|$. We have

$$\Big|\sum_{k=1}^{+\infty} \Big(\frac{1}{z - c_k} - \frac{1}{z - c_k - \psi_k}\Big)\Big| \leq \sum_{k=1}^{n-1} \frac{|\psi_k|}{|z - c_k||z - c_k - \psi_k|} +$$
$$+ \frac{|\psi_n|}{|z - c_n||z - c_n - \psi_n|} + \frac{|\psi_{n+1}|}{|z - c_{n+1}||z - c_{n+1} - \psi_{n+1}|} + \sum_{k=n+2}^{\infty} \frac{|\psi_k|}{|z - c_k||z - c_k - \psi_k|}.$$

Using conditions of the lemma we obtain for $|c_k| \geq |z|/2$

$$\min\{|z - c_k|, |z - c_k - \psi_k|\} \geq |c_n| - |c_k| - |\psi_k| \geq \sum_{m=k}^{n-1} (|c_{m+1}| - |c_m|) - \frac{K_1}{l(|c_k|)} \geq$$
$$\geq \sum_{m=k}^{n-1} \frac{h}{l(|c_{m+1}|)} - \frac{K_1}{l(|c_k|)} \geq \frac{h(n - k - \frac{1}{3})}{l(|c_n|/2)}.$$

Therefore,

$$\sum_{\substack{|c_k|\ge|z|/2\\ k\le n+1}}\frac{|\psi_k|}{|z-c_k||z-c_k-\psi_k|}\le\sum_{\substack{|c_k|\ge|z|/2\\ k\le n+1}}\frac{K_1/l(|c_k|)}{\frac{h^2(n-k-1/3)^2}{l^2(|c_n|/2)}}\le$$

$$\le\frac{K_1l(\frac{|c_n|}{2})}{h^2}\sum_{k\le n-1}\frac{1}{(n-k-\frac{1}{3})^2}\le\frac{32K_1}{h^2}l(|z|). \tag{12}$$

Applying Lemma 3 b) with $s=0$, we obtain

$$\sum_{|c_k|\le|z|/2}\frac{|\psi_k|}{|z-c_k||z-c_k-\psi_k|}\le\sum_{|c_k|\le|z|/2}\frac{8K_1}{|z|^2l(|c_k|)}\le\frac{K_2}{|z|}\le K_3l(|z|). \tag{13}$$

Next, for $k\ge n+2$ we have

$$\min\{|c_k-z|,|c_k-\psi_k-z|\}\ge|c_k|-|\psi_k|-|c_{n+1}|\ge\sum_{m=n+2}^{k}\frac{h}{l(|c_m|)}-\frac{K_1}{l(|c_k|)}\ge\frac{h(k-n-\frac{4}{3})}{l(|c_{n+2}|)}.$$

Hence, using Lemma 3 a), we get

$$\sum_{k\ge n+2}\frac{|\psi_k|}{|z-c_k||z-c_k-\psi_k|}\le\sum_{\substack{|c_k|\le2|z|\\ k\ge n+2}}\frac{\frac{K_1}{l(|c_k|)}(l(|c_{n+2}|))^2}{h^2(k-n-\frac{4}{3})^2}+\sum_{|a_k|\ge2|z|}\frac{8|\psi_k|}{|c_k|^2}\le$$

$$\le\frac{8K_1}{h^2}l(|z|)+\sum_{|c_k|\ge2|z|}\frac{K_1}{|c_k|^2l(|c_k|)}\le\frac{8K_1}{h^2}l(|z|)+\frac{K_4}{|z|}\le K_5l(|z|). \tag{14}$$

Finally, since $z\notin(G_\rho\cup G_{q\psi})$, we have

$$\frac{|\psi_n|}{|z-c_n||z-c_n-\psi_n|}+\frac{|\psi_{n+1}|}{|z-c_{n+1}||z-c_{n+1}-\psi_{n+1}|}\le\frac{K_1}{\rho}\Big(\frac{1}{|z-c_n-\psi_n|}+\frac{1}{|z-c_{n+1}-\psi_{n+1}|}\Big).$$

Whence it is easy to conclude (see [10, (8), p.127]) that

$$\frac{|\psi_n|}{|z-c_n||z-c_n-\psi_n|}+\frac{|\psi_{n+1}|}{|z-c_{n+1}||z-c_{n+1}-\psi_{n+1}|}\le K(\rho,q)l(|z|). \tag{15}$$

The assertion of the lemma follows from (12)–(15). □

Proceed the proof of Theorem 2. Fix any $j\in\{1,n_1+n_2\}$ and consider

$$\left|\frac{\pi'_{j,\psi}(z)}{\pi_{j,\psi}(z)}-\frac{\pi'_j(z)}{\pi_j(z)}\right|=\left|\sum_{n=1}^{+\infty}\frac{\psi_k^{(j)}}{(z-a_k^{(j)})(z-b_k^{(j)})}\right|.$$

We can apply Lemma 4 to $(a_k^{(j)})$ with $h=K_1/2$. It implies that

$$\left|\frac{\pi'_{j,\psi}(z)}{\pi_{j,\psi}(z)}-\frac{\pi'_j(z)}{\pi_j(z)}\right|\le P_j(\rho,q)l(|z|),\quad z\in\mathbb{C}\setminus(G_\rho(\pi_j)\cup G_q(\pi_{j,\psi})).$$

Then

$$\left|\frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)}\right| \le \sum_{j=1}^{n_1+n_2} P_j(\rho, q)l(|z|) = P(\rho, q)l(|z|) < +\infty,$$

for such z. Since $\pi(z)$ is a function of bounded l-index, by Lemma 1 we have $|\pi'(z)/\pi(z)| \le P_1(\rho)l(|z|)$, $z \notin G_\rho(\pi)$. Hence,

$$\left|\frac{\pi'_\psi(z)}{\pi_\psi(z)}\right| \le (P_1(\rho) + P(p, q))l(|z|), \quad z \in \mathbb{C} \setminus \bigcup_{k\ge1}(C'_k \cup C''_k), \tag{16}$$

where $C'_k = \{z : |z - a_k - \psi| \le q/l(|a_k|\}$, $C''_k = \{z : |z - a_k| \le \rho/l(|a_k|)\}$, Now, let $q \in (0, 3K_1/2)$ and $\rho = q/3$.

If $C''_j \subset \bigcup_{k\ge1} C'_k$ then we can omit C''_j in the exceptional set of estimate (16). If $C''_j \cap \left(\bigcup_{k\ge1} C'_k\right) = \varnothing$ then (16) holds on $\partial C''_j$, and by the maximum modulus principle it holds on C''_j. Finally, if $C''_j \cap \left(\bigcup_{k\ge1} C'_k\right) \ne \varnothing$ and $\operatorname{Int} C''_j \cap \partial\left(\bigcup_{k\ge1} C'_k\right) \ne \varnothing$ then $C''_j \cap \tilde{C}_k = \varnothing$, $\tilde{C}_k = \{z : |z - a_k - \psi_k| \le q/(3l(|a_k|)\}$. Hence, (16) holds on C''_j with $P_1(q/3) + P(q/3, q/3)$. Hence, we have (16) for all $z \in \mathbb{C} \setminus \left(\bigcup_{k\ge1} C'_k\right)$, i.e. for all $z \in \mathbb{C} \setminus G_q(\pi)$. By Lemma 1, π_ψ is of bounded index. Theorem 2 is proved. □

Theorem 3. *Let $l(r)$ be a nondecreasing function on $[0, +\infty)$ such that $l(r)/r \searrow 0$ as $r \to +\infty$, and $f(z)$ of form (3) be of bounded l-index with positive zeros. If $|\psi_k| \le \frac{K_1}{l(a_k)}$ $(k \ge 1)$, then $\pi_\psi(z)$ of form (5) is of bounded l-index.*

Proof. Let π, π_ψ be the canonical products from (3) and from (5), respectively. Then

$$\left|\frac{\pi'_\psi(z)}{\pi_\psi(z)} - \frac{\pi'(z)}{\pi(z)}\right| \le \left|\sum_{k=1}^{\infty}\left(\frac{1}{b_k} - \frac{1}{a_k} + \frac{1}{z - b_k} - \frac{1}{z - a_k}\right)\right| \le$$
$$\le \sum_{k=1}^{\infty} \frac{|\psi_k|}{|a_k||b_k|} + \left|\sum_{k=1}^{\infty} \frac{\psi_k}{(z - a_k)(z - b_k)}\right|.$$

So we deal with an extra summand $\sum\limits_{k=1}^{\infty} \frac{|\psi_k|}{|a_k||b_k|}$ and another conditions on an index $l(r)$. In view of Remark 3 we can apply the arguments similar to that in the proof of Theorem 2 to prove

$$\left|\sum_{k=1}^{\infty} \frac{\psi_k}{(z - a_k)(z - b_k)}\right| \le P(\rho, q)l(|z|), \quad z \notin G_\rho(\pi) \cup G_q(\pi_\psi).$$

Further,

$$\sum_{k=1}^{\infty} \frac{|\psi_k|}{|a_k||b_k|} \le \sum_{k=1}^{+\infty} \frac{2K_1}{|a_k|^2 l(|a_k|)} < +\infty.$$

Standard arguments finish the proof of Theorem 3. □

Corollary 2. *Let f and l satisfy the conditions of Theorem 3, $\varphi_k \in [-\pi, \pi)$, $\psi_k = a_k e^{i\varphi_k} - a_k$. Then f_ψ with zeros $b_k = a_k e^{i\varphi_k}$ is of bounded index provided that $\sup_k |a_k \varphi_k| l(|a_k|) < +\infty$.*

Remark 4. In general, we cannot change the condition $|\psi_k| = O(1/l(|a_k|))$ in Theorems 2 and 3 by the condition $|\psi_k| = O(\gamma_k/l(|a_k|))$, where (γ_k) is unbounded.

Example 2. Indeed, consider the function

$$g(z) = \prod_{k=1}^{+\infty}\Big(1 - \frac{z^2}{n^{2/\rho}}\Big).$$

It has order ρ, and is of genus 1 or 2 when $\rho \in (0,2)$, with zeros $a_n = n^{1/\rho}$, $n \in \mathbb{Z}\setminus\{0\}$. It is easy to check (cf. [4]) that g is of bounded l-index with $l(r) = r^{\rho-1}$. Hence, $l(|a_n|) = n^{\frac{\rho-1}{\rho}}$. Suppose that $\gamma_n \nearrow +\infty$ $(n \to +\infty)$. Without loss of generality we may assume that $\gamma_n = o(n)$ $(n \to \infty)$. Let $n_{k+1} > 2n_k$ $(k \geq 1)$. Since

$$a_{n_k+[\gamma_{n_k}]} = (n_k + [\gamma_{n_k}])^{\frac{1}{\rho}} \leq n_k^{\frac{1}{\rho}} + \frac{2\gamma_{n_k} n_k^{1/\rho-1}}{\rho} \leq a_{n_k} + \frac{4\gamma_{n_k+[\gamma_{n_k}]}}{\rho l(a_{n_k+[\gamma_{n_k}]})},$$

we can put $b_m = a_{n_k}$ for $m \in \{n_k+1, \dots, n_k+[\gamma_{n_k}]\}$. Then

$$|a_m - b_m| \leq a_{n_k+[\gamma_{n_k}]} - a_{n_k} \leq \gamma_m \frac{1}{l(|a_m|)}.$$

But a_{n_k} is a zero of g_ψ of the multiplicity $[\gamma_{n_k}] \to +\infty$ $(k \to +\infty)$. By Lemma 1 1) this contradicts to l-index boundedness of g_ψ.

4. Further results.

1) Evidently, the assumption that zeros of f are positive in Theorems 2 and 3 is not necessary. Of course, it is sufficient to require that zeros lay on a finite number of rays gone from the origin as well as on a finite number of curves of regular rotation, because one can prove an analogue of Lemma 2 with $h/l(|a_k|)$ instead of h. This is possible, because $r + 1/l(r) \sim r$ $(r \to +\infty)$ under our restrictions on $l(r)$.

2) One can extend the assertion of Theorem 3 on canonical products of an arbitrary genus $p \in \mathbb{N}$ with the aid of Lemma 3. The condition $l(r)/r \searrow 0$ does not hold for arbitrary p, in general. But we can replace it by $l(r)r^{-\kappa} \searrow 0$ $(r \to +\infty)$ for some $\kappa \geq p$.

REFERENCES

1. Sheremeta M. M. Analytic functions of bounded index, Lviv: VNTL Publishers, 1999, 141 pp.
2. Lepson B. *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index,* Proc. Sympos. Pure Math., V.2. Amer. Math. Soc., Providence, Rhode Island, 1968, 298–307.
3. Fricke G. H. *Functions of bounded index and their logarithmic derivative,* Math. Ann. **206** (1973), 215–223.
4. Бордуляк М. Т., Шеремета М. Н. *О существовании целых функций ограниченного l-индекса и l-регулярного роста,* Укр. мат. журн. **48** (1996), №9, 1166–1182
5. Hayman W. K. *Differential inequalities and local valency,* Pacific J. Math. **44** (1973), 117–137.
6. Sheremeta M. M. *Problems in the theory of entire functions of bounded index and functions of sine type,* Matem. Studii. **15** (2001), №2, 217–224.
7. Sheremeta M. M. *Five open problems in theory of entire functions,* Matem. studii. **6** (1996), 157–159.

8. Балашов С. К. О целых функциях конечного порядка с корнями на кривых правильного вращения, Изв. АН СССР. Сер. мат. **37** (1973), 603–629.

9. Левин Б. Я. Распределение корней целых функций, М.: Гостехиздат, 1956, 632 с.

10. Chyzhykov I.E., Sheremeta M.M. *Boundedness of l-index for entire functions of zero genus,* Mat. Studii **16** (2001), №2, 124–130.

Faculty of Mechanics and Mathematics, Lviv Ivan Franko National University

Received 10.11.2002