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ON l-INDEX BOUNDEDNESS OF THE BLASCHKE PRODUCT

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Conditions on zeros under which the Blaschke product is an analytic function of bounded l-index in the unit disc are investigated.

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Исследованы условия на нули, при которых произведение Бляшке является аналитической в единичном круге функцией ограниченного l-индекса.

1. Introduction. Let (a_k) be a sequence of numbers from $\mathbb{D} = \{z : |z| < 1\}$, $|a_k| \le |a_{k+1}|$ for all $k \ge 1$, $\sum\limits_{k=1}^{\infty}(1 - |a_k|) < +\infty$, and let

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a}_k z}$$

be the Blaschke product.

For a positive continuous function l on $[0, 1)$ such that $(1-r)l(r) > \beta > 1$ for all $r \in [0, 1)$ function B by the definition [1, p. 71] is said to be of bounded l-index if there exists $N \in \mathbb{Z}_+$ such that

$$\frac{|B^{(n)}(z)|}{n! l^n(|z|)} \le \max\left\{\frac{|B^{(k)}(z)|}{k! l^k(|z|)} : 0 \le k \le N\right\}$$

for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}$. As in [1, p. 71], for $q \in [0, \beta)$ we put

$$\lambda_1(q) = \inf\left\{\frac{l(r)}{l(r_0)} : |r - r_0| \le \frac{q}{l(r_0)},\ 0 \le r_0 < 1\right\},$$

$$\lambda_2(q) = \sup\left\{\frac{l(r)}{l(r_0)} : |r - r_0| \le \frac{q}{l(r_0)},\ 0 \le r_0 < 1\right\}$$

and say that $l \in Q_\beta(\mathbb{D})$ if $(1 - r)l(r) > \beta > 1$ for all $r \in [0, 1)$ and $0 < \lambda_1(q) \le 1 \le \lambda_2(q) < +\infty$ for every $q \in [0, \beta)$.

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In [2] it is proved that if $\varliminf\limits_{n\to\infty} \dfrac{1-|a_n|}{1-|a_{n+1}|} > 1$ then B is of bounded l-index with $l(r) = \dfrac{p}{1-r}$, $p > 1$, and if $k(1-|a_k|) \searrow 0$ $(k \to \infty)$ then B is of bounded l-index with $l(r) = \dfrac{p}{(1-r)^2}$, $p > 1$. If the sequence $\left(\dfrac{1}{1-|a_n|}\right)$ is convex (hence it follows that $n(1-|a_n|) \searrow 0$, $n \to \infty$) then in [2] it is proved that B is of bounded l-index for an arbitrary nondecreasing function $l \in Q_\beta(\mathbb{D})$, $\beta > 1$, provided the following conditions hold:

a) $l(|a_{n+1}|) = O(l(|a_n|))$, $n \to \infty$;

b) $k \ln k = O((1-|a_k|)l(|a_k|))$, $\quad k \to \infty$;

c) $\sum\limits_{k=2n}^{\infty} (1-|a_k|) = O((1-|a_n|)^2 l(|a_n|))$, $\quad n \to \infty$.

From the method applied here we see that condition a) in last statement is unnecessary. It is easy to see that condition b) holds if $l \in Q_\beta(\mathbb{D})$, $\beta > 1$, and $l(r) \asymp \dfrac{n(r)\ln n(r)}{1-r}$ as $r \to 1$, where $n(r) = \sum\limits_{|a_k|\le r} 1$ is the counting function of the sequence (a_k). For such a function l the following theorem is true.

Theorem 1. *Let $n(1-|a_n|) \searrow 0$, $n \to \infty$. In order that the Blaschke product B is of bounded l-index with the function $l \in Q_\beta(\mathbb{D})$, $\beta > 1$, such that $l(r) \asymp \dfrac{n(r)\ln n(r)}{1-r}$ as $r \to 1$, it is sufficient and in the case when the zeros are positive it is necessary that*

$$\sum_{k=2n(r)}^{\infty} (1-|a_k|) = O((1-r)n(r)\ln n(r)), \quad r \to 1. \tag{1}$$

We remark that for the sequence $a_n = 1 - 1/n^2$ we have

$$\sum_{k=2n(r)}^{\infty} (1-|a_k|) \asymp \frac{1}{n(r)}, \quad n(r) \asymp \frac{1}{\sqrt{1-r}}$$

as $r \to 1$ and condition (1) holds. If $a_n = 1 - 1/(n \ln n \ln^2 \ln n)$ then

$$\sum_{k=2n(r)}^{\infty} (1-|a_k|) \asymp \frac{1}{\ln\ln n(r)}, \quad n(r) \asymp \left\{(1-r)\ln\frac{1}{1-r}\ln^2\ln\frac{1}{1-r}\right\}^{-1}$$

as $r \to 1$ and condition (1) does not hold.

2. Auxiliary Lemmas. For $l \in Q_\beta(\mathbb{D})$ and $q \in (0, \beta)$ we put

$$G_q(B) = \bigcup_k \left\{ z : |z-a_k| \le \frac{q}{l(|a_k|)} \right\}$$

and

$$n(r, z_0, 1/B) = \sum_{|a_k - z_0| \le r} 1.$$

The following lemma is an immediate corollary of Theorem 2.1 from [1, p. 27].

Lemma 1. *For $l \in Q_\beta(\mathbb{D})$, $\beta > 1$, the Blaschke product B is of bounded l-index if and only if*

1) *for every $q \in (0,\beta)$ there exists $P(q) > 0$ such that $|B'(z)/B(z)| \le P(q)l(|z|)$ for all $z \in \mathbb{D} \setminus G_q(B)$,*
2) *for every $q \in (0,\beta)$ there exists $n^*(q) \in \mathbb{N}$ such that $n(q/l(|z_0|), z_0, 1/B) \le n^*(q)$ for each $z_0 \in \mathbb{D}$.*

From Remark 1 in [2] we see that the following lemma is true.

Lemma 2. *Condition 2) of Lemma 1 holds provided $l \in Q_\beta(\mathbb{D})$, $\beta > 1$, and $|a_{k+1}| - |a_k| > 2q_0/l(|a_k|)$ for some $q_0 \in (0,\beta)$ and all $k \ge k_0$.*

Lemma 3. *If $n(1-|a_n|) \searrow 0$, $n \to \infty$, there exists a function $l \in Q_\beta(\mathbb{D})$, $\beta > 1$, such that $l(r) \asymp \dfrac{n(r)\ln n(r)}{1-r}$ as $r \to 1$ and condition 2) of Lemma 1 holds.*

Proof. We put $n_1(r) = 1 + r/|a_1|$ for $0 \le r \le |a_1|$ and $n_1(r) = n + 1 + (r - |a_n|)/(|a_{n+1}| - |a_n|)$ for $|a_n| \le r \le |a_{n+1}|$. Then the function $n_1(r)$ is continuous, $n(r) + 1 \le n_1(r) \le n(r) + 2$ and $n_1(r)(1-r) \searrow 0$ as $r \to 1$, because for $|a_n| < r < |a_{n+1}|$

$$(n_1(r)(1-r))' = \frac{1 - 2r + |a_n| - (n+1)(|a_{n+1}| - |a_n|)}{|a_{n+1}| - |a_n|} \le \\ \le \frac{(n+1)(1-|a_{n+1}|) - n(1-|a_n|)}{|a_{n+1}| - |a_n|} \le 0$$

Therefore, if we put $l(r) = \dfrac{n_1(r)\ln n_1(r)}{1-r}$ $(r_0 \le r < 1)$ then firstly $l(r) \nearrow +\infty$, $r \to 1$, and $l(r - q/l(r)) \le l(r) \le l(r + q/l(r))$ for $q > 0$ and secondly, since

$$n_1(r + q/l(r)) \le n_1(r)(1-r)/(1 - r - q/l(r)) = (1 + o(1))n_1(r),\ r \to 1,$$

we have $l(r + q/l(r)) \le (1 + o(1))l(r)$, $r \to 1$. By analogy,

$$n_1(r - q/l(r)) \ge n_1(r)(1-r)/(1 - r + q/l(r)) = (1 + o(1))n_1(r),\ r \to 1,$$

and $l(r - q/l(r)) \ge (1 + o(1))l(r)$, $r \to 1$. Hence it follows that $l \in Q_\beta(\mathbb{D})$.

Now,

$$|a_{n+1}| - |a_n| = \frac{n(1-|a_n|)}{n} - \frac{(n+1)(1-|a_{n+1}|)}{n+1} \ge \frac{1-|a_n|}{n\ln n}\frac{n\ln n}{n+1} \ge \frac{q}{l(|a_n|)}$$

for each $q > 0$ and all $n \ge n_0(q)$. Therefore, by Lemma 2, condition 2) of Lemma 1 holds. □

Lemma 4. *If $l \in Q_\beta(\mathbb{D})$ $(\beta > 1)$, $|a_n| \le |z| \le |a_{n+1}|$, $|z - a_n| \ge q/l(|a_n|)$ and $|z - a_{n+1}| \ge q/l(|a_{n+1}|)$, $0 < q < \beta$, then*

$$S_2(z) := \frac{1}{|z - a_n|} + \frac{1}{|z - a_{n+1}|} \le P_1(q)l(|z|), \quad P_1(q) \equiv const > 0. \tag{2}$$

Proof. If $|z-a_n| \geq q/l(|z|)$ and $|z-a_{n+1}| \geq q/l(|z|)$ then (6) holds with $P_1(q) = 2/q$. Suppose that $|z-a_n| < q/l(|z|)$ and $|z-a_n| \geq q/l(|a_n|)$. Then $|z|-q/l(|z|) \leq |a_n| \leq |z|+q/l(|z|)$ and, since $l \in Q_\beta(\mathbb{D})$, we have $l(|a_n|) \leq \lambda_2(q)l(|z|)$ and, therefore, $|z-a_n| \geq q/(\lambda_2(q)l(|z|))$. By analogy, if $|z-a_{n+1}| < q/l(|z|)$ and $|z-a_{n+1}| \geq q/l(|a_{n+1}|)$, then $|z-a_{n+1}| \geq q/(\lambda_2(q)l(|z|))$. Hence inequality (6) with $P_1(q) = 2\lambda_2(q)/q$ follows. □

Lemma 5. *If $n(1-|a_n|) \searrow 0$, $n \to \infty$, and $|a_n| \leq |z| \leq |a_{n+1}|$ then*

$$S_1(z) := \sum_{k=1}^{n-1} \frac{1-|a_k|}{(|z|-|a_k|)(1-|a_k||z|)} \leq \frac{n(r)\ln\, n(r)}{1-r} \quad (r = |z|).$$

Proof. First we remark that from the condition $n(1-|a_n|) \searrow 0$, $n \to \infty$, we have for $n > k$

$$\frac{1-|a_n|}{(1-|a_k|)-(1-|a_n|)} = \frac{n(1-|a_n|)}{k(1-|a_k|)(n/k)-n(1-|a_n|)} \leq \frac{k}{n-k}.$$

Therefore,

$$S_1(z) \leq \sum_{k=1}^{n-1} \frac{1}{r-|a_k|} = \frac{1}{1-r}\sum_{k=1}^{n-1} \frac{1}{(1-|a_k|)/(1-r)-1} \leq$$
$$\leq \frac{1}{1-r}\sum_{k=1}^{n-1} \frac{1}{(1-|a_k|)/(1-|a_n|)-1} \leq \frac{1}{1-r}\sum_{k=1}^{n-1}\frac{k}{n-k} \leq \frac{n\ln n}{1-r} = \frac{n(r)\ln n(r)}{1-r}.$$

□

Lemma 6. *If $n(1-|a_n|) \searrow 0$, $n \to \infty$, and $|a_n| \leq |z| \leq |a_{n+1}|$ then*

$$S_3(z) := \sum_{k=n+2}^{2n+1} \frac{1-|a_k|}{(|a_k|-|z|)(1-|a_k||z|)} \leq \frac{3n(r)\ln\, n(r)}{1-r} \quad (r = |z|).$$

Proof. As above, we have

$$S_3(z) \leq \sum_{k=n+2}^{2n+1} \frac{1}{|a_k|-r} = \frac{1}{1-r}\sum_{k=n+2}^{2n+1} \frac{1}{1-(1-|a_k|)/(1-r))} \leq$$
$$\leq \frac{1}{1-r}\sum_{k=n+2}^{2n+1} \frac{1}{1-(1-|a_k)/(1-|a_{n+1}|))} \leq \frac{1}{1-r}\sum_{k=n+2}^{2n+1}\frac{k}{k-(n+1)} \leq \frac{3n\ln n}{1-r} =$$
$$= \frac{3n(r)\ln n(r)}{1-r}.$$

□

Lemma 7. *If $n(1-|a_n|) \searrow 0$, $n \to \infty$, and $|a_n| \leq |z| \leq |a_{n+1}|$ then*

$$S_4(z) := \sum_{k=2(n+1)}^{\infty} \frac{1-|a_k|}{(|a_k|-|z|)(1-|a_k||z|)} \asymp \frac{1}{(1-r)^2}\sum_{k=2(n+1)}^{\infty}(1-|a_k|) \quad (r = |z| \to 1).$$

Proof. Since $(|a_k|-r)(1-|a_k|r)\le(1-r)^2$, we have

$$S_4(z)\ge\frac{1}{(1-r)^2}\sum_{k=2(n+1)}^{\infty}(1-|a_k|).$$

On the other hand, for $k\ge 2(n+1)$ we have $1-|a_k|r\ge|a_k|(1-r)\ge|a_{2n+2}|(1-r)$ and

$$|a_k|-r\ge(1-r)\left(1-\frac{1-|a_k|}{1-|a_{n+1}|}\right)\ge(1-r)\left(1-\frac{n+1}{k}\right)\ge\frac{1-r}{2}.$$

Therefore,

$$S_4(z)\le\frac{2}{|a_{2n+2}|(1-r)^2}\sum_{k=2(n+1)}^{\infty}(1-|a_k|).$$

Lemma 7 is proved. □

3. Proof of Theorem. Suppose that condition (1) holds. In view of Lemma 3 we only need to prove that condition 1) of Lemma 1 holds. Since

$$\frac{B'(z)}{B(z)}=\sum_{k=1}^{\infty}\frac{1-|a_k|^2}{(z-a_k)(1-a_kz)},$$

if $|a_n|\le|z|\le|a_{n+1}|$ and $z\notin G_q(B)$ by Lemmas 4 – 7 in view of (1) we have

$$\frac{|B'(z)|}{|B(z)|}\le 2\sum_{j=1}^{4}S_j(z)\le 2(3+P_1(q))\frac{3n(r)\ln n(r)}{1-r}+\frac{2}{|a_{n+2}|(1-r)^2}\sum_{k=2(n+1)}^{\infty}(1-|a_k|)\le$$
$$\le P_2(q)\frac{3n(r)\ln n(r)}{1-r}\le P(q)l(|z|).$$

Thus, for all $|z|\ge|a_1|$ and $z\notin G_q(B)$ we obtain $|B'(z)/B(z)|\le P(q)l(|z|)$. For $|z|\le|a_1|$ and $z\notin G_q(B)$ we prove the estimate $|B'(z)/B(z)|\le P(q)l(|z|)$ (perhaps, with another constant $P(q)$) using the maximum modulus principle and positivity of l. The sufficiency of (1) is proved.

In order to prove necessity of (1) we write for $z=r>0$

$$\frac{|B'(r)|}{|B(r)|}=\left|\sum_{k=1}^{2n+1}\frac{1-a_k^2}{(r-a_k)(1-a_kr)}+\sum_{k=2(n+1)}^{\infty}\frac{1-a_k^2}{(r-a_k)(1-a_kr)}\right|\ge$$
$$\ge\sum_{k=2(n+1)}^{\infty}\frac{1-a_k}{(r-a_k)(1-a_kr)}-\left|\sum_{k=1}^{2n+1}\frac{1-a_k^2}{(r-a_k)(1-a_kr)}\right|.$$

Using Lemmas 4 – 7, we therefore obtain

$$\frac{|B'(r)|}{|B(r)|}\ge\frac{1}{(1-r)^2}\sum_{k=2(n+1)}^{\infty}(1-a_k)+O\left(\frac{n(r)\ln n(r)}{1-r}\right),\quad r\to 1.$$

Thus, if condition (1) does not hold then condition 1) of Lemma 1 for positive z does not hold. By Lemma 1, B is of unbounded l-index. The proof of Theorem is complete.

4. Remarks.

Since $\sum_{k=n(r)}^{2n(r)+1}(1-|a_k|)\le(1-r)(n(r)+1)$ and $\sum_{k=n(r)}^{\infty}(1-|a_k|)\asymp\int_r^1 n(t)dt,\quad r\to1$, it is natural to consider the function $l(r)=(1-r)^{-2}\int_r^1 n(t)dt$.

It is easy to show that the function $l\in Q_\beta(\mathbb{D})$ $(\beta>1)$ and, using auxiliary lemmas to prove the following

Proposition 1. *Let $n(1-|a_n|)\searrow 0$, $n\to\infty$. If $(1-r)n(r)\ln n(r)=O\left(\int_r^1 n(t)dt\right)$ as $r\to1$ then the Blaschke product B is of bounded l-index with $l(r)=(1-r)^{-2}\int_r^1 n(t)dt$.*

From the proof of Theorem we also see that the following proposition is true.

Proposition 2. *Let $n(1-|a_n|)\searrow 0$, $n\to\infty$, and $l\in Q_\beta(\mathbb{D})$ $(\beta>1)$ be a function such that $n(r)\ln n(r)=O((1-r)l(r))$ and $\int_r^1 n(t)dt=O((1-r)^2l(r))$ as $r\to1$. Then the Blaschke product B is of bounded l-index.*

From $\sum_{k=1}^{\infty}(1-|a_k|)<+\infty$ and $n(1-|a_n|)\searrow 0$, $n\to\infty$, we have for every $\epsilon>0$ and all $n\ge n_0(\epsilon)$

$$\epsilon\ge\sum_{\sqrt{n}\le k\le n}(1-|a_k|)=\sum_{\sqrt{n}\le k\le n}\frac{k(1-|a_k|)}{k}\ge\frac{n(1-|a_n|)\ln n}{2}$$

and, thus, $\dfrac{n(r)\ln n(r)}{1-r}=o(\dfrac{1}{(1-r)^2})$, $r\to1$. Therefore, since $\int_r^1 n(t)dt=o(1)$, $r\to1$, from Proposition 2 we obtain the following refinement of Theorem 2 from [2].

Proposition 3. *If $n(1-|a_n|)\searrow 0$, $n\to\infty$, then there exists a function $l\in Q_\beta(\mathbb{D})$ $(\beta>1)$ such that $l(r)=o((1-r)^{-2})$ as $r\to1$ and the Blaschke product B is of bounded l-index.*

On the other hand, the following proposition is true.

Proposition 4. *For any function $l\in Q_\beta(\mathbb{D})$ $(\beta>1)$ such that $l(r)=o((1-r)^{-2})$ as $r\to1$ there exists a Blaschke product B of unbounded l-index with zeroes a_n such that $n(1-|a_n|)\searrow 0$, $n\to\infty$.*

Indeed, there exists a function $\omega(x)\uparrow+\infty$, $x\to\infty$, such that

$$l(r)=\frac{1}{(1-r)^2\omega(1/(1-r))}.$$

We may assume that ω is a slowly increasing function such that $\dfrac{x\omega'(x)}{\omega(x)}\ln x\to0$, $x\to\infty$, because if $l_1(r)\le l_2(r)$ and B of unbounded l_2-index then [1, p. 23] B of unbounded l_1-index.

Since $\frac{x}{\omega(x)\ln x} \uparrow +\infty$, $x_0 \le x \to \infty$, we can choose a positive sequence (a_n) such that $n = \frac{1}{(1-a_n)\omega(1/(1-a_n))\ln(1/(1-a_n))}$. Then $n(1-|a_n|) \searrow 0$, $n \to \infty$,

$$n(r) \asymp \frac{1}{(1-r)\omega(1/(1-r))\ln(1/(1-r))}$$

and $\frac{n(r)\ln n(r)}{1-r} \asymp \frac{1}{(1-r)^2\omega(1/(1-r))} = l(r)$ as $r \to 1$.

On the other hand, for some $q > 0$ by l'Hospital rule we have

$$\varliminf_{r\to 1} \frac{\int_r^1 n(t)dt}{(1-r)n(r)\ln n(r)} \le q \varliminf_{r\to 1} \omega\left(\frac{1}{1-r}\right) \int\limits_r^1 \frac{dt}{(1-t)\omega(1/(1-t))\ln(1/(1-t))} =$$

$$= q \varliminf_{x\to+\infty} \omega(x) \int\limits_x^\infty \frac{dt}{t\omega(t)\ln t} \ge q \varliminf_{x\to+\infty} \frac{\omega(x)^2}{x\omega(x)\ln x\,\omega'(x)} = +\infty,$$

i.e. B is of unbounded l-index.

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